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On the Rigidity of the Sasakian Structure and Characterization of Cosymplectic Manifolds

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Abstract: We introduce new metric structures on a smooth manifold (called "weak" structures) that generalize the almost contact, Sasakian, cosymplectic, etc. metric structures (φ, ξ, η, g) and allow us to take a fresh look at the classical theory and find new applications. This assertion is illustrated by generalizing several well-known results. It is proved that any Sasakian structure is rigid, i.e., our weak Sasakian structure is homothetically equivalent to a Sasakian structure. It is shown that a weak almost contact structure with parallel tensor φ is a weak cosymplectic structure and an example of such a structure on the product of manifolds is given. Conditions are found under which a vector field is a weak contact vector field.

Keywords: weak contact structure; Sasakian structure; cosymplectic structure; contact vector field

MSC: 53C15; 53C25; 53D15

Introduction

The methods of contact geometry have evolved from the mathematical formalism of classical mechanics and play an important role in modern mathematics and theoretical physics. For a review of the theory of contact metric structure and related structures, see [1–4]. There are inclusions $C_3 \subset C_2 \subset C_1$ and $C_4 \subset C_1$ of well-known classes of metric structures on a manifold: almost contact C_1, contact C_2, Sasakian C_3 and cosymplectic C_4, respectively.

We introduce new metric structures on a smooth manifold (called "weak" structures, see particular case in [5] and ([6], Section 5.3.8)) that generalize the almost contact, Sasakian, cosymplectic, etc. metric structures (φ, ξ, η, g) and allow us to take a fresh look at the theory of these classical structures. To demonstrate this statement, we investigate the geometry of such weak structures and generalize several well-known results on Sasakian and cosymplectic metric structures and contact vector fields. Our metric structures: weak almost contact C_1^w, weak contact C_2^w, weak Sasakian C_3^w and weak cosymplectic C_4^w, form wider classes than the classical structures, i.e., the complex structure on im φ is replaced by a nonsingular skew-symmetric tensor, see (1) in what follows. Using the homothetical equivalence relation, we define the classes $C^w_{i/\sim}$ of our structures with natural inclusions $C^w_{4/\sim} \subset C^w_{1/\sim}$ and $C^w_{3/\sim} \subset C^w_{2/\sim} \subset C^w_{1/\sim}$. Any metric structure C_i mentioned above belongs to a family of weak structures that are homothetically equivalent to the classical structure; thus, $C_i \subset C^w_{i/\sim}$. A natural question arises: *How rich are new structures $C^w_{i/\sim}$ compared to the classical ones C_i*? In this article, we answer this question for weak Sasakian and weak cosymplectic structures.

The study of this question for weak contact metric manifolds and further generalization of classical results on the curvature and topology of contact metric manifolds and submanifolds in them and their connection with such geometrical structures as warped products and Ricci solitons is postponed to the future. The theory presented here can deepen our knowledge of the geometry of contact and symplectic manifolds as well as find new applications in theoretical physics.

This article consists of an introduction and six sections. In Section 1, we introduce new metric structures C_i^w, define their homothetical equivalence classes, and discuss the properties of our structures in order to demonstrate their similarity with the corresponding classical structures. In

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Sections 2 and 3, we introduce a new tensor $N^{(5)}$ to calculate the covariant derivative of φ for weak contact metric manifolds and study the tensor h. In Section 4 we show that any weak Sasakian structure is homothetically equivalent to a Sasakian structure, i.e., Sasakian structures are rigid in some sense. In Section 5, we study weak almost contact metric structure and show that such metric structure with parallel tensor φ is a weak cosymplectic structure, we also give an example of such a structure on the product of a manifold with a line or a circle. In Section 6, we find conditions for a vector field to be a weak contact vector field. The proofs use the properties of new tensors, as well as the constructions required in the classical case.

1. Preliminaries

Here, we define new metric structures that generalize the almost contact metric structure. A *weak almost contact structure* on a smooth odd-dimensional manifold M^{2n+1} is a set (φ, Q, ξ, η), where φ is a $(1,1)$-tensor, ξ is the characteristic vector field and η is a dual 1-form, satisfying

$$\varphi^2 = -Q + \eta \otimes Q\,\xi, \quad \eta(\xi) = 1, \tag{1}$$

and Q is a nonsingular $(1,1)$-tensor field such that

$$Q\,\xi = \nu\,\xi \tag{2}$$

for a constant $\nu > 0$, see [5] and ([6], Section 5.3.8) where $\nu = 1$. By $\eta(\xi) = 1$, the form η determines a smooth $2n$-dimensional contact distribution $\mathcal{D} := \ker\eta$, defined by the subspaces $\mathcal{D}_m = \{X \in T_m M : \eta(X) = 0\}$ for $m \in M$. We assume that the distribution $\mathcal{D}$ is φ-invariant,

$$\varphi X \in \mathcal{D}, \quad X \in \mathcal{D}, \tag{3}$$

as in the classical theory of almost contact structure [1], where $Q = \mathrm{id}_{TM}$. By (1) and (3), the distribution $\mathcal{D}$ is invariant for Q: $Q(\mathcal{D}) = \mathcal{D}$. If there is a Riemannian metric g on M such that

$$g(\varphi X, \varphi Y) = g(X, Q\,Y) - \eta(X)\,\eta(Q\,Y), \quad X, Y \in \mathfrak{X}_M, \tag{4}$$

then $(\varphi, Q, \xi, \eta, g)$ is called a *weak almost contact metric structure* on M and g is called a *compatible* metric. A weak almost contact manifold $M(\varphi, Q, \xi, \eta)$ endowed with a compatible Riemannian metric g is said to be a *weak almost contact metric manifold* and is denoted by $M(\varphi, Q, \xi, \eta, g)$.

Putting $Y = \xi$ in (4) and using $Q\,\xi = \nu\,\xi$ and $\nu \neq 0$, we get, as in the classical theory,

$$\eta = \iota_\xi\, g, \tag{5}$$

where ι (" iota") is the interior product operation. In particular, ξ is g-orthogonal to $\mathcal{D}$ for any compatible metric g.

The following statement (a) generalizes ([1], Theorem 4.1).

Proposition 1. (a) *For a weak almost contact structure on M, the tensor φ has rank $2n$ and*

$$\varphi\,\xi = 0, \quad \eta \circ \varphi = 0, \quad [Q,\, \varphi] = 0.$$

(b) *For a weak almost contact metric structure, φ is skew-symmetric and Q is self-adjoint,*

$$g(\varphi X, Y) = -g(X, \varphi Y), \quad g(QX, Y) = g(X, QY). \tag{6}$$

Proof. (a) By (1), $\varphi^2\xi = 0$, hence, either $\varphi\,\xi = 0$ or $\varphi\,\xi$ is a nontrivial vector of $\ker\varphi$. Applying (1) to $\varphi\,\xi$, we get $Q(\varphi\,\xi) = \eta(\varphi\,\xi)Q\,\xi$. If $\varphi\,\xi = \mu\,\xi$ for a nonzero function $\mu : M \to \mathbb{R}$, then $0 = \varphi^2\xi =$

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$\mu\cdot\varphi\,\xi=\mu^2\xi\ne 0$ – a contradiction. Assuming $\varphi\,\xi=\mu\,\xi+X$ for some $\mu: M\to\mathbb{R}$ and nonzero $X\in\ker\eta$, by (1) we get $QX=0$ – a contradiction to non-singularity of Q. Thus, $\varphi\,\xi=0$.

Next, since $\varphi\,\xi=0$ everywhere, rank $\varphi<2n+1$. If a vector field ξ' satisfies $\varphi\,\xi'=0$, then (1) gives $Q\,\xi'=\eta(\xi')Q\,\xi$. One may write $\xi'=\mu\xi+X$ for some $\mu: M\to\mathbb{R}$ and $X\in\ker\eta$. This yields $Q\,(\mu\xi+X)=\eta(\mu\xi+X)Q\,\xi$, i.e., $QX=0$, hence, ξ' is collinear with ξ, and so rank $\varphi=2n$.

To show $\eta\circ\varphi=0$, note that $\eta(\varphi\,\xi)=\eta(0)=0$, and, using (3), we get $\eta(\varphi X)=0$ for $X\in\mathcal{D}$. Using (1) and $\varphi(Q\,\xi)=\nu\,\varphi\,\xi=0$, we get

$$\varphi^3X=\varphi(\varphi^2X)=-\varphi\,QX+\eta(X)\,\varphi(Q\,\xi)=-\varphi\,QX,$$
$$\varphi^3X=\varphi^2(\varphi X)=-Q\,\varphi X+\eta(\varphi X)Q\,\xi=-Q\,\varphi X$$

for any $X\in\mathcal{D}$, that proves $[Q,\,\varphi]=Q\,\varphi-\varphi Q=0$.

(b) By (4), the restriction $Q_{\,|\mathcal{D}}$ is self-adjoint. This and (2) provide $(6)_2$. For any $Y\in\mathcal{D}$ there is $\tilde Y\in\mathcal{D}$ such that $\varphi\,Y=\tilde Y$. Thus, $(6)_1$ follows from (4) and $(6)_2$ for $X\in\mathcal{D}$ and $\tilde Y$. □

Remark 1. According to [5], a weak almost contact structure admits a compatible metric if φ in (1)–(3) has a skew-symmetric representation, i.e., for any $x\in M$ there exist a neighborhood $U_x\subset M$ and a frame $\{e_i\}$ on U_x, for which φ has a skew-symmetric matrix. By (4), we get $g(X,Q\,X)=g(\varphi X,\varphi X)>0$ for any nonzero vector $X\in\mathcal{D}$; thus, Q is positive definite.

The fundamental 2-form Φ of (φ,Q,ξ,η,g) is defined (as in the classical case) by

$$\Phi(X,Y)=g(X,\varphi Y),\quad X,Y\in\mathfrak{X}_M.$$

We introduce a *weak contact metric structure* as a weak almost contact metric structure satisfying

$$d\eta=\Phi, \tag{7}$$

where

$$d\eta(X,Y)=\frac{1}{2}\,\{X(\eta(Y))-Y(\eta(X))-\eta([X,Y])\},\quad X,Y\in\mathfrak{X}_M. \tag{8}$$

Recall the following formulas (with $X\in\mathfrak{X}_M$):

$$(\pounds_Z\,\varphi)(X)=[Z,\varphi X]-\varphi[Z,X], \tag{9}$$
$$(\pounds_Z\,\eta)(X)=Z(\eta(X))-\eta([Z,X]), \tag{10}$$

where $\pounds_Z$ is the Lie derivative in the Z-direction. A weak contact metric structure (φ,Q,ξ,η,g), for which ξ is Killing, i.e., $\pounds_\xi\,g=0$, is called a *weak K-contact structure*.

A weak almost contact structure (φ,Q,ξ,η) on a manifold M will be called *normal* if the following tensor (which is well-known for $Q=\mathrm{id}_{TM}$, e.g., [1]) is identically zero:

$$N^{(1)}(X,Y)=[\varphi,\varphi](X,Y)+2\,d\eta(X,Y)\,Q\,\xi,\quad X,Y\in\mathfrak{X}_M. \tag{11}$$

The Nijenhuis torsion $[\varphi,\varphi]$ of φ is given by

$$[\varphi,\varphi](X,Y)=\varphi^2[X,Y]+[\varphi X,\varphi Y]-\varphi[\varphi X,Y]-\varphi[X,\varphi Y],\quad X,Y\in\mathfrak{X}_M. \tag{12}$$

Remark 2. The Levi-Civita connection ∇ of a Riemannian metric g is given by, e.g. [6],

$$2\,g(\nabla_XY,Z)=X\,g(Y,Z)+Y\,g(X,Z)-Z\,g(X,Y)$$
$$+\,g([X,Y],Z)+g([Z,X],Y)-g([Y,Z],X), \tag{13}$$

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and has the properties $[X,Y] = \nabla_X Y - \nabla_Y X$ and $X\,g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$. Thus, (12) can be written in terms of $\nabla\varphi$ as

$$[\varphi,\varphi](X,Y) = (\varphi\nabla_Y\varphi - \nabla_{\varphi Y}\varphi)X - (\varphi\nabla_X\varphi - \nabla_{\varphi X}\varphi)Y; \tag{14}$$

in particular, since $\varphi\,\xi = 0$,

$$[\varphi,\varphi](X,\xi) = \varphi(\nabla_\xi\,\varphi)X + \nabla_{\varphi X}\,\xi - \varphi\,\nabla_X\,\xi, \quad X \in \mathfrak{X}_M. \tag{15}$$

A normal weak contact metric manifold will be called a *weak Sasakian manifold*.

Next, we define an equivalence relation on the set of weak almost contact structures C^w_1 and on its important subsets of weak contact metric structures C^w_2 and weak Sasakian structures C^w_3. The factor-spaces are denoted by $C^w_{i/\sim}$ for $i = 1,2,3$. One can easily verify that the following definition is correct (see the proof of Proposition 2).

Definition 1. (i) Two weak almost contact structures (φ, Q, ξ, η) and $(\varphi', Q', \xi, \eta)$ on M are said to be *homothetically equivalent* if

$$\varphi = \sqrt{\lambda}\,\varphi', \tag{16a}$$
$$Q\,|_{\mathcal{D}} = \lambda\,Q'\,|_{\mathcal{D}}, \quad Q\,\xi = \lambda'\,Q'\xi \tag{16b}$$

for some positive numbers λ and λ'.
(ii) Two weak contact metric structures $(\varphi, Q, \xi, \eta, g)$ and $(\varphi', Q', \xi, \eta, g')$ on M are said to be *homothetically equivalent* if they satisfy conditions (16a,b) and

$$g|_{\mathcal{D}} = \lambda^{-\frac{1}{2}}\,g'|_{\mathcal{D}}, \quad \iota_\xi\,g = \iota_\xi\,g'. \tag{16c}$$

(iii) Two weak Sasakian structures $(\varphi, Q, \xi, \eta, g)$ and $(\varphi', Q', \xi, \eta, g')$ on M are *homothetically equivalent* if they satisfy conditions (16a-c) and

$$\lambda = \lambda'.$$

In our modification of classical conditions we are motivated by the following.

Proposition 2. *Let (φ, Q, ξ, η) be a weak almost contact structure on M such that*

$$Q\,|_{\mathcal{D}} = \lambda\,\mathrm{id}_{\mathcal{D}}, \quad Q\,\xi = \nu\,\xi$$

for some $\lambda, \nu \in \mathbb{R}_+$. Then the following is true:
(i) *(φ', ξ, η) is an almost contact structure on M, where φ' is given by*

$$\varphi = \sqrt{\lambda}\,\varphi'. \tag{17a}$$

(ii) *if $(\varphi, Q, \xi, \eta, g)$ is a weak contact metric structure on M with conditions (17a) and (16c), then $(\varphi', \xi, \eta, g')$ is a contact metric structure on M.*
(iii) *if $(\varphi, Q, \xi, \eta, g)$ is a weak Sasakian structure on M with conditions (17a), (16c) and*

$$\nu = \lambda,$$

then $(\varphi', \xi, \eta, g')$ is a Sasakian structure on M.

Proof. (i) Substituting $\varphi = \gamma\,\varphi'$ in (1), for ξ we get identity, and for $X \in \mathcal{D}$ we get

$$\gamma^2(\varphi')^2 X = \lambda\, X,$$

which reduces to the classical equality when $\gamma = \sqrt{\lambda}$. Thus, (φ', ξ, η) is an almost contact structure on M.

(ii) Let $g|_{\mathcal{D}} = \beta\, g'|_{\mathcal{D}}$ and $\iota_\xi\, g = \delta\,(\iota_\xi\, g')$ for some functions β and δ on M. Using (4), we get

$$\lambda\,\beta\, g'(\varphi' X, \varphi' Y) = \lambda\,\beta\, g'(X, Q'Y) - \lambda\,\eta(X)\,\eta(Q'Y) = \lambda\,\beta\, g'(X, Q'Y), \quad X, Y \in \mathcal{D},$$

– no restrictions for β and λ. The same is when $X = \xi$ or/and $Y = \xi$. Next, we calculate

$$\delta\,(\iota_\xi\, g')(Y) \overset{(16c)}{=} (\iota_\xi\, g)(Y) = g(\xi, Y) \overset{(5)}{=} \eta(Y) \overset{(5)}{=} g'(\xi, Y) = (\iota_\xi\, g')(Y),$$

which gives $\delta = 1$. Next, substituting the values of $(\varphi, Q, \xi, \eta, g)$ in (7), we get

$$d\eta(X, Y) = \beta\gamma\, g'(X, \varphi' Y). \tag{18}$$

For $X, Y \in \mathcal{D}$, (18) is valid for $\beta\gamma = 1$, i.e., $\beta = \frac{1}{\sqrt{\lambda}}$, and for $X = \xi$ or/and $Y = \xi$ we find $d\eta(\xi, Y) = 0$, see Proposition 3 below. Hence, $(\varphi', \xi, \eta, g')$ is a contact metric structure on M.

(iii) Substituting the values of $(\varphi, Q, \xi, \eta, g)$ in the equality $N^{(1)}(X, Y) = 0$ and using the property of Lie bracket, we get

$$\lambda\,[\varphi', \varphi'](X, Y) + 2\,\nu\, d\eta(X, Y)\,\xi = 0$$

for any $X, Y \in \mathfrak{X}_M$, which reduces to the classical case when $\nu = \lambda$. Hence, $(\varphi', \xi, \eta, g')$ is a Sasakian structure on M. □

The three tensors $N^{(2)}, N^{(3)}$ and $N^{(4)}$ are well known in the classical theory, see [1]:

$$N^{(2)}(X, Y) = (\pounds_{\varphi X}\,\eta)(Y) - (\pounds_{\varphi Y}\,\eta)(X), \tag{19}$$

$$N^{(3)}(X) = (\pounds_\xi\,\varphi)X \overset{(9)}{=} [\xi, \varphi X] - \varphi[\xi, X], \tag{20}$$

$$N^{(4)}(X) = (\pounds_\xi\,\eta)(X) \overset{(10)}{=} \xi(\eta(X)) - \eta([\xi, X]) = 2\, d\eta(\xi, X). \tag{21}$$

Using (8) and $\varphi\,\xi = 0$, one can calculate (19) explicitly as

$$\begin{aligned} N^{(2)}(X, Y) &= (\varphi X)(\eta(Y)) - \eta([\varphi X, Y]) - (\varphi Y)(\eta(X)) + \eta([\varphi Y, X]) \\ &= 2\, d\eta(\varphi X, Y) - 2\, d\eta(\varphi Y, X). \end{aligned} \tag{22}$$

2. Weak contact metric manifolds

Here, we study the weak contact metric structure. Define a "small" (1,1)-tensor $\widetilde{Q}$ by

$$\tilde{Q} = Q - \mathrm{id}_{TM}, \tag{23}$$

and note that $[\tilde{Q}, \varphi] = 0$ and $\tilde{Q}\,\xi = (\nu - 1)\,\xi$, where $\nu = g(Q\,\xi, \xi)$, see (2). We also obtain

$$\varphi + \varphi^3 = -\tilde{Q}\,\varphi.$$

Note that $X^\top = X - \eta(X)\,\xi$ is the projection of the vector $X \in TM$ onto $\mathcal{D}$.

Remark 3. The notion of almost paracontact structure is an analog of that one of almost contact structure and is closely related to the almost product structure. Similarly to (1), we can define a weak para-contact structure by $\varphi^2 = Q - \eta \otimes \xi$, (see [5] and ([6], Section 5.3.8), where $\nu = -1$), which allows

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us to extend some results (in Sections 2–6) for the case of para-contact metric structures introduced in [7].

The following theorem generalizes ([1], Theorem 6.1), i.e., $Q = \mathrm{id}_{TM}$.

Theorem 1. *For a weak almost contact metric structure $(\varphi, Q, \xi, \eta, g)$, the vanishing of $N^{(1)}$ implies that $N^{(3)}$ and $N^{(4)}$ vanish and*

$$N^{(2)}(X,Y) = \eta([\widetilde{Q}X^\top,\, \varphi Y]) + g(\widetilde{Q}\,\xi,\xi)\,\eta([\varphi X, Y]).$$

Proof. By assumption: $N^{(1)}(X,Y) = 0$. Thus, taking ξ instead of Y and using the expression of Nijenhuis tensor (12), we obtain

$$\begin{aligned} 0 &= [\varphi,\varphi](X,\xi) + 2\,d\eta(X,\xi)\,Q\,\xi \\ &= \varphi^2[X,\xi] - \varphi[\varphi X,\xi] + 2\,d\eta(X,\xi)\,Q\,\xi. \end{aligned} \tag{24}$$

Taking the scalar product of (24) with ξ and using skew-symmetry of φ, $\varphi\,\xi = 0$ and $\nu \neq 0$, we get

$$\iota_\xi\, d\eta = 0; \tag{25}$$

hence, (21) yields $N^{(4)} = 0$. Next, combining (24) and (25), we get

$$0 = [\varphi,\varphi](X,\xi) = \varphi^2[X,\xi] - \varphi[\varphi X,\xi] = \varphi\,(\pounds_\xi\,\varphi)X.$$

Applying φ and using (1) and $\eta\circ\varphi = 0$, we achieve

$$\begin{aligned} 0 &= \varphi^2(\pounds_\xi\,\varphi)X = -Q(\pounds_\xi\,\varphi)X + \eta((\pounds_\xi\,\varphi)X)\,Q\,\xi \\ &= -Q(\pounds_\xi\,\varphi)X + \eta([\xi,\varphi X])\,Q\,\xi - \eta(\varphi[\xi,X])\,Q\,\xi \\ &= -Q(\pounds_\xi\,\varphi)X + \eta([\xi,\varphi X])\,Q\,\xi. \end{aligned} \tag{26}$$

Further, (25) and (8) yield

$$0 = 2\,d\eta(\varphi X,\xi) = (\varphi X)(\eta(\xi)) - \xi(\eta(\varphi X)) - \eta([\varphi X,\xi]) = \eta([\xi,\varphi X]). \tag{27}$$

Since Q is non-singular, from (26) and (27) we get $\pounds_\xi\,\varphi = 0$, i.e, $N^{(3)} = 0$.

Replacing X by φX in our assumption $N^{(1)} = 0$ and using (12) and (8), we acquire

$$\begin{aligned} 0 &= g([\varphi,\varphi](\varphi X,Y) + 2\,d\eta(\varphi X,Y)\,Q\,\xi,\ \xi) \\ &= g([\varphi^2 X,\varphi Y],\xi) + \nu\,\{(\varphi X)(\eta(Y)) - \eta([\varphi X,Y])\}. \end{aligned} \tag{28}$$

Using (1) and equality $[\varphi Y, \eta(X)\,Q\,\xi] = (\varphi Y)(\eta(X))\,Q\,\xi + \eta(X)[\varphi Y, Q\,\xi]$, we rewrite (28) as

$$0 = -g([QX,\varphi Y],\xi) + \eta(X)\eta([Q\,\xi,\varphi Y]) - (\varphi Y)(\eta(X))\nu + (\varphi X)(\eta(Y))\nu - \eta([\varphi X,Y])\nu. \tag{29}$$

Equation (27) gives $\eta([\varphi Y, Q\,\xi]) = 0$. So, (29) becomes

$$-\eta([QX,\varphi Y]) - (\varphi Y)(\eta(X))\,\nu + (\varphi X)(\eta(Y)) - \eta([\varphi X,Y])\,\nu + \eta(X)\,\eta([\widetilde{Q}\,\xi,\varphi Y]) = 0. \tag{30}$$

Finally, combining (30) with (22), we get

$$N^{(2)}(X,Y) = \eta([\widetilde{Q}X,\, \varphi Y]) - \eta(X)\,\eta([\widetilde{Q}\,\xi,\varphi Y]) + g(\widetilde{Q}\,\xi,\xi)\big((\varphi Y)(\eta(X)) + \eta([\varphi X,Y])\big),$$

from which and $X = X^\top + \eta(X)\,\xi$ the required expression of $N^{(2)}$ follows. □

Proposition 3. *For either* (i) *a weak contact metric manifold, or* (ii) *a normal weak almost contact metric manifold, we get* (25)*; moreover, the integral curves of ξ are geodesics.*

Proof. (i) Since $N^{(1)} = 0$, then $N^{(4)} = 0$, see Theorem 1. Thus, (21) provides the required (25). (ii) Equation (7) with $Y = \xi$ yields $d\eta(X,\xi) = g(X,\varphi\,\xi) = 0$ for any $X \in \mathfrak{X}_M$; thus, we get (25). Using the identity $\pounds = d\circ\iota + \iota\circ d$, from the above we also have

$$\pounds_\xi\,\eta = d(\eta(\xi)) + \iota_\xi\,d\eta = 0. \tag{31}$$

As in the proof of ([1], Theorem 4.5), we obtain $(\pounds_\xi\,\eta)(X) = g(X, \nabla_\xi\,\xi)$ for any $X \in \mathfrak{X}_M$. From this and (31) we get $\nabla_\xi\,\xi = 0$. □

Theorem 2. *For a weak contact metric structure $(\varphi, Q, \xi, \eta, g)$, the tensors $N^{(2)}$ and $N^{(4)}$ vanish; moreover, $N^{(3)}$ vanishes if and only if ξ is a Killing vector field.*

Proof. Applying (7) in (22) and using skew-symmetry of φ we get $N^{(2)} = 0$. We prove that $N^{(4)} = 0$, taking into account (21) as well as Proposition 3. Next, we find

$$(\pounds_\xi\,g)(X,\varphi Y) = \xi(g(X,\varphi Y)) - g([\xi,X],\varphi Y) - g(X,(\pounds_\xi\,\varphi)Y) - g(X,\varphi[\xi,Y]), \tag{32}$$
$$(\pounds_\xi\,d\eta)(X,Y) = \xi(d\eta(X,Y)) - d\eta([\xi,X],Y) - d\eta(X,[\xi,Y]). \tag{33}$$

Then, invoking the formula (7) in (33) and using (32), we obtain

$$(\pounds_\xi\,d\eta)(X,Y) = (\pounds_\xi\,g)(X,\varphi Y) + g(X,(\pounds_\xi\,\varphi)Y). \tag{34}$$

Since $\pounds_V = \iota_V\circ d + d\circ\iota_V$, the exterior derivative d commutes with the Lie-derivative, i.e., $d\circ\pounds_V = \pounds_V\circ d$, and using Proposition 3, we get that $d\eta$ is invariant under the action of ξ, i.e., $\pounds_\xi\,d\eta = 0$. Therefore, (34) implies that ξ is a Killing vector field if and only if $N^{(3)} = 0$. □

The following result generalizes ([1], Lemma 6.1).

Lemma 1. *For a weak almost contact metric structure $(\varphi, Q, \xi, \eta, g)$, the covariant derivative of φ is given by*

$$\begin{aligned} 2\,g((\nabla_X\,\varphi)Y,Z) &= 3\,d\Phi(X,\varphi Y,\varphi Z) - 3\,d\Phi(X,Y,Z) + g(N^{(1)}(Y,Z),\varphi X)\\ &+ N^{(2)}(Y,Z)\,\eta(X) + 2\,d\eta(\varphi Y,X)\,\eta(Z) - 2\,d\eta(\varphi Z,X)\,\eta(Y) + N^{(5)}(X,Y,Z),\end{aligned}$$

where we supplement the traditional sequence of tensors $N^{(i)}$ $(i = 1,2,3,4)$, with a new skew-symmetric with respect to Y and Z tensor $N^{(5)}(X,Y,Z)$ defined by

$$\begin{aligned} N^{(5)}(X,Y,Z) &= (\varphi Z)\,(g(X^\top,\widetilde{Q}Y)) - (\varphi Y)\,(g(X^\top,\widetilde{Q}Z))\\ &+ g([X,\varphi Z]^\top,\widetilde{Q}Y) - g([X,\varphi Y]^\top,\widetilde{Q}Z)\\ &+ g([Y,\varphi Z]^\top - [Z,\varphi Y]^\top - \varphi[Y,Z],\ \widetilde{Q}X).\end{aligned}$$

For particular values of $N^{(5)}$ we get

$$\begin{aligned} N^{(5)}(X,\xi,Z) &= g((\pounds_\xi\,\varphi)(Z)^\top,\ \widetilde{Q}X) = g(N^{(3)}(Z),\ \widetilde{Q}X),\\ N^{(5)}(\xi,Y,Z) &= g([\xi,\varphi Z]^\top,\widetilde{Q}Y) - g([\xi,\varphi Y]^\top,\widetilde{Q}Z),\\ N^{(5)}(\xi,\xi,Z) &= N^{(5)}(\xi,Y,\xi) = 0. \end{aligned} \tag{35}$$

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Proof. Using (13) and the skew-symmetry of φ, one can compute

$$\begin{aligned}2\,g((\nabla_X\,\varphi)Y,Z) &= 2\,g(\nabla_X(\varphi Y),Z)+2\,g(\nabla_X Y,\varphi Z)\\ &= X\,g(\varphi Y,Z)+(\varphi Y)\,g(X,Z)-Z\,g(X,\varphi Y)\\ &\quad+\ g([X,\varphi Y],Z)+g([Z,X],\varphi Y)-g([\varphi Y,Z],X)\\ &\quad+\ X\,g(Y,\varphi Z)+Y\,g(X,\varphi Z)-(\varphi Z)\,g(X,Y)\\ &\quad+\ g([X,Y],\varphi Z)+g([\varphi Z,X],Y)-g([Y,\varphi Z],X).\end{aligned} \tag{36}$$

Using (4), we obtain

$$\begin{aligned}g(X,Z) &= \Phi(\varphi X,Z)-g(X,\widetilde{Q}Z)+\eta(X)\,\eta(Z)+\eta(X)\,\eta(\widetilde{Q}Z)\\ &= \Phi(\varphi X,Z)+\eta(X)\,\eta(Z)-g(X^\top,\widetilde{Q}Z).\end{aligned} \tag{37}$$

Thus, and in view of the skew-symmetry of φ and applying (37) six times, (36) can be written as

$$\begin{aligned}&2\,g((\nabla_X\,\varphi)Y,Z) = X\,\Phi(Y,Z)\\ &+(\varphi Y)\,\big(\Phi(\varphi X,Z)+\eta(X)\,\eta(Z)\big)-(\varphi Y)\,g(X^\top,\widetilde{Q}Z)-Z\,\Phi(X,Y)\\ &-\Phi([X,\varphi Y],\varphi Z)+\eta([X,\varphi Y])\eta(Z)-g([X,\varphi Y]^\top,\widetilde{Q}Z)+\Phi([Z,X],Y)\\ &+\Phi([\varphi Y,Z],\varphi X)-\eta([\varphi Y,Z])\,\eta(X)+g([\varphi Y,Z]^\top,\widetilde{Q}X)+X\,\Phi(Y,Z)+Y\,\Phi(X,Z)\\ &-(\varphi Z)\,\big(\Phi(\varphi X,Y)+\eta(X)\,\eta(Y)\big)+(\varphi Z)g(X^\top,\widetilde{Q}Y)+\Phi([X,Y],Z)\\ &+g(\varphi[\varphi Z,X],\varphi Y)+\eta([\varphi Z,X])\,\eta(Y)-g([\varphi Z,X]^\top,\widetilde{Q}Y)\\ &-g(\varphi[Y,\varphi Z],\varphi X)-\eta([Y,\varphi Z])\,\eta(X)+g([Y,\varphi Z]^\top,\widetilde{Q}X).\end{aligned}$$

Recall the co-boundary formula for exterior derivative d on a 2-form Φ,

$$\begin{aligned}d\Phi(X,Y,Z) &= \frac{1}{3}\,\{X\,\Phi(Y,Z)+Y\,\Phi(Z,X)+Z\,\Phi(X,Y)\\ &\quad-\Phi([X,Y],Z)-\Phi([Z,X],Y)-\Phi([Y,Z],X)\}.\end{aligned} \tag{38}$$

We also have

$$\begin{aligned}g(N^{(1)}(Y,Z),\varphi X) &= g(\varphi^2[Y,Z]+[\varphi Y,\varphi Z]-\varphi[\varphi Y,Z]-\varphi[Y,\varphi Z],\varphi X)\\ &= g(\varphi[Y,Z],\widetilde{Q}X)+g([\varphi Y,\varphi Z]-\varphi[\varphi Y,Z]-\varphi[Y,\varphi Z]-[Y,Z],\varphi X).\end{aligned}$$

From this and (38) we get the required result. □

According to Theorem 2, on a weak contact metric manifold, we get

$$d\eta=\Phi,\quad N^{(2)}=N^{(4)}=0.$$

Thus, invoking (7) and using (7) and Proposition 3 in Lemma 1, we obtain

Corollary 1. *For a weak contact metric structure (φ,Q,ξ,η,g), the covariant derivative of φ is given by*

$$\begin{aligned}2\,g((\nabla_X\,\varphi)Y,Z) &= g(N^{(1)}(Y,Z),\varphi X)+2\,g(\varphi X,\varphi Y)\,\eta(Z)-2\,g(\varphi X,\varphi Z)\,\eta(Y)\\ &\quad+N^{(5)}(X,Y,Z).\end{aligned}$$

In particular, we have

$$2\,g((\nabla_\xi\,\varphi)Y,Z)=N^{(5)}(\xi,Y,Z). \tag{39}$$

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Example 1. Consider a weak almost contact metric structure $(\varphi, Q, \xi, \eta, g)$ with

$$Q\,|_{\mathcal{D}} = \lambda\,\mathrm{id}_{\mathcal{D}}$$

for a constant $\lambda > 0$. Using $X^\top = X - \eta(X)\,\xi$, we obtain

$$\widetilde{Q}X = (\lambda - 1)X^\top + (\nu - 1)\,\eta(X)\,\xi), \quad X \in \mathfrak{X}_M. \tag{40}$$

Using (40), we find expressions of the tensor $N^{(5)}$ (see Lemma 1)

$$\begin{aligned}
N^{(5)}(X,Y,Z) &= (\lambda - 1)\,\{(\varphi Z)\,g(X^\top, Y) - (\varphi Y)\,g(X^\top, Z) + g([X, \varphi Z]^\top, Y) \\
&\quad - g([X, \varphi Y]^\top, Z) + g([Y, \varphi Z]^\top - [Z, \varphi Y]^\top - \varphi[Y,Z], X)\},\\
N^{(5)}(\xi,Y,Z) &= (\lambda - 1)\,\{g([\xi, \varphi Z]^\top, Y) - g([\xi, \varphi Y]^\top, Z)\},\\
N^{(5)}(X,\xi,Z) &= (\lambda - 1)\,\{g([\xi, \varphi Z]^\top - \varphi[\xi, Z], X)\}.
\end{aligned}$$

3. The tensor field h

The tensor field $h = \frac{1}{2}\,£_\xi\,\varphi$ plays an important role for contact metric manifolds because of its remarkable properties, e.g., [1]. Here, we study its generalization. We define the tensor field h on a weak contact metric manifold similarly as for a contact metric manifold,

$$h = \frac{1}{2}\,N^{(3)} = \frac{1}{2}\,£_\xi\,\varphi. \tag{41}$$

We compute

$$\begin{aligned}
(£_\xi\,\varphi)X &\overset{(9)}{=} \nabla_\xi(\varphi X) - \nabla_{\varphi X}\,\xi - \varphi(\nabla_\xi X - \nabla_X\,\xi)\\
&= (\nabla_\xi\,\varphi)X - \nabla_{\varphi X}\xi + \varphi\nabla_X\,\xi.
\end{aligned} \tag{42}$$

Taking $X = \xi$ in (42) and using $(\nabla_\xi\,\varphi)\,\xi = \frac{1}{2}\,N^{(5)}(\xi, \xi, \cdot\,) = 0$, see (39) and (35) and $\nabla_\xi\,\xi = 0$ (see Proposition 3), we get

$$h\,\xi = 0. \tag{43}$$

From (42), using

$$2\,g((\nabla_\xi\,\varphi)Y, \xi) \overset{(39)}{=} N^{(5)}(\xi, Y, \xi) \overset{(35)}{=} 0,$$

we conclude that the distribution $\mathcal{D}$ is invariant with respect to h.

On a contact manifold, h is a self-adjoint linear operator that anticommutes with φ, see ([1], Lemma 6.2). The following lemma generalizes this results.

Lemma 2. *On a weak contact metric manifold, the tensor h satisfies*

$$g((h - h^*)X, Y) = \frac{1}{2}\,N^{(5)}(\xi, X, Y), \tag{44}$$

$$h\,\varphi + \varphi\,h = -\frac{1}{2}\,£_\xi\widetilde{Q}, \tag{45}$$

$$Q\,\nabla\,\xi = -\varphi\,(h^* + Q), \tag{46}$$

where h^* is the conjugate operator to h and the tensor $N^{(5)}(\xi, X, Z)$ is given in (35).

Proof. (i) The scalar product of (42) with Y for $X, Y \in \mathcal{D}$, using (43) and Corollary 1, gives

$$g((\pounds_\xi\,\varphi)X, Y) = \frac{1}{2}\{g([\xi, \varphi Y]^\top, \widetilde{Q}X) - g([\xi, \varphi X]^\top, \widetilde{Q}Y)\} + g(\varphi\nabla_X\,\xi - \nabla_{\varphi X}\xi, Y). \tag{47}$$

Similarly,

$$g((\pounds_\xi\,\varphi)Y, X) = \frac{1}{2}\{g([\xi, \varphi X]^\top, \widetilde{Q}Y) - g([\xi, \varphi Y]^\top, \widetilde{Q}X)\} + g(\varphi\nabla_Y\,\xi - \nabla_{\varphi Y}\,\xi, X). \tag{48}$$

The difference of (47) and (48), in view of $N^{(2)} = 0$ (see Theorem 2), gives (44).

(ii) Using $\nabla_\xi\,\xi = 0$ (see Proposition 3), $\nabla_\xi\,\eta = 0$ and $(\nabla_\xi\,\widetilde{Q})\,\xi = 0$, we obtain

$$\begin{aligned}\varphi(\nabla_\xi\,\varphi) + (\nabla_\xi\,\varphi)\varphi = \nabla_\xi\,(\varphi^2) = -\nabla_\xi\,\widetilde{Q} + \nabla_\xi\,(\eta\otimes Q\,\xi)\\ = -\nabla_\xi\,\widetilde{Q} + (\nabla_\xi\,\eta)\otimes Q\,\xi + \eta\otimes(\nabla_\xi\,\widetilde{Q})\,\xi = -\nabla_\xi\,\widetilde{Q}.\end{aligned}$$

By this and (42), we get

$$\begin{aligned} 2(h\varphi + \varphi h)X &= \varphi(\pounds_\xi\,\varphi)X + (\pounds_\xi\,\varphi)\varphi X \\ &= \varphi(\nabla_\xi\,\varphi)X + (\nabla_\xi\,\varphi)\varphi X + \varphi^2\nabla_X\,\xi - \nabla_{\varphi^2 X}\,\xi \\ &= -(\nabla_\xi\,\widetilde{Q})X - \widetilde{Q}\nabla_X\,\xi + \nabla_{\widetilde{Q}X}\,\xi, \end{aligned}$$

that provides (45). Note that $(h\varphi + \varphi h)\,\xi = 0$.

(iii) From Corollary 1 with $Y = \xi$, we find

$$2\,g((\nabla_X\,\varphi)\xi, Z) = g(N^{(1)}(\xi, Z), \varphi X) - 2\,d\eta(\varphi Z, X) + N^{(5)}(X, \xi, Z). \tag{49}$$

From (12) with $Y = \xi$, we get

$$[\varphi, \varphi](X, \xi) = \varphi^2[X, \xi] - \varphi[\varphi X, \xi]. \tag{50}$$

Using (4), (50) and (9), we calculate

$$\begin{aligned} g([\varphi, \varphi](\xi, Z), \varphi X) = g(\varphi^2\,[\xi, Z] - \varphi[\xi, \varphi Z], \varphi X) = -g(\varphi(\pounds_\xi\,\varphi)Z, \varphi X)\\ = -g((\pounds_\xi\,\varphi)Z, QX) + \eta(X)\,\eta(Q(\pounds_\xi\,\varphi)Z). \end{aligned} \tag{51}$$

Since $\nabla_\xi\,\xi = 0$ and $Q\xi = \nu\,\xi$, from (42) we get

$$g(Q(\pounds_\xi\,\varphi)X, \xi) = \nu\,\{g((\nabla_\xi\,\varphi)X, \xi) - g(\nabla_{\varphi X}\,\xi, \xi) + g(\varphi(\nabla_X\,\xi), \xi)\} = 0. \tag{52}$$

Since $\varphi\,\xi = 0$, we find

$$(\nabla_X\,\varphi)\,\xi = -\varphi\,\nabla_X\,\xi. \tag{53}$$

Thus, combining (49), (51) and (52), we deduce

$$-g(\varphi\,\nabla_X\,\xi, Z) = -g(hZ, QX) - g(Z, QX) + \eta(Z)\,\eta(QX) + \frac{1}{2}\,N^{(5)}(X, \xi, Z). \tag{54}$$

Since Q is symmetric and φ is skew-symmetric, replacing Z by φZ in (54) and using (1), $g(\nabla_X\,\xi, \xi) = 0$ and $\varphi\,\xi = 0$, we get

$$g(Q\,\nabla_X\,\xi, Z) = g((\varphi + h\varphi)Z, QX) - \frac{1}{2}\,N^{(5)}(X, \xi, \varphi Z).$$

From this, using (41) and (35), we achieve (46). □

4. Weak Sasakian manifolds

We are based on some classical results (see ([1], Chapter 6); [4]), and show the equality $C_3 = C^{w}_{3/\sim}$.

Lemma 3. *Let $M^{2n+1}(\varphi, Q, \xi, \eta, g)$ be a weak Sasakian manifold. Then*

$$g((\nabla_X \varphi)Y, Z) = g(QX, Y)\,\eta(Z) - g(QX, Z)\,\eta(Y) + \frac{1}{2}\,N^{(5)}(X, Y, Z). \tag{55}$$

Proof. Since $(\varphi, Q, \xi, \eta, g)$ is a weak contact metric structure with $N^{(1)} = 0$, by Corollary 1, we get

$$g((\nabla_X \varphi)Y, Z) = d\eta(\varphi Y, X)\,\eta(Z) - d\eta(\varphi Z, X)\,\eta(Y) + \frac{1}{2}\,N^{(5)}(X, Y, Z). \tag{56}$$

Using (7) and (4) in (56), we get (55). □

Since on a weak contact metric manifold, the tensor $N^{(3)}$ vanishes if and only if ξ is Killing (Theorem 2), then a weak almost contact metric structure is weak K-contact if and only if $h = 0$. Thus we have the following generalization of ([1], Corollary 6.3).

Corollary 2. *A weak Sasakian manifold is weak K-contact.*

Proof. In view of (53), Equation (55) with $Y = \xi$ becomes

$$g(\nabla_X \xi, \varphi Z) = \eta(QX)\,\eta(Z) - g(X, QZ). \tag{57}$$

Combining (54) and (57), we achieve $g(hZ, QX) = 0$, which implies $h = 0$. □

The main result in this section is the following rigidity of a Sasakian structure.

Theorem 3. *A weak almost contact metric structure $(\varphi, Q, \xi, \eta, g)$ on M^{2n+1} is weak Sasakian if and only if it is homothetically equivalent to a Sasakian structure $(\varphi', \xi, \eta, g')$ on M^{2n+1}.*

Proof. Let $M^{2n+1}(\varphi, Q, \xi, \eta, g)$ be a weak Sasakian manifold. Since $N^{(1)} = 0$, by Theorem 1, we get $N^{(3)} = 0$. By (35), we then obtain $N^{(5)}(\cdot\,, \xi, \,\cdot\,) = 0$. Take $\tilde X, \tilde Y \in \mathcal{D}$ such that $\varphi \tilde X = X^\top$ and $\varphi \tilde Y = Y^\top$. Since φ is skew-symmetric, applying (55) with $Z = \xi$ in (14), we obtain

$$\begin{aligned} g([\varphi, \varphi](X, Y), \xi) &= -g((\nabla_{\varphi Y}\, \varphi)X, \xi) + g((\nabla_{\varphi X}\, \varphi)Y, \xi) \\ &= -g(Q\,\varphi Y, X) + g(Q\,\varphi Y, \xi)\,\eta(X) + g(Q\,\varphi X, Y) - g(Q\,\varphi X, \xi)\,\eta(Y). \end{aligned} \tag{58}$$

Recall that $[Q,\, \varphi] = 0$ and $\varphi\,\xi = 0$. Thus, (58) yields

$$g([\varphi, \varphi](X, Y), \xi) = -2\,g(QX, \varphi Y).$$

From this, using (23) and definition of $N^{(1)}$, we get

$$g(N^{(1)}(X, Y), \xi) = 2\,g(X, \varphi Y)\,g(\widetilde{Q}\,\xi,\, \xi) - 2\,g(\widetilde{Q}X, \varphi Y). \tag{59}$$

By (59), since $N^{(1)} = 0$ holds, the following condition is valid for all $X, Y \in \mathfrak{X}_M$:

$$g(\widetilde{Q}X, \varphi Y) = g(X, \varphi Y)\,g(\widetilde{Q}\,\xi,\, \xi). \tag{60}$$

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In view of $g(\widetilde{Q}\,\xi,\,\xi) = \nu - 1$ and $QX = X + \widetilde{Q}X$, equation (60) reduces to

$$0 = g(\widetilde{Q}X, \varphi Y) - g(X, \varphi Y)\, g(\widetilde{Q}\,\xi,\,\xi) = g(QX - \nu X, \varphi Y),$$

thus, $Q\,|_{\mathcal{D}} = \nu\,\mathrm{id}|_{\mathcal{D}}$. By Proposition 2(iii), we conclude that our structure is homothetically equivalent to a Sasakian structure. □

One can get Corollary 2 as a consequence of Theorem 3 and Proposition 2.

Remark 4. A 3-Sasakian structure is based on a set of three Sasakian structures, see [2]. Similarly, we get a *weak 3-Sasakian structure* if each of the structures $(\phi_i, \xi_i, \eta^i, Q)$ $(i = 1, 2, 3)$ is weak Sasakian with the same tensor Q and compatible metric satisfying

$$\phi_k = \phi_i \circ \phi_j - \eta^i \otimes \xi_j = -\phi_j \circ \phi_i + \eta^j \otimes \xi_i$$

for any cyclic permutation (i, j, k) of $(1, 2, 3)$, see [5]. Thus, the results of this section on weak Sasakian manifolds yield certain results for weak 3-Sasakian manifolds.

5. Weak cosymplectic manifolds

Cosymplectic manifolds constitute an important class of almost contact manifolds, e.g., [3,8]. Here, we study wider classes of weak (almost) cosymplectic manifolds and characterize the class C_4^w in C_4 by the condition $\nabla\varphi = 0$.

Definition 2. A *weak almost cosymplectic* (or a *weak almost coKähler*) manifold is a weak almost contact metric manifold $M^{2n+1}(\varphi, Q, \xi, \eta, g)$, whose fundamental 2-form Φ and the 1-form η are closed: $d\Phi = d\eta = 0$. If a weak almost cosymplectic structure is normal, we say that M is a *weak cosymplectic* (or a *weak coKähler*) *manifold*.

Theorem 4. *For a weak almost cosymplectic structure $(\varphi, Q, \xi, \eta, g)$, the tensors $N^{(2)}$ and $N^{(4)}$ vanish and $N^{(1)} = [\varphi, \varphi]$. Moreover, $N^{(3)}$ vanishes if and only if ξ is a Killing vector field.*

Proof. By (22) and (21) and since $d\eta = 0$, the tensors $N^{(2)}$ and $N^{(4)}$ vanish on a weak almost cosymplectic structure. Moreover, by (11) and (34), respectively, the tensor $N^{(1)}$ coincides with the Nijenhuis tensor of φ, and $N^{(3)} = \pounds_\xi\,\varphi$ vanishes if and only if ξ is a Killing vector field. □

By application of the above theorem, we make the following corollary to point out some important attributes of a weak almost cosymplectic structure.

Corollary 3. *In any weak almost cosymplectic manifold the integral curves of ξ are geodesics.*

Proof. By Theorem 4, $N^{(4)} = 0$; thus, from (21) and $g(\nabla_X\xi,\,\xi) = 0$ we get for any $X \in \mathfrak{X}_M$,

$$0 = \xi(\eta(X)) - \eta([\xi, X]) = \xi(\eta(X)) - g(\nabla_\xi X, \xi) + g(\nabla_X\xi, \xi) = g(X, \nabla_\xi\,\xi).$$

Therefore, $\nabla_\xi\,\xi = 0$. □

Proposition 4. *Let $(\varphi, Q, \xi, \eta, g)$ be a weak cosymplectic structure. Then*

$$2\,g((\nabla_X\,\varphi)Y, Z) = N^{(5)}(X, Y, Z), \tag{61}$$

$$0 = N^{(5)}(X, Y, Z) + N^{(5)}(Y, Z, X) + N^{(5)}(Z, X, Y), \tag{62}$$

$$2\,g([\varphi, \varphi](X, Y), Z) = N^{(5)}(\varphi X, Y, Z) + N^{(5)}(\varphi Y, Z, X) + N^{(5)}(\varphi Z, X, Y). \tag{63}$$

Proof. For a weak almost cosymplectic structure $(\varphi, Q, \xi, \eta, g)$, we get

$$2\,g((\nabla_X\,\varphi)Y, Z) = g([\varphi,\varphi](Y,Z), \varphi X) + N^{(5)}(X,Y,Z). \tag{64}$$

From (64), using condition $[\varphi,\varphi]=0$ we get (61). Using (38) and (61), we can write

$$0 = 3\,d\Phi(X,Y,Z) = g((\nabla_X\,\varphi)Z, Y) + g((\nabla_Y\,\varphi)X, Z) + g((\nabla_Z\,\varphi)Y, X),$$

hence, (62). Using (14), (61) and the skew-symmetry of φ, we obtain

$$2g([\varphi,\varphi](X,Y),Z) = N^{(5)}(X,Y,\varphi Z) + N^{(5)}(\varphi X,Y,Z) - N^{(5)}(Y,X,\varphi Z) - N^{(5)}(\varphi Y,X,Z).$$

This and (62) with X replaced by φX provide (63). □

Remark 5. For a weak cosymplectic structure, using (61), we obtain (compare with (46))

$$2\,g(Q(\nabla_X\,\xi),\,Z) = -N^{(5)}(X,\xi,\varphi Z).$$

Recall that an almost contact metric structure (φ, ξ, η, g) is cosymplectic if and only if φ is parallel, e.g., ([1], Theorem 6.8). The following our theorem completes this result.

Theorem 5. *Any weak almost contact structure $(\varphi, Q, \xi, \eta, g)$ with the property $\nabla\varphi = 0$ is a weak cosymplectic structure with vanishing tensor $N^{(5)}$.*

Proof. Using condition $\nabla\varphi = 0$, from (14) we obtain $[\varphi,\varphi]=0$. Hence, from (11) we get $N^{(1)}(X,Y) = 2\,d\eta(X,Y)Q\xi$, and from (15) we obtain

$$\nabla_{\varphi X}\,\xi - \varphi\,\nabla_X\,\xi = 0, \quad X \in \mathfrak{X}_M. \tag{65}$$

From (38), we calculate

$$3\,d\Phi(X,Y,Z) = g((\nabla_X\,\varphi)Z, Y) + g((\nabla_Y\,\varphi)X, Z) + g((\nabla_Z\,\varphi)Y, X);$$

hence, using condition $\nabla\varphi = 0$ again, we get $d\Phi = 0$. Next,

$$N^{(2)}(Y,\xi) = -\eta([\varphi Y,\xi]) = g(\xi, \varphi\nabla_\xi Y) = 0.$$

Thus, setting $Z=\xi$ in Lemma 1 and using the condition $\nabla\varphi = 0$ and the properties $d\Phi = 0$, $N^{(2)}(Y,\xi) = 0$ and $N^{(1)}(X,Y) = 2\,d\eta(X,Y)Q\xi$, we find $0 = d\eta(\varphi Y, X) - N^{(5)}(X,\xi,Y)$. By (35) and (65), we get

$$N^{(5)}(X,\xi,Y) = g([\xi,\varphi Y]^\top - \varphi[\xi,Y],\,\widetilde{Q}X) = g(\nabla_{\varphi Y}\,\xi - \varphi\,\nabla_Y\,\xi,\,\widetilde{Q}X) = 0;$$

hence, $d\eta = 0$. By the above, $N^{(1)} = 0$. Thus, $(\varphi, Q, \xi, \eta, g)$ is a weak cosymplectic structure. Finally, from (61) and condition $\nabla\varphi = 0$ we get $N^{(5)} = 0$. □

Example 2. Let M be a $2n$-dimensional smooth manifold and $\tilde\varphi : TM \to TM$ an endomorphism of rank $2n$ such that $\nabla\tilde\varphi = 0$. To construct a weak cosymplectic structure on $M\times\mathbb{R}$ or $M\times S^1$, take any point (x,t) of either space and set $\xi = (0, d/dt)$, $\eta = (0, dt)$ and

$$\varphi(X,Y) = (\tilde\varphi X, 0), \quad Q(X,Y) = (-\tilde\varphi^2 X, \nu\,\xi).$$

where $X \in M_x$, $Y \in \{\mathbb{R}_t \text{ or } S^1_t\}$ and $\nu \in \mathbb{R}_+$. Then (1) holds and Theorem 5 can be applied.

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6. Weak contact vector fields

Contact vector fields (and contact infinitesimal transformations) is a fruitful tool in contact manifold geometry. Here, we are based on some classical results, e.g., ([1], Section 5.2).

Definition 3. A vector field X on a weak contact metric manifold $M^{2n+1}(\varphi, Q, \xi, \eta, g)$ is called a *weak contact vector field* (or, a *weak contact infinitesimal transformation*), if the flow of X preserves the form η, i.e., there exists a smooth function $\sigma : M \to \mathbb{R}$ such that

$$\pounds_X \eta = \sigma \eta, \tag{66}$$

and if $\sigma = 0$, then the vector field X is said to be *strict weak contact vector field*.

The following our result generalizes ([1], Theorem 5.7).

Theorem 6. *A vector field X on a weak contact metric manifold $M^{2n+1}(\varphi, Q, \xi, \eta, g)$ is a weak contact infinitesimal transformation if and only if there exists a function f on M such that*

$$QX = -\frac{1}{2}\,\varphi\,\nabla f + \nu\, f\,\xi. \tag{67}$$

Moreover, a weak contact vector field X is strict if and only if $\xi(f) = 0$.

Proof. (a) One can explicitly write $(\pounds_X \eta)(Y) = X(\eta(Y)) - \eta([X,Y])$ $(Y \in \mathfrak{X}_M)$. Using (8), we rewrite (66) as

$$2\,d\eta(X,Y) + Y(f) = \sigma\,\eta(Y), \tag{68}$$

where $f := \eta(X)$. Using $d\eta = \Phi$, see (7), in (68), we get $-2\,g(\varphi X, Y) + Y(f) = \sigma\,\eta(Y)$, which provides

$$-2\,\varphi X + \nabla f = \sigma\,\xi. \tag{69}$$

Applying φ and using (1), we obtain (67). Multiplying (69) by ξ gives $\sigma = \xi(f)$, that proves the assertion about the "strict" property of a vector field X.

(b) Conversely, assume the condition (67) for X. The scalar product of (67) with ξ gives

$$\nu\,\eta(X) = g(Q\,\xi, X) = \nu\, f;$$

therefore, $f = \eta(X)$. Multiplying (67) by φ and using (1), gives

$$Q\,\varphi X = -\frac{1}{2}\,\varphi^2(\nabla f) = \frac{1}{2}\,Q\,\nabla f - \frac{1}{2}\,\eta(\nabla f)\,Q\,\xi.$$

Then, in view of non-degeneracy of Q, we get

$$\nabla f - 2\,\varphi X = \eta(\nabla f)\,\xi = \xi(f)\,\xi.$$

Thus, using (4), (7) and (8), one can calculate

$$
\begin{aligned}
(\pounds_X\,\eta)(Y) &= 2\,d\eta(X,Y) + Y(\eta(X)) = 2\,d\eta(QX - \widetilde{Q}X, Y) + Y(f)\\
&= 2\,g(-\frac{1}{2}\,\varphi\nabla f + f\,\nu\,\xi, \varphi Y) + Y(f) - 2\,g(\widetilde{Q}X, \varphi Y)\\
&= -g(\varphi\nabla f, \varphi Y) + Y(f) - 2\,g(\widetilde{Q}X, \varphi Y)\\
&= -(QY)(f) + \nu\,\xi(f)\,\eta(Y) + Y(f) - 2\,g(\widetilde{Q}X, \varphi Y)\\
&= \nu\,\xi(f)\,\eta(Y) - g(\widetilde{Q}(\nabla f - 2\,\varphi X), Y)\\
&= \nu\,\xi(f)\,\eta(Y) - (\nu - 1)\,\xi(f)\,\eta(Y) = \xi(f)\,\eta(Y).
\end{aligned}
$$

By this, (66) with $\sigma = \xi(f)$ is valid. □

7. Concluding remarks

The theory of contact and symplectic manifolds have been significantly developed and successfully applied to theoretical physics. In this article, we investigated new metric structures on a smooth manifold, which allow us to take a fresh look at the classical theory and find new applications. We expect weak contact manifolds and submanifolds in them to be fruitful in theoretical physics as well. We have shown that
(i) a weak Sasakian structure is homothetically equivalent to a Sasakian structure,
(ii) a weak almost contact structure with parallel structural tensor is weak cosymplectic,
(iii) conditions are found under which a vector field is a weak contact vector field.

We propose a further study of the geometry of weak contact manifolds, in particular, when such manifolds equipped with a warped products or Ricci-type soliton structure carry a canonical (e.g., with constant sectional curvature or Einstein type) metric.

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