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Another Approach to the Analysis of Isotropic Rectangular Thin Plates Subjected to External Bending Moments Using the Fourier Series

Valentin Fogang

Civil Engineer, C/o BUNS Sarl, P.O Box 1130, Yaounde, Cameroon; valentin.fogang@bunscameroun.com

Abstract: The object of this paper is the analysis of isotropic rectangular thin plates supported (simply supported or clamped) along two opposite edges whereby the other edges have arbitrary support conditions; the plates are subjected to external bending moments, concentrated or distributed, and perpendicular to the supported edges. The standard approach to this problem is to replace the bending moment with a couple of forces infinitely close and to use the known expressions of efforts and deformations for the plate subjected to concentrated forces; the results are then related to the first derivatives of those efforts and deformations with respect to the position of application of the load. In this study, the external bending moment was expanded into a Fourier series, leading to a distributed external bending moment, and the boundary conditions and continuity equations were applied. Various types of rectangular plates were analyzed, as well as plates of infinite length whose results were identical to those in the literature. In addition, results for cantilever plates of infinite length were presented.

Keywords: isotropic rectangular thin plate; concentrated/distributed bending moment; plates of infinite length; Lévy solution; Fourier sine series

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1. Introduction

The Kirchhoff–Love plate theory (KLPT) was developed in 1888 by Love using assumptions proposed by Kirchhoff [1]. The KLPT is governed by the Germain–Lagrange plate equation; this equation was first derived by Lagrange in December 1811 in correcting the work of Germain [2] who provided the basis of the theory. Lévy [3] proposed in 1899 an approach for rectangular plates simply supported along two opposite edges; the applied load and the deflection were expressed in terms of Fourier components and simple trigonometric series, respectively. Many exact solutions for isotropic elastic thin plates were developed by Timoshenko [4] and Girkmann [5]; the simple trigonometric series of Lévy was mostly considered. Jian et al. [6] presented the equations for the lateral buckling analysis of fixed rectangular plates under the lateral concentrated load whereby the critical buckling strength of the plate calculated by finite element method was analyzed. Xu et al. [7] got exact solutions for rectangular anisotropic plates with four clamped edges through the state space method whereby the Fourier series in exponential form were adopted. Onyia et al. [8] presented the elastic buckling analysis of rectangular thin plates using the single finite Fourier sine integral transform method. Imrak et al. [9] presented an exact solution for a rectangular plate clamped along all edges in which each term of the series is trigonometric and hyperbolic, and identically satisfies the boundary conditions on all four edges. Fogang [10] used the flexibility method and a modified Lévy solution to analyze arbitrarily loaded isotropic rectangular thin plates with two opposite edges supported (simply supported or clamped), from which one or both are clamped, and the other edges with arbitrary support conditions. Mama et al. [11] presented the single finite Fourier sine integral transform method for the flexural analysis of rectangular Kirchhoff plate with opposite edges simply supported, and the other edges clamped for the case of triangular load distribution on the plate domain. Kamel [12] described the operational properties of the finite Fourier transform method, with the purpose of solving boundary value problems of partial differential equations.

In this paper, isotropic rectangular thin plates were analyzed; they were simply supported or clamped along two opposite edges with the other edges having arbitrary support conditions, and were subjected to external bending moments, concentrated or distributed and perpendicular to the supported edges. The external bending moment was expanded into a Fourier series, leading to a distributed external bending moment, and the boundary conditions and continuity equations were applied.

2. Materials and methods

2.1. Governing equations of the plate

The Kirchhoff–Love plate theory (KLPT) [1] is used for thin plates whereby shear deformations are not considered. The spatial axis convention (X, Y, Z) is represented in Figure 1 below.

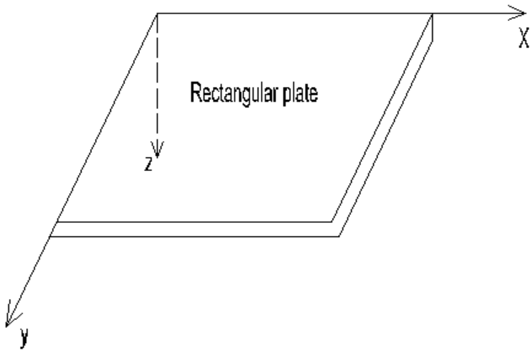

Figure 1. Spatial axis convention X, Y, Z.

The equations of the present section are related to the KLPT. The governing equation of the isotropic Kirchhoff plate, derived by Lagrange, is given by

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{q(x,y)}{D} \tag{1}$$

where $w(x,y,z)$ is the displacement in z-direction, $q(x,y)$ the applied transverse load per unit area, and D the flexural rigidity of the plate.

The bending moments per unit length m_{xx} and m_{yy}, and the twisting moments per unit length m_{xy} are given by

$$m_{xx}=-D\times\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right), \qquad m_{yy}=-D\times\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right),$$

$$m_{xy}=-D\times(1-\nu)\times\frac{\partial^2 w}{\partial x \partial y}, \tag{2a-d}$$

$$D=\frac{Eh^3}{12(1-\nu^2)}$$

The shear forces per unit length are given by

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$$Q_x = -D\times\left(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x \partial y^2}\right), \qquad Q_y = -D\times\left(\frac{\partial^3 w}{\partial y^3}+\frac{\partial^3 w}{\partial x^2 \partial y}\right). \qquad (3a\text{-}b)$$

The Kirchhoff shear forces per unit length used along the free edges combine shear forces and twisting moments, and are expressed as follows:

$$V_x = -D\times\left(\frac{\partial^3 w}{\partial x^3}+(2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right), \qquad V_y = -D\times\left(\frac{\partial^3 w}{\partial y^3}+(2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y}\right). \qquad (4a\text{-}b)$$

In these equations, E is the elastic modulus of the plate material, h is the plate thickness, and ν is the Poisson's ratio.

2.2. Rectangular plate supported along two opposite edges and subjected to external concentrated bending moments

The plate dimensions in x- and y-direction are denoted by a and b, respectively. The rectangular plate is assumed simply supported or clamped along the opposite edges x = 0 and x = a. The external concentrated bending moment denoted by Mx0 is applied at the position (x_0, y_0) and oriented along the + x-axis (see Figure 2).

For rectangular plates simply supported along the edges x = 0 and x = a, bending moment loading parallel to those edges are satisfactorily treated in the literature and will not be analyzed in this paper.

First, the standard solution to this problem will be recalled. Second, the approach of this study will be presented whereby plates with two opposite edges simply supported and plates of infinite length will be considered. Third, plates with two opposite edges clamped will be treated.

2.2.1. Standard solution to the problem

Let the external concentrated moment Mx0 be applied at the position (x_0, y_0) as shown in Figure 2

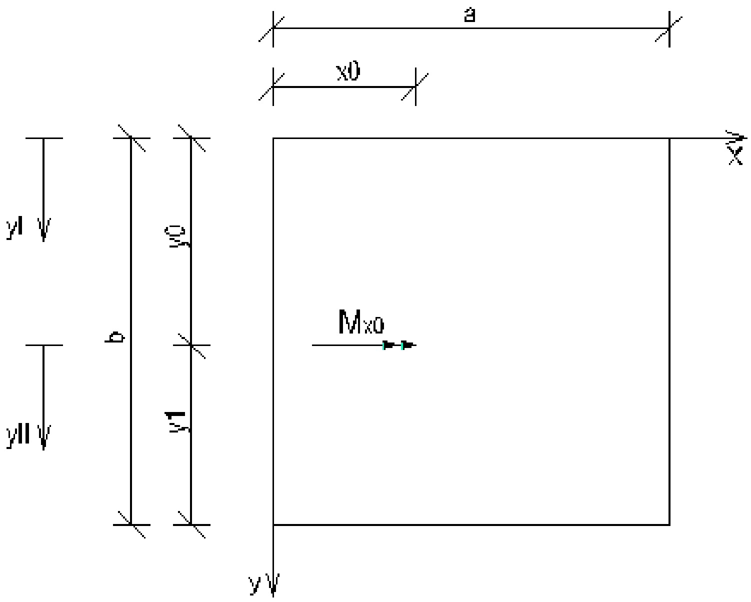

Figure 2. Rectangular plate subjected to a concentrated moment Mx0 at (x_0, y_0).

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The standard approach is to replace the bending moment with a couple of forces -P and P applied at the positions (x_0, y_0) and (x_0, $y_0 + c$), respectively, with c approaching zero and $M_{x0} = P \times c$.

Let $S(x, y, x_0, y_0)$ and $S^*(x, y, x_0, y_0)$ be values of interest (efforts, deformations …) at positions (x, y) for the plate subjected to a load P and a unit load, respectively, at a position (x_0, y_0). The values of interest are determined by combining the effect of the forces -P and P as follows

$$\begin{aligned} &S(x,y,x_0,y_0+c)-S(x,y,x_0,y_0) \\ &= P\times S^*(x,y,x_0,y_0+c)-P\times S^*(x,y,x_0,y_0) \\ &= P\frac{\partial S^*(x,y,x_0,y_0)}{\partial y_0}c \\ &= M_{x0}\frac{\partial S^*(x,y,x_0,y_0)}{\partial y_0} \end{aligned} \tag{5}$$

Hence the determination of a quantity of interest for the case of the plate subjected to an external concentrated moment M_{x0} at a position (x_0, y_0) requires the analytical formulation of the quantity of interest for the case of the plate subjected to a unit load at the position (x_0, y_0) and its first derivative with respect to y_0; this result can be found in Girkmann [5].

2.2.2. Rectangular plate with two opposite edges simply supported and subjected to an external concentrated bending moment M_{x0}

In this paper the external concentrated bending moment M_{x0} (see Figure 2) is expanded into a Fourier series as follows

$$m_{x0}(x)=\frac{2M_{x0}}{a}\sum_m \sin\frac{m\pi x_0}{a}\sin\frac{m\pi x}{a} \tag{6}$$

Therefore, the external concentrated moment M_{x0} is replaced with the distributed external moment $m_{x0}(x)$ along the line $y = y_0$.

Referring further to Figure 2, the efforts and deformations are represented with the subscripts I and II for the plate zones $0 \le y_I \le y_0$ and $0 \le y_{II} \le y_1$, respectively. The continuity equations along the line $y = y_0$ express the continuity of deflection w and slope $\partial w/\partial y$ and the equilibrium of bending moment m_{yy} and shear force Q_y; observing that the position $y = y_0$ corresponds to $y_I = y_0$ and $y_{II} = 0$, these equations are given by

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$$w_I(x,y_I)\big|_{y_I=y_0} = w_{II}(x,y_{II})\big|_{y_{II}=0}$$
$$\left.\frac{\partial w_I(x,y_I)}{\partial y_I}\right|_{y_I=y_0} = \left.\frac{\partial w_{II}(x,y_{II})}{\partial y_{II}}\right|_{y_{II}=0}$$

(7a-d)

$$m_{yy,I}(x,y_I)\big|_{y_I=y_0} - m_{yy,II}(x,y_{II})\big|_{y_{II}=0} = -m_{x0}(x)$$
$$Q_{y,I}(x,y_I)\big|_{y_I=y_0} - Q_{y,II}(x,y_{II})\big|_{y_{II}=0} = 0$$

Assuming the rectangular plate simply supported along the edges x = 0 and x = a, the solution by Lévy [4] that satisfies the boundary conditions at these edges is considered for the deflection curve w(x,y) as follows:

$$w(x,y) = \frac{1}{D}\sum_m F_m(y)\sin\alpha_m x, \qquad \alpha_m = \frac{m\pi}{a} \tag{7e}$$

The solution to Equation (7e) as derived in the literature (e.g., Timoshenko [4], Girkmann [5]) is given by

$$w(x,y) = \frac{1}{D}\sum_m \begin{pmatrix} A_m\cosh\alpha_m y + B_m\alpha_m y\sinh\alpha_m y + \\ C_m\sinh\alpha_m y + D_m\alpha_m y\cosh\alpha_m y \end{pmatrix}\sin\alpha_m x \tag{7f}$$

Hence the efforts and deformations needed for the continuity equations for the plate zone I ($0 \le y_I \le y_0$) can be expressed using Equations (7f), (2b), and (3b) as follows

$$w_I(x,y_I) = \frac{1}{D}\sum_m \begin{pmatrix} A_{mI}\cosh\alpha_m y_I + B_{mI}\alpha_m y_I\sinh\alpha_m y_I + \\ C_{mI}\sinh\alpha_m y_I + D_{mI}\alpha_m y_I\cosh\alpha_m y_I \end{pmatrix}\sin\frac{m\pi x}{a}$$

$$\frac{\partial w_I(x,y_I)}{\partial y_I} = \frac{1}{D}\sum_m \alpha_m \begin{bmatrix} A_{mI}\sinh\alpha_m y_I + B_{mI}\left(\sinh\alpha_m y_I + \alpha_m y_I\cosh\alpha_m y_I\right) + \\ C_{mI}\cosh\alpha_m y_I + D_{mI}\left(\cosh\alpha_m y_I + \alpha_m y_I\sinh\alpha_m y_I\right) \end{bmatrix}\sin\alpha_m x$$

(8a-d)

$$m_{yy,I} = -\sum_m \alpha_m^2(1-\nu)\begin{bmatrix} A_{mI}\cosh\alpha_m y_I + B_{mI}\left(\frac{2}{1-\nu}\cosh\alpha_m y_I + \alpha_m y_I\sinh\alpha_m y_I\right) + \\ C_{mI}\sinh\alpha_m y_I + D_{mI}\left(\frac{2}{1-\nu}\sinh\alpha_m y_I + \alpha_m y_I\cosh\alpha_m y_I\right) \end{bmatrix}\sin\alpha_m x$$

$$Q_{y,I} = -\sum_m \alpha_m^3\left[2B_{mI}\sinh\alpha_m y_I + 2D_{mI}\cosh\alpha_m y_I\right]\sin\alpha_m x$$

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The efforts and deformations for the plate zone II ($0 \le y_{II} \le y_1$) are expressed by replacing the subscript I with II in Equations (8a-d). Recalling that the position $y = y_0$ corresponds to $y_I = y_0$ and $y_{II} = 0$, the continuity equations can be formulated using Equations (7a-d) and (8a-d) as follows

$$A_{mI}\cosh\alpha_m y_0 + B_{mI}\alpha_m y_0 \sinh\alpha_m y_0 + C_{mI}\sinh\alpha_m y_0 + D_{mI}\alpha_m y_0\cosh\alpha_m y_0 - A_{mII} = 0$$

$$A_{mI}\sinh\alpha_m y_0 + B_{mI}\left(\sinh\alpha_m y_0 + \alpha_m y_0\cosh\alpha_m y_0\right) + C_{mI}\cosh\alpha_m y_0 + D_{mI}\left(\cosh\alpha_m y_0 + \alpha_m y_0\sinh\alpha_m y_0\right) - C_{mII} - D_{mII} = 0$$

(9a-d)

$$A_{mI}\cosh\alpha_m y_0 + B_{mI}\left(\frac{2}{1-\nu}\cosh\alpha_m y_0 + \alpha_m y_0\sinh\alpha_m y_0\right) + C_{mI}\sinh\alpha_m y_0 + D_{mI}\left(\frac{2}{1-\nu}\sinh\alpha_m y_0 + \alpha_m y_0\cosh\alpha_m y_0\right) - A_{mII} - \frac{2}{1-\nu}B_{mII} = \frac{1}{\alpha_m^2(1-\nu)}\frac{2M_{x0}}{a}\sin\frac{m\pi x_0}{a}$$

$$B_{mI}\sinh\alpha_m y_0 + D_{mI}\cosh\alpha_m y_0 - D_{mII} = 0$$

The Kirchhoff shear forces needed for boundary conditions at free edges are expressed using Equations (4b) and (8a)

$$V_{y,I} = -\sum_m \alpha_m^3(\nu-1)\left[\begin{array}{l} A_{mI}\sinh\alpha_m y_I + B_{mI}\left(\dfrac{\nu+1}{\nu-1}\sinh\alpha_m y_I + \alpha_m y_I\cosh\alpha_m y_I\right) + \\ C_{mI}\cosh\alpha_m y_I + D_{mI}\left(\dfrac{\nu+1}{\nu-1}\cosh\alpha_m y_I + \alpha_m y_I\sinh\alpha_m y_I\right)\end{array}\right]\sin\alpha_m x \quad (10)$$

In summary, the coefficients A_{mI}, B_{mI}, C_{mI}, and D_{mI} and A_{mII}, B_{mII}, C_{mII}, and D_{mII} are determined by satisfying the boundary conditions at $y = 0$ and $y = b$ and the continuity conditions at $y = y_0$. They are calculated for various boundary conditions at $y = 0$ and $y = b$ in Appendix A whereby the cases of bending moment applied at the edge $y = 0$ are also considered. Then, the bending moments m_{yy} are calculated using Equation (8c), and the bending moments m_{xx} and twisting moments m_{xy} are calculated using Equations (2a, c) and (8a) as follows

$$m_{xx,I} = -\sum_m \alpha_m^2(\nu-1)\left[\begin{array}{l} A_{mI}\cosh\alpha_m y_I + B_{mI}\left(\dfrac{2\nu}{\nu-1}\cosh\alpha_m y_I + \alpha_m y_I\sinh\alpha_m y_I\right) + \\ C_{mI}\sinh\alpha_m y_I + D_{mI}\left(\dfrac{2\nu}{\nu-1}\sinh\alpha_m y_I + \alpha_m y_I\cosh\alpha_m y_I\right)\end{array}\right]\sin\alpha_m x$$

(11a-b)

$$m_{xy,I} = -(1-\nu)\times\sum_m \alpha_m^2\left[\begin{array}{l} A_{mI}\sinh\alpha_m y_I + B_{mI}\left(\sinh\alpha_m y_I + \alpha_m y_I\cosh\alpha_m y_I\right) + \\ C_{mI}\cosh\alpha_m y_I + D_{mI}\left(\cosh\alpha_m y_I + \alpha_m y_I\sinh\alpha_m y_I\right)\end{array}\right]\cos\alpha_m x.$$

Load case "Concentrated bending moment applied at the edge x = 0"

Let now the external concentrated moment M_{y0} be applied at the position $(0, y_0)$ as represented in Figure 3

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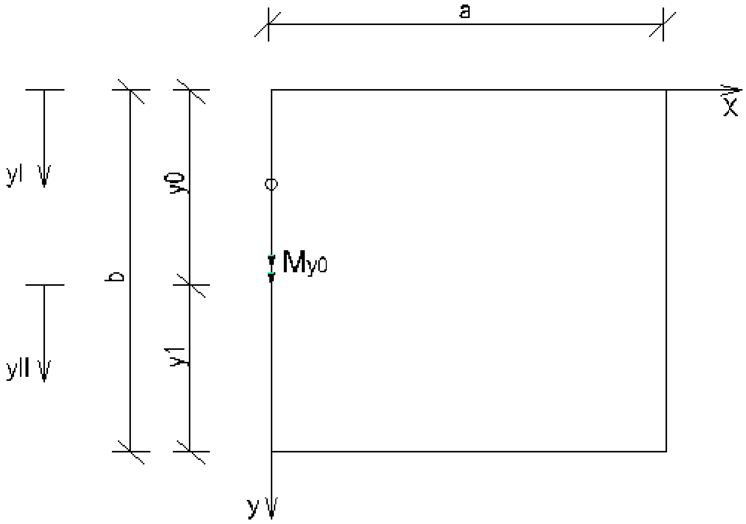

Figure 3. Rectangular plate subjected to an external concentrated moment My0 at (0, y_0).

The concentrated moment assumed applied at a position (x_0, y_0), with x_0 later set to zero, is replaced with a couple of forces -P and P applied at the positions (x_0 + c, y_0) and (x_0, y_0), respectively, with c approaching zero and My0 = P×c. The combined forces can then be expanded into a Fourier series along the line y = y_0 as follows

$$\begin{aligned} q_{y0}(x) &= \frac{2P}{a}\sum_m\left(-\sin\frac{m\pi(x_0+c)}{a}+\sin\frac{m\pi x_0}{a}\right)\sin\frac{m\pi x}{a} \\ &= -\frac{2P}{a}\sum_m\frac{m\pi c}{a}\cos\frac{m\pi x_0}{a}\sin\frac{m\pi x}{a} \\ &= -\frac{2\pi M_{y0}}{a^2}\sum_m m\cos\frac{m\pi x_0}{a}\sin\frac{m\pi x}{a} \end{aligned} \tag{11c}$$

Observing that x_0 = 0 it yields the following distributed load along the line y = y_0 that replaces My0

$$q_{y0}(x) = -\frac{2\pi M_{y0}}{a^2}\sum_m m\sin\frac{m\pi x}{a} \tag{11d}$$

The continuity equations along y = y_0 can then be formulated using Equations (7a-d) whereby (7c-d) are modified as follows

$$\begin{aligned} m_{yy,I}(x,y_I)\Big|_{y_I=y_0} - m_{yy,II}(x,y_{II})\Big|_{y_{II}=0} &= 0 \\ Q_{y,I}(x,y_I)\Big|_{y_I=y_0} - Q_{y,II}(x,y_{II})\Big|_{y_{II}=0} &= q_{y0}(x) \end{aligned} \tag{11e-f}$$

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Equation (11e) corresponds to (9c) by setting to zero the term on the right-hand side of (9c) and (11f) is expressed using (8d), (11d), and (11f) as follows

$$B_{mI}\sinh\alpha_m y_0 + D_{mI}\cosh\alpha_m y_0 - D_{mII} = \frac{M_{y0}}{a\alpha_m^2} \tag{11g}$$

Load case "Distributed bending moment applied along the edge x = 0"

Assuming a distributed moment $m_{y0}(y)$ applied along the edge x = 0 as represented in Figure 4

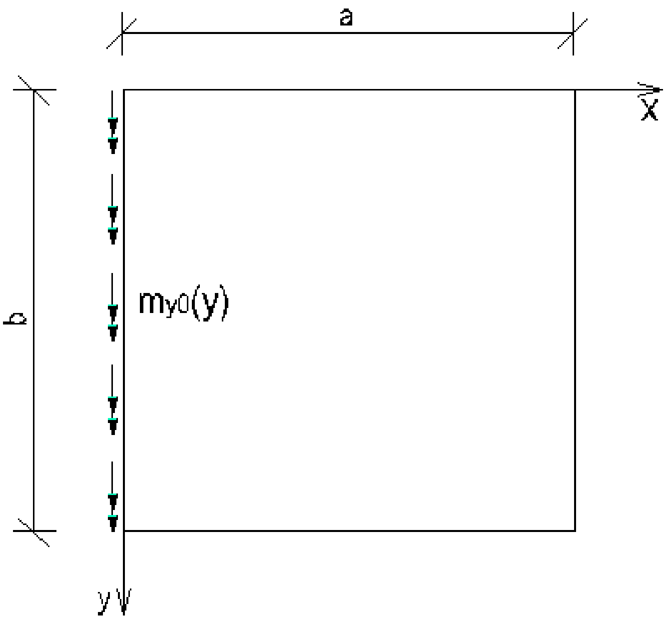

Figure 4. Rectangular plate subjected to a distributed bending moment along the edge x = 0.

Inspired by Fogang [10], the deflection curve can be taken

$$w(x,y) = \frac{1}{D}\sum_m F_m(y)\sin\alpha_m x - \frac{a^2}{6D}\left[1-\frac{x}{a}-\left(1-\frac{x}{a}\right)^3\right]m_{yo}(y) \tag{11h}$$

It is noted that Equation (11h) satisfies the boundary conditions at edges x = 0 and x = *a*. Substituting Equation (11h) into (1) yields

$$\begin{aligned}&\frac{1}{D}\sum_m\left[\frac{d^4F_m(y)}{dy^4} - 2\alpha_m^2\frac{d^2F_m(y)}{dy^2} + \alpha_m^4F_m(y)\right]\sin\alpha_m x + \\ &+\frac{2}{D}\left(1-\frac{x}{a}\right)\frac{d^2m_{y0}(y)}{dy^2} - \frac{a^2}{6D}\left[1-\frac{x}{a}-\left(1-\frac{x}{a}\right)^3\right]\frac{d^4m_{y0}(y)}{dy^4} = 0\end{aligned} \tag{11i}$$

The following functions contained in Equation (11i) are expanded in Fourier sine series

$$1-\frac{x}{a}=\sum_m \frac{2}{m\pi}\sin\alpha_m x, \qquad 1-\frac{x}{a}-\left(1-\frac{x}{a}\right)^3=\sum_m \frac{12}{m^3\pi^3}\sin\alpha_m x \tag{11j}$$

Substituting Equations (11j) into (11i) and given that the latter holds for any value of x, it results the following differential equation

$$\frac{d^4F_m(y)}{dy^4}-2\alpha_m^2\frac{d^2F_m(y)}{dy^2}+\alpha_m^4F_m(y)=\frac{2a^2}{m^3\pi^3}\frac{d^4m_{y0}(y)}{dy^4}-\frac{4}{m\pi}\frac{d^2m_{y0}(y)}{dy^2} \tag{11k}$$

The solution to Equation (11k) is

$$F_m(y)=A_m\cosh\alpha_m y+B_m\alpha_m y\sinh\alpha_m y+C_m\sinh\alpha_m y+D_m\alpha_m y\cosh\alpha_m y+F_{mp}(y) \tag{11l}$$

where the coefficients A_m, B_m, C_m, and D_m are determined by satisfying the boundary conditions at y = 0 and y = *b*, and $F_{mp}(y)$ is a particular solution to Equation (11k). Substituting Equation (11l) into (11h) yields

$$\begin{aligned} w(x,y)=\frac{1}{D}\sum_m&\begin{bmatrix} A_m\cosh\alpha_m y+B_m\alpha_m y\sinh\alpha_m y+\\ C_m\sinh\alpha_m y+D_m\alpha_m y\cosh\alpha_m y+F_{mp}(y)\end{bmatrix}\sin\alpha_m x+\\ &-\frac{a^2}{6D}\left[1-\frac{x}{a}-\left(1-\frac{x}{a}\right)^3\right]m_{yo}(y)\end{aligned} \tag{11m}$$

To satisfy the boundary conditions at y = 0 and y = *b*, the Fourier series of Equation (11j) is used; so Equation (11m) becomes

$$w(x,y)=\frac{1}{D}\sum_m\begin{bmatrix} A_m\cosh\alpha_m y+B_m\alpha_m y\sinh\alpha_m y+C_m\sinh\alpha_m y+\\ D_m\alpha_m y\cosh\alpha_m y+F_{mp}(y)-\frac{2a^2}{m^3\pi^3}m_{yo}(y)\end{bmatrix}\sin\alpha_m x \tag{11n}$$

With respect to the boundary conditions at y = 0 and y = *b* the equations for the slope ∂w/∂y, the bending moment m_{yy} and the Kirchhoff shear forces are formulated using Equations (2b) and (4b) as follows

$$\frac{\partial w(x,y)}{\partial y}=\frac{1}{D}\sum_m\alpha_m\begin{bmatrix} A_m\sinh\alpha_m y+B_m\left(\sinh\alpha_m y+\alpha_m y\cosh\alpha_m y\right)+\\ C_m\cosh\alpha_m y+D_m\left(\cosh\alpha_m y+\alpha_m y\sinh\alpha_m y\right)+\\ \frac{1}{\alpha_m}\frac{dF_{mp}(y)}{dy}-\frac{2a^3}{m^4\pi^4}\frac{dm_{yo}(y)}{dy}\end{bmatrix}\sin\alpha_m x \tag{11o}$$

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$$m_{yy} = -\sum_m \alpha_m^2 (1-\nu) \left[\begin{array}{l} A_m \cosh\alpha_m y + B_m \left(\dfrac{2}{1-\nu}\cosh\alpha_m y + \alpha_m y \sinh\alpha_m y \right) + \\ C_m \sinh\alpha_m y + D_m \left(\dfrac{2}{1-\nu}\sinh\alpha_m y + \alpha_m y \cosh\alpha_m y \right) + \\ \dfrac{1}{\alpha_m^2 (1-\nu)} \left[\dfrac{d^2 F_{mp}(y)}{dy^2} - \nu \alpha_m^2 F_{mp}(y) \right] + \\ \dfrac{1}{\alpha_m^2 (1-\nu)} \left[-\dfrac{2a^2}{m^3 \pi^3} \dfrac{d^2 m_{yo}(y)}{dy^2} + \dfrac{2\nu}{m\pi} m_{yo}(y) \right] \end{array} \right] \sin\alpha_m x$$

$$V_y = -\sum_m \alpha_m^3 (\nu-1) \left[\begin{array}{l} A_m \sinh\alpha_m y + B_m \left(\dfrac{\nu+1}{\nu-1}\sinh\alpha_m y + \alpha_m y \cosh\alpha_m y \right) + \\ C_m \cosh\alpha_m y + D_m \left(\dfrac{\nu+1}{\nu-1}\cosh\alpha_m y + \alpha_m y \sinh\alpha_m y \right) + \\ \dfrac{1}{\alpha_m^3 (\nu-1)} \left[\dfrac{d^3 F_{mp}(y)}{dy^3} - (2-\nu) \alpha_m^2 \dfrac{dF_{mp}(y)}{dy} \right] + \\ \dfrac{1}{\alpha_m^3 (\nu-1)} \left[-\dfrac{2a^2}{m^3 \pi^3} \dfrac{d^3 m_{yo}(y)}{dy^3} + \dfrac{2(2-\nu)}{m\pi} \dfrac{dm_{yo}(y)}{dy} \right] \end{array} \right] \sin\alpha_m x \qquad (11p\text{-}q)$$

The special case of a constant bending moment my0 applied along the edge x = 0 will be treated in the Results section.

2.2.3. Rectangular plate of infinite length loaded near its end and having two opposite edges simply supported

The case of loading around the plate middle will be treated in the Results section.

Let us analyze here a plate of infinite length with two opposite edges x = 0 and x = a simply supported and subjected near its end to an external concentrated moment Mx0, as shown in Figure 5

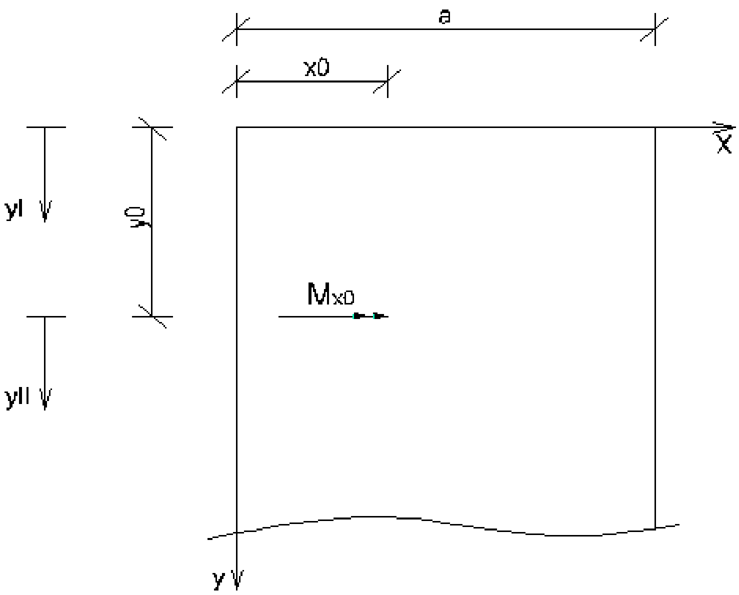

Figure 5. Rectangular plate of infinite length subjected near its end to an external concentrated moment Mx0.

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The efforts and deformation in plate zone I ($0 \leq y_I \leq y_0$) are formulated according to Section 2.2.2.

The edges x = 0 and x = *a* being simply supported, the deflection curve for the plate of infinite length (plate zone $y_{II} \geq 0$ in Figure 5) as derived in the literature (e.g., Timoshenko [4], Girkmann [5]) is given by

$$w_{II}(x, y_{II}) = \frac{1}{D}\sum_m \left(A_{mII} + B_{mII}\alpha_m y_{II}\right) e^{-\alpha_m y_{II}} \sin\alpha_m x \tag{12}$$

It yields as follows the slope $\partial w_{II}/\partial y_{II}$, the bending moment m_{yy} and the shear force Q_y needed for the continuity conditions along $y = y_0$

$$\frac{\partial w_{II}(x, y_{II})}{\partial y_{II}} = \frac{1}{D}\sum_m \alpha_m \left(B_{mII} - A_{mII} - B_{mII}\alpha_m y_{II}\right) e^{-\alpha_m y_{II}} \sin\alpha_m x$$

$$m_{yy,II}(x, y_{II}) = -\sum_m \alpha_m^2 \left[(1-\nu) A_{mII} - 2B_{mII} + (1-\nu) B_{mII}\alpha_m y_{II}\right] e^{-\alpha_m y_{II}} \sin\alpha_m x \tag{13a-c}$$

$$Q_{y,II}(x, y_{II}) = -\sum_m 2\alpha_m^3 B_{mII}\, e^{-\alpha_m y_{II}} \sin\alpha_m x$$

The Kirchhoff shear force needed for boundary conditions at free edges is given by

$$V_{y,II} = -\sum_m \alpha_m^3 \left[(1-\nu) A_{mII} + (1+\nu) B_{mII} + (1-\nu) B_{mII}\alpha_m y_{II}\right] e^{-\alpha_m y_{II}} \sin\alpha_m x \tag{14}$$

Reminding that the position $y = y_0$ corresponds to $y_I = y_0$ and $y_{II} = 0$, the continuity equations between the plate zone I ($0 \leq y_I \leq y_0$) and the plate zone II of infinite length ($y_{II} \geq 0$) can be expressed using Equations (7a-d), (8a-d), (12), and (13a-c)

$$A_{mI}\cosh\alpha_m y_0 + B_{mI}\alpha_m y_0 \sinh\alpha_m y_0 + C_{mI}\sinh\alpha_m y_0 + D_{mI}\alpha_m y_0 \cosh\alpha_m y_0 - A_{mII} = 0$$

$$\begin{aligned}
&A_{mI}\sinh\alpha_m y_0 + B_{mI}\left(\sinh\alpha_m y_0 + \alpha_m y_0\cosh\alpha_m y_0\right) + C_{mI}\cosh\alpha_m y_0 + \\
&D_{mI}\left(\cosh\alpha_m y_0 + \alpha_m y_0 \sinh\alpha_m y_0\right) + A_{mII} - B_{mII} = 0 \\
&A_{mI}\cosh\alpha_m y_0 + B_{mI}\left(\frac{2}{1-\nu}\cosh\alpha_m y_0 + \alpha_m y_0\sinh\alpha_m y_0\right) + C_{mI}\sinh\alpha_m y_0 + \\
&D_{mI}\left(\frac{2}{1-\nu}\sinh\alpha_m y_0 + \alpha_m y_0\cosh\alpha_m y_0\right) - A_{mII} + \frac{2}{1-\nu}B_{mII} = \frac{1}{\alpha_m^2(1-\nu)}\frac{2M_{x0}}{a}\sin\frac{m\pi x_0}{a}
\end{aligned} \tag{15a-d}$$

$$2B_{mI}\sinh\alpha_m y_0 + 2D_{mI}\cosh\alpha_m y_0 - 2B_{mII} = 0$$

In summary, the coefficients A_{mI}, B_{mI}, C_{mI}, D_{mI}, A_{mII}, and B_{mII} are determined by satisfying the boundary conditions at $y = 0$ and the continuity conditions at $y = y_0$. They are calculated for various support conditions at y = 0 in Appendix B.

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Then, in the plate zone of infinite length the bending moments myy are calculated using Equation (13b), and the bending moments mxx and twisting moments mxy are calculated using Equations (2a, c) and (12) as follows

$$m_{xx,II}(x, y_{II}) = \sum_m \alpha_m^2 \left[(1-\nu) A_{mII} + 2\nu B_{mII} + (1-\nu) B_{mII} \alpha_m y_{II} \right] e^{-\alpha_m y_{II}} \sin \alpha_m x$$

(15e-f)

$$m_{xy,II} = -(1-\nu) \times \sum_m \alpha_m^2 \left[-A_{mII} + B_{mII} - B_{mII} \alpha_m y_{II} \right] e^{-\alpha_m y_{II}} \cos \alpha_m x$$

2.2.4. Rectangular plate of infinite length loaded around the plate middle and having two opposite edges arbitrarily supported

A rectangular plate with dimensions *a* and *b* in x- and y- direction, respectively, and loaded around the plate middle or at a large distance from the edges x = 0 and x = *a* is analyzed. The plate with a very high aspect ratio a/b is then considered of infinite length while the edges y = 0 and y = *b* are arbitrarily supported, as represented in Figure 6

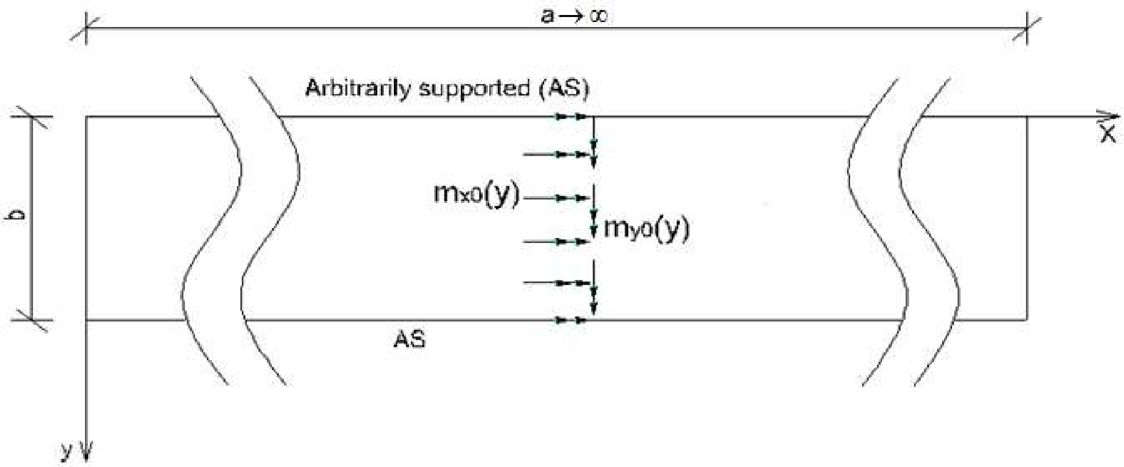

Figure 6. Rectangular plate of infinite length loaded around the plate middle.

The edges x = 0 and x = *a* are at a large distance from the loading. Therefore, the structural behavior of the plate subjected to this loading does not depend on the support conditions at these edges; then the latter can be assumed simply supported and so the Lévy solution (Equation (7e)) can be applied. Finally the results are obtained by letting *a* tend to infinity. This reasoning can be found in Courbon et al. [13].

Furthermore, it is observed that the structural behavior of the plate of infinite length subjected to an external concentrated bending moments at a large distance from the edges x= 0 and x = *a* is the same as if the loads were applied at the middle. The y_0- axis of application of the loads is then considered as the middle of the plate.

Load case "Distributed bending moment mx0(y) applied along x = a/2"

The distributed bending moment $m_{x0}(y)$ is replaced at any position *y* with a couple of forces and so it yields the distributed load $-dm_{x0}(y)/dy$ along x = a/2. The Fourier series expansion of the load applied at x_0 yields the following load per unit area throughout the plate

$$q_z(x, y) = -\frac{2}{a} \frac{dm_{x0}(y)}{dy} \sum_m \sin \frac{m\pi x_0}{a} \sin \frac{m\pi x}{a} \quad (15g)$$

The deflection curve follows Equation (7e) and observing that x_0 = a/2 the differential equation is given by

$$\frac{d^4F_m(y)}{dy^4} - 2\alpha_m^2 \frac{d^2F_m(y)}{dy^2} + \alpha_m^4 F_m(y) = -\frac{2}{a}\frac{dm_{x0}(y)}{dy}\sin\frac{m\pi}{2} \quad (15h)$$

Then the analysis continues using Equations (11l-q) whereby $m_{y0}(y)$ is set to zero.

Load case "Distributed bending moment my0(y) applied along x = a/2"

Using Equation (11c) the Fourier series expansion of the load applied at x_0 yields the following load per unit area throughout the plate

$$q_z(x,y) = -\frac{2\pi m_{y0}(y)}{a^2}\sum_m m\cos\frac{m\pi x_0}{a}\sin\frac{m\pi x}{a} \quad (15i)$$

The deflection curve follows Equation (7e) and observing that x_0 = a/2 the differential equation is given by

$$\frac{d^4F_m(y)}{dy^4} - 2\alpha_m^2 \frac{d^2F_m(y)}{dy^2} + \alpha_m^4 F_m(y) = -\frac{2\pi m_{y0}(y)}{a^2} m\cos\frac{m\pi}{2} \quad (15j)$$

Load case "External concentrated bending moments applied at (x = a/2, y = b)"

These load cases will be treated in the Results section.

2.2.5. Solution of this study: Rectangular plate with one or two opposite edges clamped

It is assumed that from the two supported opposite edges x = 0 and x = a, one or both are clamped. The analysis can then be conducted using the flexibility method according to Fogang [10]. The primary system is the rectangular plate simply supported along the edges x = 0 and x = a, and is treated according to the previous sections. The flexibilities δ_{j0} (slopes at positions j of the opposite edges where the compatibility equations will be set) for the primary problem are calculated for an ordinary plate and a plate of infinite length, respectively, as follows

$$\delta_{j0} = \left.\frac{\partial w(x,y)}{\partial x}\right|_{\substack{x=x_j\\y=y_j}} = \frac{1}{D}\sum_m \alpha_m \begin{bmatrix} A_{mI}\cosh\alpha_m y_j + B_{mI}\alpha_m y_j \sinh\alpha_m y_j + \\ C_{mI}\sinh\alpha_m y_j + D_{mI}\alpha_m y_j\cosh\alpha_m y_j \end{bmatrix}\cos\alpha_m x_j$$

(16a-b)

$$\delta_{j0} = \left.\frac{\partial w(x,y)}{\partial x}\right|_{\substack{x=x_j\\y=y_j}} = \frac{1}{D}\sum_m \alpha_m \left(A_{mI} + B_{mI}\alpha_m y_j\right)e^{-\alpha_m y_j}\cos\alpha_m x_j$$

The redundant system is the plate simply supported along the opposite edges and subjected to bending moments along those edges.

3. Results and discussion

3.1. Plate of infinite length subjected to an external concentrated moment at the middle

The plate of infinite length is subjected to an external concentrated moment at the middle or at a large distance from the y edges, as represented in Figure 7.

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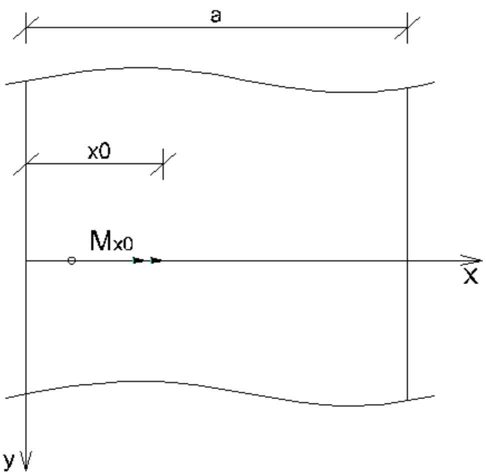

Figure 7. Rectangular plate of infinite length subjected to an external concentrated moment Mx0 at the middle.

Given the anti-symmetrical nature of the load and the symmetry of the system, the deflections are zero along the line y = 0. And since the load is equally divided between the two halves of the plate it yields $m_{yy}(x, y = 0) = m_{xo}(x)/2$. Substituting these conditions into Equations (12), (13b), and (6) yields for y ≥ 0

$$w(x, y) = \frac{M_{x0} y}{2\pi D} \sum_m \frac{1}{m} e^{-\alpha_m y} \sin \alpha_m x_0 \sin \alpha_m x \qquad (17)$$

This result can be found in Girkmann [5].

3.2. Plate of infinite length subjected to a concentrated moment at its end

The plate of infinite length is subjected at its end y = 0 to an external concentrated moment, as represented in Figure 8.

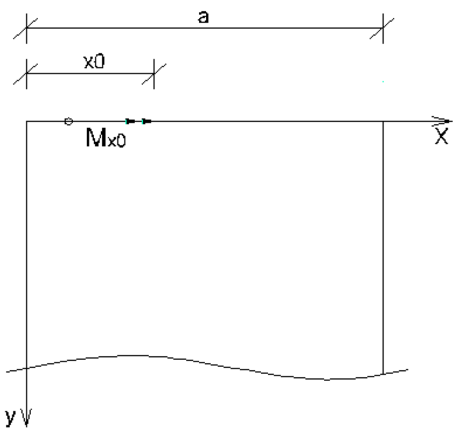

Figure 8. Rectangular plate of infinite length subjected at its end to an external concentrated bending moment Mx0.

Case 1: Edge y = 0 simply supported

The deflections are zero along the line y = 0 and the bending moment myy at y = 0 is equal to the distributed moment m_{xo} (x) (the concentrated moment Mx0 expanded into a Fourier series according to Equation (6)). Substituting these conditions into Equations (12) and (13b) yields

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$$w(x,y) = \frac{M_{x0}y}{\pi D}\sum_m \frac{1}{m} e^{-\alpha_m y} \sin\alpha_m x_0 \sin\alpha_m x \quad (18)$$

Case 2: Edge y = 0 free

The Kirchhoff shear force is zero along the line y = 0 and the bending moment myy at y = 0 is equal to the distributed moment (the concentrated moment M_{x0} expanded into a Fourier series according to Equation (6)). Substituting these conditions into Equations (13b) and (14) yields

$$w(x,y) = \frac{2M_{x0}}{D}\sum_m \left(\frac{-(1+\nu)a}{(1-\nu)(3+\nu)\pi^2 m^2} + \frac{y}{(3+\nu)\pi m} \right) e^{-\alpha_m y} \sin\alpha_m x_0 \sin\alpha_m x \quad (19)$$

3.3. Rectangular plate with the edges x = 0 and x = a simply supported and subjected to a constant bending moment loading along x = 0

The plate simply supported along the edges x = 0 and x = *a* is subjected at x = 0 to a constant bending moment loading m_{yo}, as represented in Figure 9. The edges y = 0 and y = *b* are taken both simply supported and both clamped; therefore, for simplification purpose the x-axis can be shifted to the middle of the plate

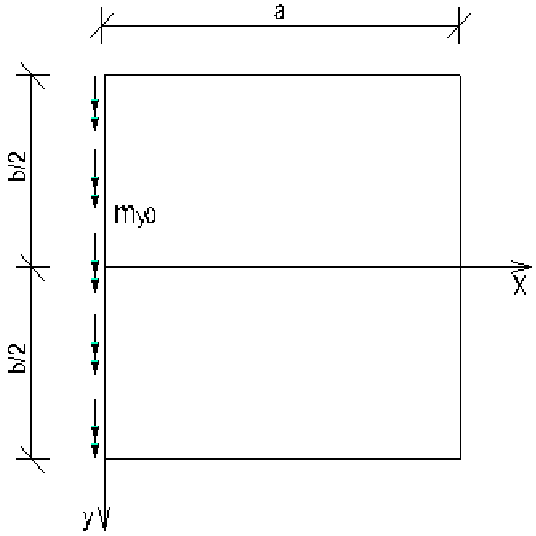

Figure 9. Rectangular plate subjected at the edge x = 0 to a constant bending moment loading m_{yo}.

Case y = 0 and y = b simply supported

Using Equation (11k) yields the particular solution $F_{mp}(y)$ is zero. From the conditions of symmetry $C_m = D_m = 0$.

The satisfaction of the boundary conditions yields the deflection curve as follows

$$w(x,y) = \frac{1}{D}\sum_m \left[A_m \cosh\alpha_m y + B_m \alpha_m y \sinh\alpha_m y\right]\sin\alpha_m x - \frac{a^2}{6D}\left[1 - \frac{x}{a} - \left(1-\frac{x}{a}\right)^3\right] m_{yo}$$

$$A_m = \frac{a^2\left(2+Y_m \tanh Y_m\right)}{m^3\pi^3 \cosh Y_m} m_{yo}, \qquad B_m = \frac{-a^2}{m^3\pi^3\cosh Y_m} m_{yo}, \qquad Y_m = \frac{m\pi b}{2a} \quad (20)$$

Then the bending moments per unit length m_{xx} and m_{yy}, and the twisting moments per unit length m_{xy} are calculated using Equations (2a-c) and (20).

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Case y = 0 and y = b clamped

Similarly, the deflection curve is as follows

$$w(x,y)=\frac{1}{D}\sum_m\left[A_m\cosh\alpha_m y+B_m\alpha_m y\sinh\alpha_m y\right]\sin\alpha_m x-\frac{a^2}{6D}\left[1-\frac{x}{a}-\left(1-\frac{x}{a}\right)^3\right]m_{yo} \tag{21}$$

$$A_m=\frac{2a^2}{m^3\pi^3}\frac{\sinh Y_m+Y_m\cosh Y_m}{Y_m+1/2\sinh 2Y_m}m_{yo},\qquad B_m=\frac{-2a^2}{m^3\pi^3}\frac{\sinh Y_m}{Y_m+1/2\sinh 2Y_m}m_{yo},\qquad Y_m=\frac{m\pi b}{2a}$$

3.4. Cantilever plate of infinite length subjected to external concentrated moments at the middle

The cantilever plate of infinite length is subjected to concentrated bending moments at the middle, as shown in Figure 10

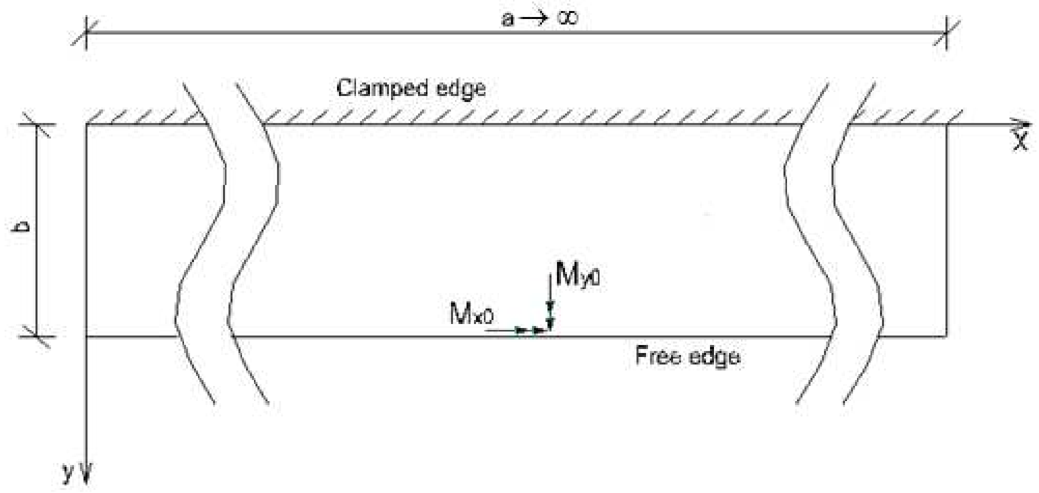

Figure 10. Cantilever plate of infinite length subjected to external concentrated bending moments at the middle.

Load case "Concentrated bending moment Mx0 applied at (x = a/2, y = b)"

The dispositions of Section 2.2.4 are applied whereby the Poisson's ratio is not considered for simplification purpose. The following boundary conditions are set using Equations (6), (8a-c), and (10)

$$w(x,y)\big|_{y=0}=0,\quad \left.\frac{\partial w(x,y)}{\partial y}\right|_{y=0}=0,\quad m_{yy}\big|_{y=b}=-m_{x0}(x),\quad V_y\big|_{y=b}=0 \tag{22a}$$

Then the coefficients A_m, B_m, C_m, and D_m are determined as follows

$$A_m=0,\qquad B_m=\frac{2M_{x0}}{a\alpha_m^2}\frac{2\cosh Y_m-Y_m\sinh Y_m}{Y_m^2+3\cosh^2 Y_m+1}\sin\frac{m\pi}{2},$$
$$C_m=-D_m,\qquad D_m=\frac{-2M_{x0}}{a\alpha_m^2}\frac{\sinh Y_m-Y_m\cosh Y_m}{Y_m^2+3\cosh^2 Y_m+1}\sin\frac{m\pi}{2},\qquad Y_m=\frac{m\pi b}{a} \tag{22b}$$

Setting $Y_m = m\rho$ with $\rho = \pi b/a \to 0$, the bending moments m_{yy} at any position (x = a/2, y = kb) are

$$m_{yy}\Big|_{\substack{x=a/2\\y=kb}}=-\frac{2M_{x0}}{\pi b}\lim_{\rho\to 0}\sum_{m=1,3,5\ldots}\left[\begin{array}{l}\rho\dfrac{2\cosh m\rho-m\rho\sinh m\rho}{m^2\rho^2+3\cosh^2 m\rho+1}\left(2\cosh km\rho+km\rho\sinh km\rho\right)-\\ \rho\dfrac{\sinh m\rho-m\rho\cosh m\rho}{m^2\rho^2+3\cosh^2 m\rho+1}\left(\sinh km\rho+km\rho\cosh km\rho\right)\end{array}\right] \tag{22c}$$

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The summation F(k) in Equation (22c) for ρ tending to zero is evaluated depending on k and the values are calculated in the Supplementary Material "Cantilever plate of infinite length under bending moment Mxo" and listed in Table 1

Table 1. Coefficient F(k) of the bending moment m_{yy} at a position y = kb along x = a/2.

k =	**0,00**	**0,20**	**0,40**	**0,50**	**0,60**	**0,70**	**0,80**	**0,90**
F(k) =	0,3820	0,4992	0,6289	0,7302	0,8853	1,1514	1,7021	3,4303

So, the bending moments m_{yy} at the middle of the clamped edge and at y = b/2, respectively, are

$$m_{yy}\Big|_{\substack{x=a/2\\y=0}} = -\frac{0.764M_{x0}}{\pi b}, \qquad m_{yy}\Big|_{\substack{x=a/2\\y=b/2}} = -\frac{1.460M_{x0}}{\pi b} \tag{22d}$$

At the point of application of the load (k = 1.0) the summation F(k) did not converge. This result is in agreement with Timoshenko [4].

Moreover, the calculated moments in the vicinity of the load (k = 0.80, 0.90) showed a rapid increase and must be verified. For this purpose the external concentrated moment is replaced with a distributed moment M_{x0} /2c acting over a length 2c; this distributed moment is expanded in Fourier series as follows

$$m_{x0}(x) = \frac{2M_{x0}}{\pi c}\sum_{m}\frac{1}{m}\sin\frac{m\pi c}{a}\sin\frac{m\pi x_0}{a}\sin\frac{m\pi x}{a} \tag{22e}$$

Equation (22e) replaces then (6) in the analysis whereby x_0 = a/2. Setting c = λb and ρ = πb/a → 0, the bending moments m_{yy} at positions (x = a/2, y = kb) are expressed as follows

$$m_{yy}\Big|_{\substack{x=a/2\\y=kb}} = -\frac{2M_{x0}}{\pi b}\lim_{\rho\to 0}\sum_{m=1,3,5...}\left[\begin{array}{l}\dfrac{\sin\lambda m\rho}{\lambda m}\dfrac{2\cosh m\rho - m\rho\sinh m\rho}{m^2\rho^2+3\cosh^2 m\rho+1}\left(2\cosh km\rho + km\rho\sinh km\rho\right) - \\ \dfrac{\sin\lambda m\rho}{\lambda m}\dfrac{\sinh m\rho - m\rho\cosh m\rho}{m^2\rho^2+3\cosh^2 m\rho+1}\left(\sinh km\rho + km\rho\cosh km\rho\right)\end{array}\right] \tag{22f}$$

The summation F(k, λ) in Equation (22f) for ρ tending to zero is evaluated depending on k for λ = **0.05** and **0.10**, and the values are calculated in the above mentioned supplementary material and displayed in Table 2.

Table 2. Coefficients F(k, λ) of the bending moments m_{yy} at (x = a/2, y = kb) for λ = **0.05, 0.10.**

k =	**0,00**	**0,20**	**0,40**	**0,50**	**0,60**	**0,70**	**0,80**	**0,90**
F(k, λ = 0.05) =	0,3826	0,4992	0,6288	0,7300	0,8853	1,1510	1,6970	3,3397
F(k, λ = 0.10) =	0,3842	0,4994	0,6287	0,7297	0,8846	1,1484	1,6779	3,0979

At the point of application of the load (k = 1.0) the summation F(k, λ) did not converge; however the bending moment at this position is a boundary condition namely m_{yy} = -M_{x0} /2c.

In the vicinity of the load (k = 0.80, 0.90) the results for a distributed moment acting over a length 2c (c = 0.05b) were in good agreement with those of a concentrated moment.

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In addition, the results showed that away from the point of application of the load the results for concentrated load are identical to those with the load distributed over a small length: this is in agreement with the Saint Venant's principle.

The values of deflection along the axis x = a/2 of application of the concentrated moment for positions y = kb are

$$w\Big|_{\substack{x=a/2\\y=kb}} = \frac{2bM_{x0}}{\pi D}\lim_{\rho\to 0}\sum_{m=1,3,5...}\left(\begin{array}{l}\dfrac{k}{m}\dfrac{2\cosh m\rho - m\rho\sinh m\rho}{m^2\rho^2+3\cosh^2 m\rho+1}\sinh km\rho - \\ \dfrac{1}{\rho m^2}\dfrac{\sinh m\rho - m\rho\cosh m\rho}{m^2\rho^2+3\cosh^2 m\rho+1}\left(km\rho\cosh km\rho - \sinh km\rho\right)\end{array}\right) \quad (22g)$$

The summation F(k) in Equation (22g) for ρ tending to zero is evaluated depending on the position k and the values calculated in the above mentioned supplementary material are listed in Table 3

Table 3. Coefficient F(k) of the deflection at a position y = kb along the axis x = a/2.

k =	**0,20**	**0,40**	**0,50**	**0,60**	**0,70**	**0,80**	**0,90**	**1,00**
F(k) =	0,0085	0,0370	0,0603	0,0910	0,1307	0,1821	0,2513	0,3610

So, the maximal deflection is at the tip and its value is

$$w\Big|_{\substack{x=a/2\\y=b}} = 0.23\frac{bM_{x0}}{D} \quad (22h)$$

Load case "Concentrated bending moment My0 applied at (x = a/2, y = b)"

Using Equation (11c) the concentrated moment M_{yo} is replaced with a distributed load whereby x_0 = a/2. The boundary conditions are identical to Equation (22a) with following modifications

$$m_{yy}\Big|_{y=b} = 0,\quad V_y\Big|_{y=b} = -\frac{2\pi M_{y0}}{a^2}\sum_m m\cos\frac{m\pi}{2}\sin\frac{m\pi x}{a} \quad (23a)$$

Then the coefficients A_m, B_m, C_m, and D_m are determined as follows

$$A_m = 0,\quad B_m = \frac{-2M_{y0}}{a\alpha_m^2}\frac{\sinh Y_m + Y_m\cosh Y_m}{Y_m^2+3\cosh^2 Y_m+1}\cos\frac{m\pi}{2},$$
$$C_m = -D_m,\quad D_m = \frac{2M_{y0}}{a\alpha_m^2}\frac{2\cosh Y_m + Y_m\sinh Y_m}{Y_m^2+3\cosh^2 Y_m+1}\cos\frac{m\pi}{2},\quad Y_m = \frac{m\pi b}{a} \quad (23b)$$

The bending moments m_{yy} along the clamped edge, at a distance x_0 from the middle, are given by

$$m_{yy}\Big|_{\substack{x=a/2+x0\\y=0}} = \sum_m \frac{4M_{y0}}{a}\frac{\sinh Y_m + Y_m\cosh Y_m}{Y_m^2+3\cosh^2 Y_m+1}\cos\frac{m\pi}{2}\sin\left(\frac{m\pi}{2}+\frac{m\pi x_0}{a}\right),\quad Y_m = \frac{m\pi b}{a} \quad (23c)$$

The bending moment m_{yy} is zero along the axis x = a/2 (x_0 = 0) of application of the concentrated moment.

Setting Y_m = mρ with ρ = πb/a → 0, and x_0 = kb, the bending moment m_{yy} along the clamped edge is

$$m_{yy}\Big|_{\substack{x=a/2+kb\\y=0}} = \frac{4M_{y0}}{\pi b}\lim_{\rho\to 0}\sum_{m=2,4...}\rho\frac{\sinh m\rho + m\rho\cosh m\rho}{m^2\rho^2+3\cosh^2 m\rho+1}\sin\left(km\rho\right) \quad (23d)$$

The summation F(k) in Equation (23d) for ρ tending to zero is evaluated depending on k and the values calculated in Supplementary Material "Cantilever plate of infinite length under bending moment Myo" are listed in Table 4

Table 4. Coefficient F(k) of the bending moment m_{yy} at a position x0 = kb of the clamped edge.

k =	**0,00**	**0,25**	**0,50**	**0,75**	**1,00**	**1,50**	**2,50**
F(k) =	0,000	0,1846	0,2686	0,2577	0,2070	0,1081	0,0264
k =	**0,45**	**0,5**	**0,6**				
F(k) =	0,2610	0,2686	0,2725				

The maximal/minimal value of F(k) is obtained by a parabolic interpolation between k = 0.45 and k = 0.60: it yields the value F(k),max,min = ±0.2730 for k = ±0.5759.

So the maximal/minimal bending moments m_{yy} are found ± 0.5759×b from the middle of the plate and the values are

$$m_{yy},_{\max,\min} = \pm \frac{1.09 M_{y0}}{\pi b} \tag{23e}$$

3.5. Plate of infinite length subjected to a concentrated moment Mx0 at (x = a/2, y = b)

Case y = 0 clamped and y = b simply supported

The plate of infinite length is subjected to a concentrated moment Mx0 at the middle of the simply supported edge.

The coefficients A_m, B_m, C_m, and D_m are determined as follows

$$\begin{aligned} &A_m = 0, \quad B_m = \frac{2M_{x0}}{a\alpha_m^2} \frac{Y_m \cosh Y_m - \sinh Y_m}{2Y_m - \sinh 2Y_m} \sin\frac{m\pi}{2}, \\ &C_m = -D_m, \quad D_m = -\frac{2M_{x0}}{a\alpha_m^2} \frac{Y_m \sinh Y_m}{2Y_m - \sinh 2Y_m} \sin\frac{m\pi}{2}, \quad Y_m = \frac{m\pi b}{a} \end{aligned} \tag{24a}$$

Setting Ym = mρ with ρ = πb/a → 0, the bending moments m_{yy} at positions (x = a/2, y = kb) are

$$m_{yy}\Big|_{\substack{x=a/2\\ y=kb}} = -\frac{2M_{x0}}{\pi b} \lim_{\rho\to 0} \sum_{m=1,3,5...} \left[\begin{aligned} &\rho \frac{m\rho \cosh m\rho - \sinh m\rho}{2m\rho - \sinh 2m\rho} \left(2\cosh km\rho + km\rho \sinh km\rho\right) - \\ &\rho \frac{m\rho \sinh m\rho}{2m\rho - \sinh 2m\rho} \left(\sinh km\rho + km\rho \cosh km\rho\right) \end{aligned} \right] \tag{24b}$$

The summation F(k) in Equation (24b) for ρ tending to zero is evaluated depending on k and the values are calculated in the Supplementary Material "Plate of infinite length clamped simply supported under moment Mxo" and listed in Table 5

Table 5. Coefficient F(k) of the bending moment m_{yy} at a position y = kb along x = a/2.

k =	**0,00**	**0,30**	**0,40**	**0,50**	**0,60**	**0,70**	**0,80**	**0,90**
F (k) =	-0,7258	-0,1205	0,0369	0,2018	0,3984	0,6782	1,1577	2,6953

So, the bending moments m_{yy} at the middle of the clamped edge and in the middle of the y-dimension *b*, respectively, are

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$$m_{yy}\Big|_{\substack{x=a/2\\y=0}} = \frac{1.452M_{x0}}{\pi b}, \qquad m_{yy}\Big|_{\substack{x=a/2\\y=b/2}} = -\frac{0.404M_{x0}}{\pi b} \tag{24c}$$

Case y = 0 and y = b simply supported

The coefficients A_m, B_m, C_m, and D_m are determined as follows

$$A_m = 0, \quad B_m = 0$$

$$C_m = -\frac{M_{x0}}{a\alpha_m^2}\frac{Y_m \cosh Y_m}{\sinh^2 Y_m}\sin\frac{m\pi}{2}, \quad D_m = \frac{M_{x0}}{a\alpha_m^2}\frac{1}{\sinh Y_m}\sin\frac{m\pi}{2}, \quad Y_m = \frac{m\pi b}{a} \tag{24d}$$

Setting $Y_m = m\rho$ with $\rho = \pi b/a \to 0$, the bending moments m_{yy} at positions ($x = a/2$, $y = kb$) are given by

$$m_{yy}\Big|_{\substack{x=a/2\\y=kb}} = \frac{M_{x0}}{\pi b}\lim_{\rho\to 0}\sum_{m=1,3,5...}\left[\frac{m\rho^2\cosh m\rho}{\sinh^2 m\rho}\sinh km\rho - \frac{\rho}{\sinh m\rho}\left(2\sinh km\rho + km\rho\cosh km\rho\right)\right] \tag{24e}$$

The summation F(k) in Equation (24e) for ρ tending to zero is evaluated depending on k and the values are calculated in the Supplementary material "Plate of infinite length simply supported under bending moment Mxo"and listed in Table 6

Table 6. Coefficient F(k) of the bending moment m_{yy} at a position y = kb along x = a/2.

k =	**0,00**	**0,30**	**0,40**	**0,50**	**0,60**	**0,70**	**0,80**	**0,90**
F (k) =	0,0000	-0,4002	-0,5707	-0,7858	-1,0839	-1,5619	-2,5538	-5,4557

4. Conclusions

In this paper, isotropic rectangular thin plates were analyzed; they were simply supported or clamped along two opposite edges with the other edges having arbitrary support conditions, and were subjected to external bending moments perpendicular to the supported edges. Bending moments parallel to the supported opposite edges are satisfactorily treated in the literature and were less analyzed in this study. The standard approach to this problem is to replace the bending moment with a couple of forces infinitely close and to use the known expressions of efforts and deformations for the plate subjected to concentrated forces; the results are then related to the first derivatives of these efforts and deformations with respect to the position of application of the load. In this study the external bending moment was expanded into a Fourier series, leading to a distributed external bending moment, and the boundary conditions and continuity equations were applied. Various types of rectangular plates were so analyzed and also plates of infinite length whose results were identical to those in the literature.

The following aspect was not addressed in this study but could be analyzed in the future: Rectangular anisotropic plate

Supplementary Materials: The following files were uploaded during submission: "Cantilever plate of infinite length under bending moment Mxo"; "Cantilever plate of infinite length under bending moment Myo"; "Plate of infinite length clamped simply supported under moment Mxo"; "Plate of infinite length clamped simply supported under moment Mxo.".

Conflicts of Interest: The author declares no conflict of interest.

Appendix A. Coefficients A_{mI}, B_{mI}, C_{mI}, and D_{mI} and A_{mII}, B_{mII}, C_{mII}, and D_{mII} for various support conditions at y = 0 and y = b

Edges y = 0 and y = b clamped

The plate is represented in Figure 2. We set

$$Y_0 = \alpha_m y_0, \qquad Y_1 = \alpha_m y_1, \qquad m^*_{x0m} = \frac{1}{\alpha_m^2 (1-\nu)} \frac{2M_{x0}}{a} \sin\frac{m\pi x_0}{a}$$

The boundary conditions and continuity equations are expressed in matrices form as follows, whereby the first two rows and the last two rows represent the boundary conditions at the edges y = 0 and y = b, respectively.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \cosh Y_0 & Y_0 \sinh Y_0 & \sinh Y_0 & Y_0 \cosh Y_0 & -1 & 0 & 0 & 0 \\ \sinh Y_0 & \sinh Y_0 + Y_0 \cosh Y_0 & \cosh Y_0 & \cosh Y_0 + Y_0 \sinh Y_0 & 0 & 0 & -1 & -1 \\ \cosh Y_0 & \frac{2\cosh Y_0}{1-\nu} + Y_0 \sinh Y_0 & \sinh Y_0 & \frac{2\sinh Y_0}{1-\nu} + Y_0 \cosh Y_0 & -1 & -\frac{2}{1-\nu} & 0 & 0 \\ 0 & 2\sinh Y_0 & 0 & 2\cosh Y_0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & \cosh Y_1 & Y_1 \sinh Y_1 & \sinh Y_1 & Y_1 \cosh Y_1 \\ 0 & 0 & 0 & 0 & \sinh Y_1 & \sinh Y_1 + Y_1 \cosh Y_1 & \cosh Y_1 & \cosh Y_1 + Y_1 \sinh Y_1 \end{bmatrix} \times \begin{bmatrix} A_{mI} \\ B_{mI} \\ C_{mI} \\ D_{mI} \\ A_{mII} \\ B_{mII} \\ C_{mII} \\ D_{mII} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ m^*_{x0m} \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(A1)}$$

Edge y = 0 simply supported and edge y = b free

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{2}{1-\nu} & 0 & 0 & 0 & 0 & 0 & 0 \\ \cosh Y_0 & Y_0 \sinh Y_0 & \sinh Y_0 & Y_0 \cosh Y_0 & -1 & 0 & 0 & 0 \\ \sinh Y_0 & \sinh Y_0 + Y_0 \cosh Y_0 & \cosh Y_0 & \cosh Y_0 + Y_0 \sinh Y_0 & 0 & 0 & -1 & -1 \\ \cosh Y_0 & \frac{2\cosh Y_0}{1-\nu} + Y_0 \sinh Y_0 & \sinh Y_0 & \frac{2\sinh Y_0}{1-\nu} + Y_0 \cosh Y_0 & -1 & -\frac{2}{1-\nu} & 0 & 0 \\ 0 & 2\sinh Y_0 & 0 & 2\cosh Y_0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & \cosh Y_1 & \frac{2\cosh Y_1}{1-\nu} + Y_1 \sinh Y_1 & \sinh Y_1 & \frac{2\sinh Y_1}{1-\nu} + Y_1 \cosh Y_1 \\ 0 & 0 & 0 & 0 & \sinh Y_1 & \frac{\nu+1}{\nu-1}\sinh Y_1 + Y_1 \cosh Y_1 & \cosh Y_1 & \frac{\nu+1}{\nu-1}\cosh Y_1 + Y_1 \sinh Y_1 \end{bmatrix} \times \begin{bmatrix} A_{mI} \\ B_{mI} \\ C_{mI} \\ D_{mI} \\ A_{mII} \\ B_{mII} \\ C_{mII} \\ D_{mII} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ m^*_{x0m} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(A2)}$$

Plates with other combinations of support conditions at y = 0 and y = b can be analyzed similarly. The first two rows and last two rows are modified accordingly.

Edge y = 0 simply supported and edge y = b free: bending moment acting at (x_0, y_0 = 0)

The plate is represented in Figure 2 with $y_0 = 0$. We set

$$Y_1 = \alpha_m b, \qquad m^*_{x0m} = \frac{1}{\alpha_m^2 (1-\nu)} \frac{2M_{x0}}{a} \sin \frac{m\pi x_0}{a}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{2}{1-\nu} & 0 & 0 \\ \cosh Y_1 & \frac{2\cosh Y_1}{1-\nu} + Y_1 \sinh Y_1 & \sinh Y_1 & \frac{2\sinh Y_1}{1-\nu} + Y_1 \cosh Y_1 \\ \sinh Y_1 & \frac{\nu+1}{\nu-1}\sinh Y_1 + Y_1 \cosh Y_1 & \cosh Y_1 & \frac{\nu+1}{\nu-1}\cosh Y_1 + Y_1 \sinh Y_1 \end{bmatrix} \times \begin{bmatrix} A_{mII} \\ B_{mII} \\ C_{mII} \\ D_{mII} \end{bmatrix} = \begin{bmatrix} 0 \\ -m^*_{x0m} \\ 0 \\ 0 \end{bmatrix} \quad \text{(A3)}$$

Edge y = 0 free and edge y = b clamped: bending moment acting at (x_0, y_0 = 0)

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$$\begin{bmatrix} 0 & 0 & 1 & \frac{\nu+1}{\nu-1} \\ 1 & \frac{2}{1-\nu} & 0 & 0 \\ \cosh Y_1 & Y_1 \sinh Y_1 & \sinh Y_1 & Y_1 \cosh Y_1 \\ \sinh Y_1 & \sinh Y_1 + Y_1 \cosh Y_1 & \cosh Y_1 & \cosh Y_1 + Y_1 \sinh Y_1 \end{bmatrix} \times \begin{bmatrix} A_{mII} \\ B_{mII} \\ C_{mII} \\ D_{mII} \end{bmatrix} = \begin{bmatrix} 0 \\ -m^*_{x0m} \\ 0 \\ 0 \end{bmatrix} \quad \text{(A4)}$$

Plates with other support conditions at y = b can be analyzed similarly, the last two rows being modified accordingly. Then, the bending moments myy are calculated using Equation (8c), and the bending moments mxx and twisting moments mxy are calculated using Equations (11a-b).

Appendix B. Plate of infinite length: coefficients A_{mI}, B_{mI}, C_{mI}, D_{mI}, A_{mII}, and B_{mII} for various support conditions at y = 0

Edge y = 0 simply supported

The plate is represented in Figure 2. We set

$$Y_0 = \alpha_m y_0, \qquad m^*_{x0m} = \frac{1}{\alpha_m^2 (1-\nu)} \frac{2M_{x0}}{a} \sin \frac{m\pi x_0}{a}$$

The boundary conditions and continuity equations are expressed in matrices form as follows, whereby the first two rows represent the boundary conditions at the edge y = 0.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{2}{1-\nu} & 0 & 0 & 0 & 0 \\ \cosh Y_0 & Y_0 \sinh Y_0 & \sinh Y_0 & Y_0 \cosh Y_0 & -1 & 0 \\ \sinh Y_0 & \sinh Y_0 + Y_0 \cosh Y_0 & \cosh Y_0 & \cosh Y_0 + Y_0 \sinh Y_0 & 1 & -1 \\ \cosh Y_0 & \frac{2\cosh Y_0}{1-\nu} + Y_0 \sinh Y_0 & \sinh Y_0 & \frac{2\sinh Y_0}{1-\nu} + Y_0 \cosh Y_0 & -1 & \frac{2}{1-\nu} \\ 0 & 2\sinh Y_0 & 0 & 2\cosh Y_0 & 0 & -2 \end{bmatrix} \times \begin{bmatrix} A_{mI} \\ B_{mI} \\ C_{mI} \\ D_{mI} \\ A_{mII} \\ B_{mII} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ m^*_{x0m} \\ 0 \end{bmatrix} \quad \text{(B1)}$$

Edge y = 0 clamped

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$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ \cosh Y_0 & Y_0 \sinh Y_0 & \sinh Y_0 & Y_0 \cosh Y_0 & -1 & 0 \\ \sinh Y_0 & \sinh Y_0 + Y_0 \cosh Y_0 & \cosh Y_0 & \cosh Y_0 + Y_0 \sinh Y_0 & 1 & -1 \\ \cosh Y_0 & \frac{2\cosh Y_0}{1-\nu} + Y_0 \sinh Y_0 & \sinh Y_0 & \frac{2\sinh Y_0}{1-\nu} + Y_0 \cosh Y_0 & -1 & \frac{2}{1-\nu} \\ 0 & 2\sinh Y_0 & 0 & 2\cosh Y_0 & 0 & -2 \end{bmatrix} \times \begin{bmatrix} A_{mI} \\ B_{mI} \\ C_{mI} \\ D_{mI} \\ A_{mII} \\ B_{mII} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ m^*_{x0m} \\ 0 \end{bmatrix} \quad \text{(B2)}$$

Edge y = 0 free

$$\begin{bmatrix} 0 & 0 & 1 & \frac{\nu+1}{\nu-1} & 0 & 0 \\ 1 & \frac{2}{1-\nu} & 0 & 0 & 0 & 0 \\ \cosh Y_0 & Y_0 \sinh Y_0 & \sinh Y_0 & Y_0 \cosh Y_0 & -1 & 0 \\ \sinh Y_0 & \sinh Y_0 + Y_0 \cosh Y_0 & \cosh Y_0 & \cosh Y_0 + Y_0 \sinh Y_0 & 1 & -1 \\ \cosh Y_0 & \frac{2\cosh Y_0}{1-\nu} + Y_0 \sinh Y_0 & \sinh Y_0 & \frac{2\sinh Y_0}{1-\nu} + Y_0 \cosh Y_0 & -1 & \frac{2}{1-\nu} \\ 0 & 2\sinh Y_0 & 0 & 2\cosh Y_0 & 0 & -2 \end{bmatrix} \times \begin{bmatrix} A_{mI} \\ B_{mI} \\ C_{mI} \\ D_{mI} \\ A_{mII} \\ B_{mII} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ m^*_{x0m} \\ 0 \end{bmatrix} \quad \text{(B3)}$$

References

1. Kirchhoff, G. Über das Gleichgewicht und die Bewegung einer elastischen Scheibe. J. für die Reine und Angew. Math.; vol. 18, no. 40, pp. 51-88, 1850.
2. Germain, S. Remarques sur la nature, les bornes et l'étendue de la question des surfaces élastiques et équation générale de ces surfaces. impr. de Huzard-Courcier, paris, 1826.
3. Lévy, M. Sur l'équilibre élastique d'une plaque rectangulaire. Comptes rendus l'Académie des Sci. Paris, vol. 129, no. 1, pp. 535-539, 1899.
4. Timoshenko, S., Woinowsky-Krieger, S. Theory of plates and shells. Second ed., McGraw Hill, New York (1959)
5. Girkmann, K. Flächentragwerke. Springer-Verlag. Wien, New York. 1986.
6. Jiang, R., Jiang, L. Trends in Developing Critical Elastic Buckling Formula for Fixed Rectangular Plate Subjected To a Concentrated Load. Trends in Civil Engineering and its Architecture. Volume 1 – Issue 2. 2018. DOI: 10.32474/TCEIA.2018.01.000109
7. Xu, Y., Wu, Zhangjan. Exact solutions for rectangular anisotropic plates with four clamped edges. Mechanics of Advanced Materials and Structures, Volume 29, 2022 - Issue 12. https://doi.org/10.1080/15376494.2020.1838007
8. Onyia, M.E., Rowland-Lato, E.O., Ike, C.C. Elastic buckling analysis of SSCF and SSSS rectangular thin plates using the single finite Fourier sine integral transform method. IJERT, Vol 13, No 6, pp 1147 – 1158, july 2020. DOI: 10.37624/IJERT/13.6.2020.1147-1158
9. Imrak, C.E., Gerdemeli, I. An Exact Solution for the Deflection of a Clamped Rectangular Plate under Uniform Load. Applied Mathematical Sciences, Vol. 1, 2007, no. 43, 2129 - 2137
10. Fogang, V. Analysis of Arbitrarily Loaded Rectangular Thin Plates with Two Opposite Edges Supported, One or Both of them Clamped, Using the Flexibility Method and a Modified Lévy Solution. Preprints.org 2023, 2023052121. https://doi.org/10.20944/preprints202305.2121.v2

 doi:10.20944/preprints202307.0112.v2

11. Mama, B.O., Oguaghamba, O.A., Ike, C.C. Single Finite Fourier Sine Integral Transform Method for the Flexural Analysis of Rectangular Kirchhoff Plate with Opposite Edges Simply Supported, Other Edges Clamped for the Case of Triangular Load Distribution. IJERT, Vol. 13, 7 (2020), pp. 1802-1813. DOI: 10.37624/IJERT/13.7.2020.1802-1813
12. Kamel, A-K. Finite Fourier transform for solving potential and steady-state temperature problems. Advances in Difference Equations (2018), 2018:98. https://doi.org/10.1186/s13662-018-1552-8
13. Courbon, J., Theillout, JN. Formulaire de résistance des matériaux. Techniques de l'Ingénieur, Traité Construction, pages 106, 107. 1987. doi.org/10.51257/a-v1-c2060