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Three Remarks On Asset Pricing

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Abstract: This paper considers the theoretical framework of the consumption-based asset-pricing model and derives successive approximations of the modified basic pricing equation using Taylor series expansions of the investor's utility function during the averaging time interval Δ. For linear and quadratic Taylor approximations we derive new expressions for the mean asset price, mean payoff, their volatilities, skewness and amount of asset that delivers max to investor's utility. We introduce new market-based price probability determined by statistical moments of the market trade values and volumes. We show that the market-based price probability results zero correlations between time-series of n-th power of price p^n and trade volume U^n, but doesn't cause statistical independence and we derive correlation between time-series of price p and squares of trade volume U^2. The market-based treatment of the random trade price describes impact of the size of market trade values and volumes on price probability. Predictions of the market-based price probability at horizon T should match forecasts of statistical moments of the trade values and volumes at the same horizon T. The market-based price probability emphasizes direct dependence on random properties of market trades.

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1. Introduction

Predictions of asset prices are the most desired results for investor and the same time the most complex problems of economic and financial theory. The literature on asset pricing is huge and boundless and our references present not a historic review but our personal preferences only. We mention only tiny part of endless publications on asset pricing, starting with CAPM by Sharpe (1964), which was followed by various modifications as Intertemporal CAPM by Merton (1973), the Arbitrage Theory of Capital Asset Pricing by Ross (1976), consumption-based asset pricing model described by Duffie and Zame (1989), Cochrane (2001), Campbell (2002) and many others. In his paper Cochrane (2001) demonstrates that consumption-based asset-pricing frame gives the unified approach for description of most variations of pricing models. We rely on that Cochrane's statement and consider methods in the roots of the consumption-based model as the key tools of current pricing theories.

One may consider most pricing theories as fruitful derivation products of CAPM, which mostly have certain common assumptions, foundations and limitations. Sharpe (1964) mentioned, that his assumptions in the foundation of CAPM as "common pure rate of interest" and "homogeneity of investor expectations: investors are assumed to agree on the prospects of various investments" are "highly restrictive and undoubtedly unrealistic assumptions". CAPM and consumption-based model are based on assumption of general market equilibrium and they use utility functions that model investors' market decisions. Maximum condition of investor's utility function results the basic pricing equation that describes most current results in pricing models. These and other initial assumptions are in the basis of modern asset pricing models. On the one hand these assumptions support description of definite relations between current asset price, expected payoffs, discount factors and etc. On the other hand "highly restrictive and undoubtedly unrealistic assumptions" carry the threats of failures and inconsistencies between predictions of pricing theories and real market price dynamics.

We don't study how conventional assumptions in the foundation of the pricing models impact their predictions or limit their applications. Instead, we take the consumption-based frame and consider how few remarks generated by market trade reality could impact consequences and

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performance of the asset pricing. As main reference to the consumption-based model we chose Cochrane (2001) and even use its main notations. If one agrees with Cochrane's statement that the consumption-based model and the basic pricing equation describe results of most variants of pricing theories, then our remarks, approximations and results make sense and could be applied to other asset pricing theories.

Our pure theoretical paper considers the frame of the consumption-based asset pricing model as a general economic problem and investigates its compliance with major economic issues. Actually, any economic and financial model describes processes and relations only as approximation that captures certain averaging, smoothness, coarsening of the economic reality. Thus our first remark concerns importance to consider particular time averaging interval Δ as a determining factor of pricing model. Indeed, current stock markets support initial time axis division determined by the time-series of the trades performed at moments t_i with time shift $\varepsilon = t_i - t_{i-1}$ between trades. For simplicity in this paper we consider the time shift ε as constant. As usual, the time shift ε is sufficiently small and can be equal 1 second or even fraction of a second. That is not much useful for modelling asset prices at horizon 1 month, quarter or year. However, records of market trade time-series with time shift ε determine initial market time axis division and define the discrete nature of all initial market trade data. To evaluate any reasonable pricing model at a given horizon T that can be equal month, quarter, year and so on, one should chose the averaging time scale Δ, that should obey $\varepsilon \ll \Delta < T$. The choice of the averaging time interval Δ is a key factor of any pricing model. It determines the scale of averaging of initial trade times-series and thus performs transition from initial market time division multiple of ε to averaged time division multiple of Δ. Below we show how the choice of the averaging interval Δ results in modification of the consumption-based utility function and the basic pricing equation.

Our second remark indicates that the choice of the averaging interval Δ allows expand utility function and the basic pricing equation into Taylor series near average values of price and payoff and then average the fluctuating terms of series during Δ. Mathematical expectations of linear and quadratic Taylor approximations of basic pricing equation by price and payoff variations during Δ give new expressions of the mean price and payoff, their volatilities, skewness and other factors. Actually, even linear Taylor expansion demonstrates that the famous statement: "price equals expected discounted payoff" with which Cochrane (2001) and Brunnermeier (2015) begin their papers, describes only markets with zero price volatility during current and "next" periods, which makes almost no economic sense. As we show in Sec. 4, Taylor expansion of modified basic pricing equation determines relations between mean price and price volatility during current period and mean payoff and payoff volatility during the "next" period. Further in Section 4 we derive additional relations, which expend results of the consumption-based model.

Our third remark concerns the economic origin, definition, approximations and forecasting of the asset price probability as the major problem of any pricing model and financial economics as a whole. Furthermore, we consider the assessments and predictions of the finite number of price statistical moments, which establish basis for approximations of price probability and its forecasts, as principal and most complex problems of financial economics and the key problem of any asset pricing models in particular. We introduce the market-based probability of asset price, which is determined by statistical moments of market trade values and volumes during Δ. Conventional treatment considers frequency-based assessment of price probability which is proportional to number of trades at price p. Actually, any particular market trade at time t_i is determined by its trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ which follow trivial equation:

$$C(t_i) = p(t_i)U(t_i) \tag{1.1}$$

One should mention that it impossible independently define probabilities of three variables - trade value $C(t_i)$, volume $U(t_i)$, and price $p(t_i)$ – those match equation (1.1). We consider trade value $C(t_i)$ and volume $U(t_i)$ time-series as major random variables during the interval Δ, which completely determine random price $p(t_i)$ properties. It is well known that properties of a random variable can be equally described by its probability measure, characteristic function or by a set of statistical moments (Shephard, 1991; Shiryaev, 1999; Klyatskin, 2005). The random market trade time-series during Δ

determine assessments of n-th statistical moments of trade value $C(t;n)$ (5.7) and volume $U(t;n)$ (5.8) and we use them to define n-th statistical moments $p(t;n)$ (5.10) of price. We compare the frequency-based and the market-based *n-th* statistical moments of price and explain why the conventional frequency-based treatment of price probability has too little economic sense.

We propose that readers are familiar with Cochrane (2001) and refer his monograph for any notions and clarifications. In Sec.2 we briefly remind main notions of asset pricing according to Cochrane (2001). In Sec.3 we consider remarks on the averaging interval Δ and explain necessity for modification of the consumption-based basic pricing equation. In Sec.4 we discuss Taylor series expansion of the utility functions and derive successive approximations of the modified basic equation in linear and quadratic approximations by the price and payoff variations. In Sec.5 we introduce the market-based price statistical moments and briefly consider implications for asset pricing. Sec.6 – Conclusion. In App.A we collect some calculations that define maximum of investor's utility. In App.B we present simple approximations of the price characteristic function.

Equation (4.5) means equation 5 in the Sec. 4 and (A.2) – notes equation 2 in Appendix A. We assume that readers are familiar with basic notions of probability, statistical moments, characteristic functions and etc.

2. Brief Notations

In this Sec. we briefly remind main notations and assumptions of asset pricing used by Cochrane (2001). The consumption-based basic pricing equation has form:

$$p = E[m\,x] \tag{2.1}$$

In (2.1) p denotes the asset price at date t, $x=p_{t+1}+d_{t+1}$ – payoff, p_{t+1} - price and d_{t+1} - dividends at date $t+1$, m - the stochastic discount factor and $E[..]$ – mathematical expectation at day $t+1$ made by the forecast under the information available at date t. Cochrane (2001) considers equation (2.1) in various forms to show that most asset pricing models can be described by the similar equations. For convenience we briefly reproduce derivation of the consumption-based basic pricing equation (2.1). Cochrane models investors by a utility function $U(c_t; c_{t+1})$ defined over current c_t and future c_{t+1} values of consumption at date t and $t+1$.

$$U(c_t;\, c_{t+1}) = u(c_t) + \beta E[u(c_{t+1})] \tag{2.2}$$

$$c_t = e_t - p\xi \quad ; \quad c_{t+1} = e_{t+1} + x\xi \tag{2.3}$$

$$x = p_{t+1} + d_{t+1} \tag{2.4}$$

In (2.2) $u(c_t)$ and $u(c_{t+1})$ – utility functions at date t and $t+1$; in (2.3) e_t and e_{t+1} "denotes original consumption level (if the investor bought none of the asset), and ξ denotes the amount of the asset he chooses to buy" (Cochrane, 2001). Cochrane calls β as "subjective discount factor that captures impatience of future consumption". The first-order maximum condition for (2.2) by amount of assets ξ is fulfilled by putting derivative of (2.2) by ξ equals zero (Cochrane, 2001):

$$max_{\xi}\, U(c_t;\, c_{t+1}) \;\leftrightarrow\; \frac{\partial}{\partial \xi} U(c_t;\, c_{t+1}) = 0 \tag{2.5}$$

From (2.2-2.5) one obtains:

$$p = \beta E\left[\frac{u'(c_{t+1})}{u'(c_t)}\, x\right] = E[mx] \quad ; \quad m = \beta \frac{u'(c_{t+1})}{u'(c_t)} \quad ; \quad u'(c) \equiv \frac{d}{dc} u(c) \tag{2.6}$$

and (2.6) reproduces (2.1) for m (2.6). We refer Cochrane (2001) for any further details.

3. Remarks on Time Scales

We start with simple remarks on averaging of economic and financial time-series. Any economic or financial model, and asset pricing in particular, approximates real processes, which are averaged during certain time interval. To describe market asset pricing one should take into account that market trade time-series are the only source of price variations. Market trade causes price $p(t_i)$ time-series with time shift $\varepsilon = t_i - t_{i-1}$ between trades at time t_i and t_{i-1}. Interval ε can be very small and can be equal 1 second or even fraction of a second. Trade time-series introduce initial discreteness of the time axis division of any market-pricing problem. However, initial market price time-series $p(t_i)$ with time-shift ε are very irregular and not much useful for modelling and forecasting asset prices at any reasonable time horizon T that can be equal week, month, year, etc. To derive reasonable description of asset price one should enlarge initial time axis discreteness to obtain more smooth variations of market prices. To perform such a transition from the initial time discreteness ε one should choose time interval Δ such as $\varepsilon<<\Delta<T$ and average price time-series $p(t_i)$ during Δ. Time shift $\Delta = t(k) - t(k-1)$ of averaged prices $p(t(k))$ at times $t(k)$ introduces new division of the time axis. The averaging interval Δ, that can be equal 1 hour, 1 day, 1 week or whatever, defines the new averaged time axis division of the problem under consideration. After averaging during Δ time-series of average price $p(t(k))$ are determined at times $t(k)$:

$$t(k) = t(0) + k\,\Delta \quad ; \quad k = 0,\ 1, 2, .. \tag{3.1}$$

One can consider time $t=t(0)$ as the moment "to-day". Let us underline, that division of the time axis "to-day" at moment t and the "next-day" at $t+1$ must be the same. Indeed, time scale divisions can't be measured "to-day" in hours and "next-day" in weeks. The utility (2.2) "to-day" at moment t and the "next-day" at $t+1$ should have the same time divisions. Averaging of any time-series at the "next-day" at $t+1$ during the interval Δ undoubtedly implies averaging "to-day" at date t during same time interval Δ and vise-versa. Thus, if the utility (2.2) is averaged at $t+1$ during the interval Δ, then the utility (2.2) also should be averaged at date t during same interval Δ and (2.2) should take form:

$$U(c_t;\ c_{t+1}) = E_t[u(c_t)] + \beta E[u(c_{t+1})] \tag{3.2}$$

We denote $E_t[..]$ in (3.2) as mathematical expectation "to-day" at date t during Δ. It does not matter how one considers the market price time-series "to-day" – as random or as irregular. Mathematical expectation $E_t[..]$ performs smoothing of the random or irregular time-series via aggregating data during Δ under particular probability measure. Mathematical expectations $E_t[..]$ at t and $E[..]$ at $t+1$ during same averaging intervals Δ establish identical time division of the problem at dates t and $t+1$ in (3.2). Hence, relations similar to (2.5; 2.6) should cause modification of the basic pricing equation (2.1; 2.6) in the form (3.3):

$$E_t[p\, u'(c_t)] = \beta E\, [x\, u'(c_{t+1})] \tag{3.3}$$

Cochrane (2001) takes "subjective discount factor" β as non-random and we follow his assumption. Mathematical expectation $E_t[..]$ averages $pu'(c_t)$ over random price p fluctuations during Δ "to-day". In the right side $E[xu'(c_{t+1})]$ averages $xu'(c_{t+1})$ over random payoff fluctuations during Δ "next day" on base of data available at date t "to-day".

4. Remarks on Taylor series

Relation (2.5) presents the first-order condition for amount of assets ξ_{max} that delivers maximum to investor's utility (2.2) or (3.2). Let us choose the averaging interval Δ and take the price p at date t during Δ and the payoff x at date $t+1$ during Δ as:

$$p\ =\ p_0 + \delta p\,; \quad x = x_0 + \delta x\ ; \quad E_t[p] = p_0\,;\ E[x] = x_0 \tag{4.1}$$

$$E_t[\delta p] = E[\delta x] = 0\,;\ \sigma^2(p) = E_t[\delta^2 p]\ ;\ \sigma^2(x) = E[\delta^2 x] \tag{4.2}$$

Relations (4.1; 4.2) denote the average price p_0 and its volatility $\sigma^2(p)$ at date t and the average payoff x_0 and its volatility $\sigma^2(x)$ at date $t+1$. We consider δp and δx as random fluctuations of price and payoff during Δ. We underline, that we consider averaging during Δ as averaging of a random or as smoothing of an irregular variable. Thus $E_t[p]$ – at date t smooth random or irregular price p (4.1) during Δ and $E[x]$ – averages the random payoff x (4.1) during Δ at date $t+1$. We present the derivatives of utility functions in (3.3) by Taylor series in a linear approximation by δp and δx during Δ:

$$u'(c_t) = u'(c_{t;0}) - \xi u''(c_{t;0})\delta p \quad ; \quad u'(c_{t+1}) = u'(c_{t+1;0}) + \xi u''(c_{t+1;0})\delta x \tag{4.3}$$

$$c_{t;0} = e_t - p_0\xi \quad ; \quad c_{t+1;0} = e_{t+1} + x_0\xi$$

Now substitute (4.3) into (3.3) and due to (4.2) obtain equation (4.4):

$$u'(c_{t;0})p_0 - \xi u''(c_{t;0})\sigma^2(p) = \beta u'(c_{t+1;0})x_0 + \beta\xi u''(c_{t+1;0})\sigma^2(x) \tag{4.4}$$

Taylor series are simple mathematical tools and Cochrane (2001) also used them. We underline: Taylor series and (4.1-4.4) are determined by duration of Δ. The change of Δ can implies change of the mean price p_0, the mean payoff x_0 and their volatilities $\sigma^2(p)$, $\sigma^2(x)$ (4.2). Equation (4.4) is a linear

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approximation by the price and payoff fluctuations of the first-order max conditions (2.5) and assesses the root ξ_{max} that delivers maximum to the utility $U(c_t;c_{t+1})$ (3.2)

$$\xi_{max} = \frac{u'(c_{t;0})p_0 - \beta u'(c_{t+1;0})x_0}{u''(c_{t;0})\sigma^2(p) + \beta u''(c_{t+1;0})\,\sigma^2(x)} \qquad (4.5)$$

We note that (4.5) is not an "exact" solution for ξ_{max} as derivatives of utilities u' and u'' also depend on ξ_{max} as it follows from (4.3). However, (4.5) gives an assessment of ξ_{max} in a linear approximation by Taylor series δp and δx averaged during Δ. Let underline that the ξ_{max} (4.5) depends on the price volatility $\sigma^2(p)$ at date t and on the forecast of payoff volatility $\sigma^2(x)$ at date $t+1$ (4.2).

It is clear that sequential iterations may give more accurate approximations of ξ_{max}. Nevertheless, our approach and (4.5) give a new look on the basic equation (2.6; 3.3). If one follows the standard derivation of (2.6) (Cochrane, 2001) and neglects the averaging at date t in the left side (3.3), then (2.6; 4.5) give

$$\xi_{max} = \frac{u'(c_t)p - \beta u'(c_{t+1;0})x_0}{\beta u''(c_{t+1;0})\sigma^2(x)} \qquad (4.6)$$

Relations (4.6) show that even the standard form of the basic equation (2.6) hides dependence of the amount of assets ξ_{max} on the payoff volatility $\sigma^2(x)$ at date $t+1$. If one has the independent assessment of ξ_{max} then one can present (4.6) in a way alike to the basic equation (2.6):

$$p = \frac{u'(c_{t+1;0})}{u'(c_t)}\beta x_0 + \xi_{max}\frac{u''(c_{t+1;0})}{u'(c_t)}\beta\sigma^2(x) \qquad (4.7)$$

Otherwise, if there are no independent assessments of ξ_{max}, then one should consider (4.6) as the solution of the first order maximum condition (2.5), which presents the root ξ_{max} of the amount of assets, determined for the given values in the right hand of (4.6). In that case the basic pricing equations (2.1; 2.6; 4.7) have almost no sense, as the value of ξ_{max} in (4.7) is not determined. We consider this misstep – usage of the maximum condition (2.5) to determine the basic pricing equation (2.1; 4.7) instead of defining ξ_{max} as a root of the maximum condition (2.5) as a significant oversight of the consumption-based asset pricing model, which requires essential clarifications.

One can transform (4.7) alike to (2.6):

$$p = m_0 x_0 + \xi_{max} m_1 \sigma^2(x) \qquad (4.8)$$

$$m_0 = \frac{u'(c_{t+1;0})}{u'(c_t)}\beta \;\;;\;\; m_1 = \frac{u''(c_{t+1;0})}{u'(c_t)}\beta \qquad (4.9)$$

For the given ξ_{max} equation (4.8) in a linear approximation by Taylor series describes dependence of the price p at date t (3.1) on the mean discount factors m_0 and m_1 (4.9), the mean payoff x_0 (4.1) and the payoff volatility $\sigma^2(x)$ during Δ. Let underline that while the mean discount factor $m_0>0$, the mean discount factor $m_1<0$ because utility $u'(c_t)>0$ and $u''(c_t)<0$ for all t. Hence, irremovable payoff volatility $\sigma^2(x)$ at day $t+1$ states that price p at day t always less than discounted mean payoff x_0:

$$p < m_0 x_0 \quad ; \quad \xi_{max} m_1 \sigma^2(x) < 0$$

One can consider (4.8) as a linear Taylor expansion of (2.1; 2.6). However, equation (4.4) presents dependence of mean price p_0 at day t on price volatility $\sigma^2(p)$ at day t, mean payoff x_0 and payoff volatility $\sigma^2(x)$ at day $t+1$. That definitely enlarges the conventional statement "price equals expected discounted payoff". We underline that (4.6-4.9) have sense for the given value of ξ_{max}. As the price p in (4.8) should be positive hence ξ_{max} should obey inequality (4.10):

$$0 < \xi_{max} < -\frac{u'(c_{t+1;0})}{u''(c_{t+1;0})}\frac{x_0}{\sigma^2(x)} \qquad (4.10)$$

For the conventional power utility (Cochrane, 2001) (A.2) from (4.3) obtain for (4.10):

$$u(c) = \frac{1}{1-\alpha}c^{1-\alpha} \;\;;\;\; \frac{u'(c)}{u''(c)} = -\frac{c}{\alpha} \;\;;\;\; 0 < \alpha \le 1$$

inequality (4.10) valid always if

$$\alpha\,\sigma^2(x) < x_0^2$$

For this approximation (4.10) limits the value of ξ_{max}. For (4.4; 4.5) obtain equations similar to (4.8; 4.9):

$$m_0 = \frac{u'(c_{t+1;0})}{u'(c_{t;0})}\beta > 0\,;\;\; m_1 = \frac{u''(c_{t+1;0})}{u'(c_{t;0})}\beta < 0\,;\;\; m_2 = \frac{u''(c_{t;0})}{u'(c_{t;0})} < 0 \qquad (4.11)$$

$$p_0 = m_0 x_0 + \xi_{max}[m_1\sigma^2(x) + m_2\sigma^2(p)] \qquad (4.12)$$

We use the same notions m_0, m_1 to denote the discount factors taking into account replacement of $u'(c_t)$ in (4.9) by $u'(c_{t;0})$ in (4.11; 4.12). Modified basic equation (4.12) at date t describes dependence

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of the mean price p_0 on the price volatility $\sigma^2(p)$ at date t, the mean payoff x_0 and payoff volatility $\sigma^2(x)$ at date $t+1$ averaged during Δ.

Equation (4.12) illustrates well-known practice that high volatility $\sigma^2(p)$ of the price at date t and forecast of high volatility $\sigma^2(x)$ of payoff at date $t+1$ may cause decline of the mean price p_0 at date t. We leave the detailed analysis of (4.4-4.12) for the future.

4.1. The Idiosyncratic Risk

Here we follow (Cochrane, 2001) and briefly consider usage of Taylor series for his example of the idiosyncratic risk for which the payoff x in (2.6) is not correlated with the discount factor m at moment $t+1$:

$$cov(m,x) = 0 \qquad (4.13)$$

In this case equation (2.6) takes form:

$$p = E[mx] = E[m]E[x] + cov(m,x) = E[m]x_0 = \frac{x_0}{R_f} \qquad (4.14)$$

The risk-free rate R_f in (4.14) is known ahead (Cochrane, 2001). Taking into account (4.3) in a linear approximation by δx Taylor series for derivative of the utility $u'(c_t+1)$:

$$u'(c_{t+1}) = u'(c_{t+1;0}) + u''(c_{t+1;0})\xi\delta x \qquad (4.15)$$

Hence, the discount factor m (2.6) takes form:

$$m = \beta\frac{u'(c_{t+1})}{u'(c_t)} = \frac{\beta}{u'(c_t)}[u'(c_{t+1;0}) + u''(c_{t+1;0})\xi\delta x]$$

$$E[m] = \bar{m} = \beta\frac{u'(c_{t+1;0})}{u'(c_t)}$$

$$\beta E\left[\frac{u'(c_{t+1})}{u'(c_t)}\right]x_0 = \frac{x_0}{R_f} \quad ; \quad E[u'(c_{t+1})x] = 0$$

and

$$\delta m = m - \bar{m} = \frac{\beta}{u'(c_t)}u''(c_{t+1;0})\xi\delta x$$

Hence, (4.13) implies:

$$cov(m,x) = E[\delta m\delta x] = \beta\frac{u''(c_{t+1;0})}{u'(c_t)}\xi_{max}\sigma^2(x) = 0 \qquad (4.16)$$

That causes zero payoff volatility $\sigma^2(x)=0$. Of course, zero payoff volatility does not model market reality, but (4.16) reflects restrictions of the linear approximation (4.15). To overcome this discrepancy let us take into account Taylor series up to the second degree by $\delta^2 x$:

$$u'(c_{t+1}) = u'(c_{t+1;0}) + u''(c_{t+1;0})\xi\delta x + \frac{1}{2}u'''(c_{t+1;0})\xi^2\delta^2 x \qquad (4.17)$$

$$m = \beta\frac{u'(c_{t+1})}{u'(c_t)} = \frac{\beta}{u'(c_t)}\left[u'(c_{t+1;0}) + u''(c_{t+1;0})\xi\delta x + \frac{1}{2}u'''(c_{t+1;0})\xi^2\delta^2 x\right] \qquad (4.18)$$

For this case the mean discount factor $E[m]$ takes form:

$$E[m] = \bar{m} = \frac{\beta}{u'(c_t)}[u'(c_{t+1;0}) + \frac{1}{2}u'''(c_{t+1;0})\xi^2\sigma^2(x)] \qquad (4.19)$$

and variations of the discount factor δm:

$$\delta m = m - \bar{m} = \frac{\beta}{u'(c_t)}\,[u''(c_{t+1;0})\xi\delta x + \frac{1}{2}u'''(c_{t+1;0})\xi^2\{\delta^2 x - \sigma^2(x)\}$$

Thus Taylor series approximation up to the second degree by $\delta^2 x$ gives:

$$cov(m,x) = E[\delta m\delta x] = \left[u''(c_{t+1;0})\xi\sigma^2(x) + \frac{1}{2}u'''(c_{t+1;0})\xi^2\,\gamma^3(x)\right] = 0 \quad (4.20)$$

$$\gamma^3(x) = E[\delta^3 x] \quad ; \quad Sk(x) = \frac{\gamma^3(x)}{\sigma^3(x)} \qquad (4.21)$$

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Sk(x) – denotes normalized payoff skewness at date *t+1* treated as the measure of asymmetry of the probability distribution during Δ. For approximation (4.18) from (4.20; 4.21) obtain relations on the skewness *Sk(x)* and ξ_{max}:

$$\xi_{max}\, Sk(x)\sigma(x) = -2\frac{u''(c_{t+1;0})}{u'''(c_{t+1;0})} \qquad (4.22)$$

For the conventional power utility (A.2)

$$u(c) = \frac{1}{1-\alpha}c^{1-\alpha}$$

and (4.3) relations (4.22) take form

$$\xi_{max} = \frac{2e_{t+1}}{(1+\alpha)Sk(x)\sigma(x)-2x_0} \qquad (4.23)$$

It is assumed that second derivative of utility $u''(c_{t+1})<0$ always negative and third derivative $u'''(c_{t+1})>0$ is positive and hence the right side in (4.22) is positive. Hence to get positive ξ_{max} for (4.23) for the power utility (A.2) the payoff skewness *Sk(x)* should obey inequality (4.24) that defines the lower limit of the payoff skewness *Sk(x):*

$$Sk(x) > \frac{2x_0}{(1+\alpha)\sigma(x)} \qquad (4.24)$$

In (4.14) R_f denotes risk-free rate. Hence, (4.19; 4.22; 4.24) define relations:

$$\frac{\beta}{u'(c_t)}\left[u'(c_{t+1;0}) + \frac{1}{2}u'''(c_{t+1;0})\xi_{max}{}^2\sigma^2(x)\right] = \frac{1}{R_f}$$

$$\frac{1}{2}\xi_{max}{}^2\sigma^2(x) = \frac{1}{\beta R_f}\frac{u'(c_t)}{u'''(c_{t+1;0})} - \frac{u'(c_{t+1;0})}{u'''(c_{t+1;0})}$$

$$Sk^2(x) = \frac{R_f}{1-m_0R_f}\frac{m_1^2}{m_3} > \frac{4x_0^2}{(1+\alpha)^2\sigma^2(x)} \quad ; \quad m_0 < 1/R_f$$

$$\frac{\sigma^2(x)}{4x_0^2} > \frac{m_3}{m_1^2}\frac{1-m_0R_f}{(1+\alpha)^2R_f} \qquad (4.25)$$

Inequality (4.25) establishes the lower limit on the payoff volatility $\sigma^2(x)$ normalized by the square of the mean payoff x_0^2. The lower limit in the right side of (4.25) is determined by the discount factors (4.26), the risk-free rate R_f and the conventional power utility factor α (A.2).

$$m_0 = \beta\frac{u'(c_{t+1;0})}{u'(c_t)} \; ; \; m_1 = \beta\frac{u''(c_{t+1;0})}{u'(c_t)} \; ; \; m_3 = \beta\frac{u'''(c_{t+1;0})}{u'(c_t)} \qquad (4.26)$$

The coefficients in (4.26) differ a little from (4.1) as (4.26) takes the denominator $u'(c_t)$ instead of $u'(c_{t;0})$ in (4.11) but we use the same letters to avoid extra notations. The similar calculations for (3.2; 3.3) describe both the price volatility $\sigma^2(p)$ and price skewness *Sk(p)* at date *t* and the payoff volatility $\sigma^2(x)$ and payoff skewness *Sk(x)* at date *t+1*. Further approximations by Taylor series of the utility derivative $u'(c_t)$ up to δ^3p and $u'(c_{t+1})$ up to δ^3x similar to (4.17) could give assessments of kurtosis of the price probability at date *t* and kurtosis of the payoff probability at date *t+1* estimated during interval Δ.

4.2. The Utility Maximum

Relations (2.5) define the first-order condition that determines the amount of asset ξ_{max} that delivers the max to the utility $U(c_t;c_{t+1})$ (2.2; 3.2). To confirm that function $U(c_t;c_{t+1})$ has max at ξ_{max}, the first order condition (2.5) must be supplemented by condition:

$$\frac{\partial^2}{\partial\xi^2}U(c_t;\, c_{t+1}) < 0 \qquad (4.27)$$

Usage of (4.27) gives interesting consequences. From (2.2–2.4) and (4.27) obtain:

$$p^2 > -\frac{\beta}{u''(c_t)}E\left[x^2\, u''(c_{t+1})\right] \qquad (4.28)$$

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Take the linear Taylor series expansion of the second derivative of the utility $u''(c_{t+1})$ by δx

$$u''(c_{t+1}) = u''(c_{t+1;0}) + u'''(c_{t+1;0})\xi\delta x$$

Then (4.28) takes form:

$$p^2 > -\beta\frac{u''(c_{t+1;0})}{u''(c_t)}[x_0{}^2 + \sigma^2(x)] - \beta\frac{u'''(c_{t+1;0})}{u''(c_t)}\xi_{max}\,[2x_0\sigma^2(x) + \gamma^3(x)] \qquad (4.29)$$

For the power utility (A.2) simple calculations (see App.A) give relations on (4.27; 4.29). If the payoff volatility $\sigma^2(x)$ multiplied by factor $(1+2\alpha)$ is less then mean payoff $x_0{}^2$ (4.30; A.5)

$$(1+2\alpha)\sigma^2(x) < x_0{}^2 \quad ; \quad \frac{1}{3} \le \frac{1}{1+2\alpha} < 1 \qquad (4.30)$$

Then (4.29) is always valid. If payoff volatility $\sigma^2(x)$ is high (A.6)

$$(1+2\alpha)\sigma^2(x) > x_0{}^2$$

Then (4.29) valid only for ξ_{max} (A.6):

$$\xi_{max} < \frac{e_{t+1}[x_0{}^2 + \sigma^2(x)]}{x_0\,[(1+2\alpha)\sigma^2(x) - x_0{}^2]}$$

However, this upper limit for ξ_{max} can be high enough. The same but more complex considerations can be presented for (3.2).

$$E_t[p^2 u''(c_t)] < -E[\beta x^2\, u''(c_{t+1})]$$

5. Remarks on the Price Probability

As usual the problems that are the most common and "obvious" hide the most complexities. The price probability is exactly the case of such hidden complexity. We warn that some letter designations in this section can coincide with the ones in the previous Sections, but hope that readers able distinguish and adopt both. Conventional asset pricing models assume that it is possible forecast probability of random price p and payoff x at horizon T. We consider the choice and prediction of the price probability as most interesting and complex problems of financial economics.

The usual treatment of the price probability "is based on the probabilistic approach and using A. N. Kolmogorov's axiomatic of probability theory, which is generally accepted now" (Shiryaev, 1999). The conventional definition of the price probability is based on the frequency of trades at a price p during the averaging interval Δ. The economic foundation of such choice is simple: it is assumed that each of N trades during Δ have equal probability ~ $1/N$. If there are $m(p)$ trades at the price p then probability $P(p)$ of the price p during Δ is assessed as $m(p)/N$. Usage the frequency of the particular event is absolutely correct, general and conventional approach to probability definition. The conventional frequency-based approach to price probability checks how almost all standard probability measures (Walck, 2007; Forbes et.al., 2011) fit description of the market random price. Parameters, which define standard probabilities permit calibrate each in a manner that increase the plausibility and consistency with the observed random price time-series. For different assets and markets different standard probabilities are tested and applied to fit and predict the random price dynamics as well as possible.

However, one may ask a simple question: does the conventional frequency-based approach to the price probability fit random market pricing? Indeed, the asset price is a result of the market trade and it seems reasonable that the market trade randomness should conduct the price stochasticity. We propose the new definition of the market-based price probability that is different from the conventional frequency-based probability and is entirely determined by the statistical moments of the market trade values and volumes.

Let us remind that almost 30 years ago the volume weighted average price (VWAP) was introduced and is widely used now (Berkowitz et.al., 1988; Buryak and Guo, 2014; Busseti and Boyd, 2015; Duffie and Dworczak, 2018; CME Group, 2020). Definition of the VWAP $p(t;1)$ that matches equation (1.1) during Δ is follows. Let us take that during Δ (5.3) there are N market trades at moments t_i, $i=1,\ldots N$. Let denote $E[..]$ as mathematical expectation. Then the VWAP $p(t;1)$ (5.1) that match (1.1) during Δ (5.3) at moment t equals

$$p(t;1) \equiv E[p(t_i)] = \frac{1}{\sum_{i=1}^{N} U(t_i)} \sum_{i=1}^{N} p(t_i)U(t_i) \equiv \frac{C_\Sigma(t;1)}{U_\Sigma(t;1)} \qquad (5.1)$$

$$C_\Sigma(t;1) \equiv \sum_{i=1}^{N} C(t_i) \equiv \sum_{i=1}^{N} p(t_i)\,U(t_i) \quad ; \quad U_\Sigma(t;1) \equiv \sum_{i=1}^{N} U(t_i) \qquad (5.2)$$

$$\Delta = \left[t - \frac{\Delta}{2}, t + \frac{\Delta}{2}\right] \quad ; \quad t_i \in \Delta, \; i = 1, \dots N \qquad (5.3)$$

We consider time-series of the trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ as random variables during Δ (5.3). Equation (1.1) at moment t_i defines the price $p(t_i)$ of market trade value $C(t_i)$ and volume $U(t_i)$. The sum $C_\Sigma(t;1)$ of values $C(t_i)$ (5.2) and sum $U_\Sigma(t;1)$ of volumes $U(t_i)$ (5.2) of N trades during Δ (5.3) define the VWAP $p(t;1)$ (5.1).

We hope that readers able distinguish the difference between notations of consumption c_t (2.2; 2.3) and utility U (2.2) in Sections 2-4 and trade value $C(t_i)$ and volume $U(t_i)$ (5.1) in current Section.

It is obvious, that VWAP (5.1) can be equally determined (5.4) by the mean value $C(t;1)$ (5.5) and the mean volume $U(t;1)$ (5.6) of N trades during Δ:

$$C(t;1) = p(t;1)\,U(t;1) \qquad (5.4)$$

The mean trade value $C(t;1)$ and volume $U(t;1)$ are assessed by finite number N of trades during Δ (5.3) through the conventional frequency-based approach:

$$C(t;1) \equiv E[C(t_i)] \sim \frac{1}{N} \sum_{i=1}^{N} C(t_i) \qquad (5.5)$$

$$U(t;1) \equiv E[U(t_i)] \sim \frac{1}{N} \sum_{i=1}^{N} U(t_i) \qquad (5.6)$$

The notion ~ underlines that (5.5; 5.6) give only assessments of mean trade value $C(t;1)$ and mean volume $U(t;1)$ by finite number N of trades during Δ (5.3). VWAP $p(t;1)$ (5.4) is a coefficient between the mean value $C(t;1)$ (5.5) and the mean volume $U(t;1)$ (5.6).

Actually, the trade equation (1.1) imposes constraints on probabilities of trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ time-series. Given probabilities of trade value $C(t_i)$ and volume $U(t_i)$ time-series during Δ that match (1.1) should determine the price probability. However, VWAP $p(t;1)$ and relations (5.1-5.6) are not sufficient to define all random properties of price as a random variable during Δ (5.3). Actually, it is well known that properties of a random variable can be equally described by probability measure, characteristic function and by a set of statistical moments (Shephard, 1991; Shiryaev, 1999; Klyatskin, 2005). To approximate properties of the market trade value and volume as random variables during Δ (5.3) one could assess their n-th statistical moments of the trade value $C(t;n)$ and volume $U(t;n)$:

$$C(t;n) \equiv E[C^n(t_i)] \sim \frac{1}{N} \sum_{i=1}^{N} C^n(t_i) \qquad (5.7)$$

$$U(t;n) \equiv E[U^n(t_i)] \sim \frac{1}{N} \sum_{i=1}^{N} U^n(t_i) \qquad (5.8)$$

Let us mention that n-th power of (1.1) for each particular trade at time t_i gives:

$$C^n(t_i) = p^n(t_i)U^n(t_i) \quad ; \quad n = 1, 2, \dots \qquad (5.9)$$

We use (5.7-5.9) to determine price n-th statistical moments $p(t;n)$ for n=1,2,3,… via n-th statistical moments of the trade value $C(t;n)$ (5.7) and volume $U(t;n)$ (5.8). We extend definition of the VWAP (5.1; 5.2) and use (5.7; 5.8; 5.11) to introduce n-th statistical moment $p(t;n)$ of price in a way similar to VWAP (5.1) as n-th power volume averaged:

$$p(t;n) \equiv E[p^n(t_i)] = \frac{1}{\sum_{i=1}^{N} U^n(t_i)} \sum_{i=1}^{N} p^n(t_i)U^n(t_i) = \frac{C_\Sigma(t;n)}{U_\Sigma(t;n)} = \frac{C(t;n)}{U(t;n)} \qquad (5.10)$$

$$C_\Sigma(t;n) \equiv \sum_{i=1}^{N} C^n(t_i) = \sum_{i=1}^{N} p^n(t_i)U^n(t_i) \quad ; \quad U_\Sigma(t;n) \equiv \sum_{i=1}^{N} U^n(t_i) \qquad (5.11)$$

We underline that definitions (5.10) use equation (5.9) and that results expression (5.12) of price n-th statistical moments $p(t;n)$ through n-th statistical moments of the market trade value $C(t;n)$ and volume $U(t;n)$:

$$C(t;n) = p(t;n)U(t;n) \qquad (5.12)$$

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Definitions of price *n*-th statistical moments *p(t;n)* (5.10; 5.12) for all *n=1,2,…* match equation (5.9) for *n*-th degree of price $p^n(t_i)$ at time t_i during *Δ* (5.3). It is important that price *n*-th statistical moments *p(t;n)* (5.10; 5.12) for all *n=1,2,…* completely determine properties of market price as a random variable during *Δ* (5.3).

Let us outline important unnoticed consequence of the VWAP *p(t;1)* (5.1) and similar consequences of our definition of price *n*-th statistical moments *p(t;n)* (5.10; 5.12). Definition of VWAP *p(t;1)* (5.1) results in zero correlations between time-series of price $p(t_i)$ and trade volume $U(t_i)$ during *Δ* (5.3). Indeed, from (1.1; 5.1; 5.5; 5.6) obtain:

$$E[C(t_i)] \sim \frac{1}{N}\sum_{i=1}^{N} C(t_i) = \frac{1}{N}\sum_{i=1}^{N} p(t_i)U(t_i) \sim E[p(t_i)U(t_i)] \sim$$

$$\sim \frac{1}{\sum_{i=1}^{N} U(t_i)} \sum_{i=1}^{N} p(t_i)U(t_i) \cdot \frac{1}{N}\sum_{i=1}^{N} U(t_i) \sim E[p(t_i)]E[U(t_i)] \qquad (5.13)$$

Hence, from (5.13) obtain correlation $corr\{p(t_i)U(t_i)\}$ between time-series of price $p(t_i)$ and trade volume $U(t_i)$, which are averaged during *Δ* (5.3):

$$corr\{p(t_i)U(t_i)\} \equiv E[p(t_i)U(t_i)] - E[p(t_i)]E[U(t_i)] = 0 \qquad (5.14)$$

Zero correlations (5.14) between price-volume time-series impact results of many publications those "observe" positive or negative correlations between price and trading volume (Tauchen and Pitts, 1983; Karpoff, 1987; Campbell et.al., 1993; Llorente et.al., 2001; DeFusco et.al., 2017). These papers describe correlations determined by the frequency-based treatment of price probability. Actually, assessments of correlations between any time-series follow definitions of their averaging procedures. Usage of VWAP (5.1; 5.2; 5.13; 5.14) states no correlations between trade volume and price and some papers on price-volume relations could be reconsidered.

Our definitions of price *n*-th statistical moments *p(t;n)* (5.7-5.12) for all *n=1,2,3,..* cause zero correlations $corr\{p^n(t_i)U^n(t_i)\}$ between time-series of *n*-th power of price $p^n(t_i)$ and volume $U^n(t_i)$ during *Δ* (5.3). One can easy reproduce (5.13; 5.14) for *n=1,2,3,…*:

$$E[C^n(t_i)] \sim \frac{1}{N}\sum_{i=1}^{N} C^n(t_i) \sim E[p^n(t_i)U^n(t_i)] \sim \frac{1}{N}\sum_{i=1}^{N} p^n(t_i)U^n(t_i) =$$

$$= \frac{1}{\sum_{i=1}^{N} U^n(t_i)} \sum_{i=1}^{N} p^n(t_i)U^n(t_i) \cdot \frac{1}{N}\sum_{i=1}^{N} U^n(t_i) \sim E[p^n(t_i)]E[U^n(t_i)] \qquad (5.15)$$

$$corr\{p^n(t_i)U^n(t_i)\} \equiv E[p^n(t_i)U^n(t_i)] - E[p^n(t_i)]E[U^n(t_i)] = 0 \qquad (5.16)$$

Thus, the market-based definition of price *n*-th statistical moments *p(t;n)* (5.7-5.12) causes zero correlations between time-series of *n*-th power of price $p^n(t_i)$ and volume $U^n(t_i)$ during *Δ* but doesn't imply statistical independence between time series of $p(t_i)$ and volume $U(t_i)$. For example we derive correlation $corr\{p(t_i)U^2(t_i)\}$ between time-series of price $p(t_i)$ and squares of trade volume $U^2(t_i)$ during *Δ:*

$$E[p(t_i)U^2(t_i)] \equiv E[C(t_i)U(t_i)] = E[C(t_i)]E[U(t_i)] + corr\{C(t_i)U(t_i)\}$$

$$E[p(t_i)U^2(t_i)] = E[p(t_i)]E[U^2(t_i)] + corr\{p(t_i)U^2(t_i)\}$$

$$corr\{p(t_i)U^2(t_i)\} = E[C(t_i)U(t_i)] - p(t;1)U(t;2)$$

Thus, from above (5.4-5.6; 5.13) one easy obtains:

$$corr\{p(t_i)U^2(t_i)\} = corr\{C(t_i)U(t_i)\} - p(t;1)\sigma^2(U) \qquad (5.17)$$

Correlation $corr\{C(t_i)U(t_i)\}$ (5.17) between time-series of trade value and volume could be assessed by (5.5; 5.6) and (5.17.1):

$$corr\{C(t_i)U(t_i)\} \equiv E[C(t_i)U(t_i)] - E[C(t_i)]E[U(t_i)]$$

$$E[C(t_i)U(t_i)] \sim \frac{1}{N}\sum_{i=1}^{N} C(t_i)\,U(t_i) \qquad (5.17.1)$$

In (5.17) we denote as $\sigma^2(U)$ – the volatility of the trade volume (5.18):

$$\sigma^2(U) \equiv U(t;2) - U^2(t;1) \qquad (5.18)$$

It is obvious that market-based price statistical moments *p(t;n)* (5.10; 5.12) differ from statistical moments *π(t;n)* generated by frequency-based price probability *P(p)* (5.19):

$$P(p)\sim\frac{m(p)}{N} \quad ; \quad \pi(t;n)\sim\frac{1}{N}\sum_{i=1}^{N}p^n(t_i) \tag{5.19}$$

$$\pi(t;n)\sim\frac{1}{N}\sum_{i=1}^{N}p^n(t_i)=\frac{1}{N}\sum_{i=1}^{N}\frac{C^n(t_i)}{U^n(t_i)}\neq\frac{\sum_{i=1}^{N}C^n(t_i)}{\sum_{i=1}^{N}U^n(t_i)}=\frac{C_\Sigma(t;n)}{U_\Sigma(t;n)}=\frac{C(t;n)}{U(t;n)}=p(t;n) \tag{5.20}$$

The difference between the frequency-based $\pi(t;n)$ and market-based $p(t;n)$ price statistical moments determine the economic distinctions between two approaches to definition of the price probability. Statistical moments $\pi(t;n)$ equal $p(t;n)$ only if all trade volumes equal unit $U(t_i)=1$ during Δ (5.3).

To clarify the economic origin of the difference between two assessments of the price statistical moments determined by the market-based (5.10; 5.12) and by the conventional frequency-based approach (5.19; 5.20) we mention, that in a general case n-th statistical moments $p_\mu(t;n)$ of the given price time-series $p(t_i)$, $i=1,..N$ during interval Δ (5.3) can be assessed via weighted functions $\mu_i(t;n)$:

$$\mu_i(t;n)\geq 0 \quad ; \quad \sum_{i=1}^{N}\mu_i(t;n)=1 \quad ; \quad p_\mu(t;n)=\sum_{i=1}^{N}\mu_i(t;n)\,p^n(t_i) \tag{5.21}$$

The frequency-based price statistical moments (5.19) correspond to all $\mu_i(t;n)=1/N$ and the market-based statistical moments (5.10) of price take $\mu_i(t;n)$ as (5.22):

$$\mu_i(t;n)=\frac{1}{\sum_{i=1}^{N}U^n(t_i)}U^n(t_i) \tag{5.22}$$

From equations (1.1; 5.9) and due to n-th statistical moments of trade values and volumes (5.7; 5.8) one obtains that n-th statistical moments $p_\mu(t;n)$ of price determined by weighted functions $\mu_i(t;n)$ (5.21) cause correlations $corr_\mu\{p^n(t_i)U^n(t_i)\}$ (5.23) between n-th degrees of price $p^n(t_i)$ and trade volume $U^n(t_i)$ time-series:

$$corr_\mu\{p^n(t_i)U^n(t_i)\}=\frac{1}{N}\sum_{i=1}^{N}p^n(t_i)U^n(t_i)-\sum_{i=1}^{N}\mu_i(t;n)\,p^n(t_i)\frac{1}{N}\sum_{i=1}^{N}U^n(t_i) \tag{5.23}$$

For $n=1$ relations (5.23) and $\mu_i(t;n)=1/N$ determine the frequency-based price-volume $corr\{p(t_i)U(t_i)\}$ correlation during Δ (5.3):

$$corr\{p(t_i)U(t_i)\}=\frac{1}{N}\sum_{i=1}^{N}p(t_i)U(t_i)-\frac{1}{N}\sum_{i=1}^{N}p(t_i)\frac{1}{N}\sum_{i=1}^{N}U(t_i)$$

This form of correlation corresponds to frequency-based price probability (5.19) and was studied by (Tauchen and Pitts, 1983; Karpoff, 1987; Campbell et.al., 1993; Llorente et.al., 2001; DeFusco et.al., 2017). However, market-based price statistical moments (5.7; 5.8), which are described by (5.22), results zero correlations (5.16) for all n=1,2,…

We repeat the essence of the difference between frequency-based and the market based price statistical moments. Frequency-based approach (5.19) assumes that all N trades during Δ have equal probability and hence price n-th statistical moments for all $n=1,2,..$ are assessed via $\mu_i(t;n)=1/N$. Contrary to that, the market-based assessments of price n-th statistical moments $p(t;n)$ (5.10; 5.12) states that due to (5.22) the price, which is linked with large trading volume, has higher weight $\mu_i(t;n)$ and makes more contribution into the composition of price statistical moments. Price is a result of market trades and obeys trade equations (1.1; 5.9). Thus random properties of trade values (5.7) and volumes (5.8) should determine random properties of price (5.10; 5.12).

We don't consider these reasons as "a rigorous proof" of the preferred choice of the market-based statistical moments (5.10; 5.12) in comparison with the frequency-based (5.19). Investors, traders and researchers may prefer time-tested, generally accepted, familiar and comprehensive to anyone since school years the frequency-based price probability. Economics is a social science and faith of the mass could turn upside down any economic law and states a new one. However, our argumentation in favor of the market-based approach may make some investors ponder and then follow our models. Excess financial gains of followers would be the best "rigorous proof".

Now let us consider the approximations of price characteristic functions and probability measures by finite sets of price statistical moments. The set of price n-th statistical moments $p(t;n)$ (5.10; 5.12) for all $n=1,2,3,…$ determines Taylor series of the price characteristic function $F(t;x)$ (Shephard, 1991; Shiryaev, 1999; Klyatskin, 2005):

$$F(t;x) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} p(t;n)\, x^n \qquad (5.24)$$

In (5.24) i denotes imaginary unit $i^2 = -1$. However, any records of market trades during Δ (5.3) assess only finite number of statistical moments of the trade value $C(t;n)$ (5.7) and volume $U(t;n)$ (5.8) Hence, one can assess only finite number of price statistical moments $p(t;n)$ (5.10; 5.12). In App.B we consider simple successive approximations of the price characteristic function $F_K(t;x)$ that takes into account finite number K of the Taylor series terms (5.24) and corresponding K-approximations of the price probability measure $\eta_K(t;p)$ derived as Fourier transforms of characteristic function $F_K(t;x)$:

$$\eta_K(t;p) = \frac{1}{\sqrt{2\pi}} \int dx\, F_K(t;x)\, exp(-ixp) \qquad (5.25)$$

Relations (5.25) define successive approximations of the price probability measure $\eta_K(t;p)$. Assessments of finite number K of the market trade and price statistical moments result that one can forecast only approximations of price characteristic function or price probability measure those match finite number K of price statistical moments $p(t;n)$ (5.10; 5.12).

$$p(t;n) = \frac{d^n}{(i)^n dx^n} F_K(t;x)|_{x=0} = \int dp\, \eta_K(t;p) p^n \quad ; \quad n \le K \qquad (5.26)$$

Any hypothesis on the form of the price probability measure $\eta_K(t;p)$ during Δ (5.3) and predictions of the price probability at horizon T should match relations (5.10; 5.12; 5.26) at $t+T$. Thus one should predict K statistical moments of the trade value $C(t;n)$ (5.7) and volume $U(t;n)$ (5.8) at $t+T$ for $n \le K$. That equals prediction of the K-approximations of the market trade probabilities at horizon T. In simple words: accuracy of price probability predictions depends on precision of forecasts of market trade statistical moments. For example, consider the market-based price volatility $\sigma^2(t;p)$ (Olkhov, 2020):

$$\sigma^2(t;p) \equiv E\left[\left(p(t_i) - p(t;1)\right)^2\right] = p(t;2) - p^2(t;1) = \frac{C(t;2)}{U(t;2)} - \frac{C^2(t;1)}{U^2(t;1)} \qquad (5.27)$$

From (5.7; 5.8; 5.11) one can express market-based price volatility $\sigma^2(t;p)$ as:

$$\sigma^2(t;p) = \frac{C(t;2)}{U(t;2)} - \frac{C^2(t;1)}{U^2(t;1)} = \frac{C_\Sigma(t;2)}{U_\Sigma(t;2)} - \frac{C_\Sigma^2(t;1)}{U_\Sigma^2(t;1)} \qquad (5.28)$$

Prediction of the price volatility $\sigma^2(t;p)$ at horizon T during Δ requires forecasts of the market trade statistical moments $C(t;1)$, $C(t;2)$ (5,7) and $U(t;1)$, $U(t;2)$ (5.8) at the same horizon T. Accuracy of the price probability forecasts is determined by accuracy of the market trade probabilities predictions. In simple words: to predict price probability one should be able predict market trade values and volumes probabilities and that is almost the same as "predict the future of the entire economy".

6. Conclusion

Each economic theory and asset pricing in particular should directly indicate the time scales Δ of the model under consideration. Time-series of the market trades with time shift ε introduce initial division of the time axis multiple of ε. Asset pricing model should take into account these initial data as only source for averaged market time-series. Any averaging of market time-series presume usage of particular time averaging interval $\Delta \gg \varepsilon$. Averaging of initial market time-series during Δ introduces transition from initial time axis division multiple of ε to new division multiple of Δ. To consider utility function and price dynamics "today" and "next day" one should use the same time axis division "today" and "next day" and hence the same averaging interval Δ. Averaging of investor's utility function "today" and "next day" introduces modification of the investor's utility and basic pricing equation. The choice of interval Δ allows consider Taylor series expansions of the modified investor's utility and basic pricing equation by price and payoff fluctuations and subsequent averaging of fluctuations. For linear and quadratic approximations of the basic pricing equation that give relations, which describe mean price, price volatility, mean payoff, payoff volatility and etc. In linear Taylor approximation (4.12) presents dependence of mean price p_0 "to-day" during Δ, on price volatility $\sigma^2(p)$ "to-day"

$$p_0 = m_0 x_0 + \xi_{max}[m_1 \sigma^2(x) + m_2 \sigma^2(p)] \qquad (6.1)$$

and on mean payoff x_0 and payoff volatility $\sigma^2(x)$ "next day" and the amount of assets ξ_{max} that delivers max to investor's utility and equals the root of the equation (3.3). On the one hand (6.1) modifies conventional statement "price equals expected discounted payoff" and demonstrates dependence on price volatility $\sigma^2(p)$ "to-day". On the other hand (6.1) uncovers direct dependence of the mean price p_0 "to-day" on the amount of assets ξ_{max} that delivers max to investor's utility. That direct dependence doesn't add confidence in the impeccability of the consumption-based model' frame and further argumentation required to solve the troubles, which arises with direct dependence of (6.1) on ξ_{max}. We consider recent paper (Cochrane, 2022) as implicit confirmation that current pricing models are of little help for investors in real stock markets.

Usage of averaging interval Δ as mandatory factor of any financial model, results in introduction of the market-based probability of asset price. Indeed, aggregations of market time-series during Δ permit consider total values and volumes of market trades during Δ as important variables, which govern the variations of market price. As we show, n-th statistical moments of the trade value $C(t;n)$ (5.7) and volume $U(t;n)$ (5.8) project impact of the size of the trading values and volumes on statistical moments of market price. With growing n the impact of large trades on market price grows up.

Time-series of the performed market trades assess only finite number K of statistical moments $C(t;n)$ (5.7) and $U(t;n)$ (5.8) and determine K-approximations of the market price probability. Any predictions of the price probability at horizon T should match forecasts of $n \le K$ n-th trade statistical moments at the same horizon T.

We define the market-based price statistical moments $p(t;n)$ (5.10; 5.12) as extensions of VWAP (5.1). Their usage results in zero correlations $corr\{p^n(t_i)U^n(t_i)\}=0$ (5.14; 5.16) between time series of n-th power of price $p^n(t_i)$ and trading volume $U^n(t_i)$. In particular, VWAP causes zero correlations (5.14) between time-series of price $p(t_i)$ and trading volume $U(t_i)$. That impact studies on price-volume correlations based on usage of frequency-based definition of the mean price. Zero correlations (5.16) between n-th power of price $p^n(t_i)$ and trade volume $U^n(t_i)$ don't cause statistical independence between price and volume random variables during Δ (5.3). We derive expression for correlation $corr\{p(t_i)U^2(t_i)\}$ (5.17) between price and squares of volume during Δ (5.3).

This trinity – the averaging interval Δ, Taylor series and the market-based price probability can provide successive approximations for other versions of asset pricing, financial and economic models. This method was used to describe Value-at-Risk problems, volatility, option pricing, market-based price and payoff autocorrelations and the market-based probability of stock returns (Olkhov, 2020-2023).

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Appendix A. Max of Utility

We start with (4.29):

$$p^2 > -\beta \frac{u''(c_{t+1;0})}{u''(c_t)} [x_0{}^2 + \sigma^2(x)] - \beta \frac{u'''(c_{t+1;0})}{u''(c_t)} \xi_{max} [2x_0\sigma^2(x) + \gamma^3(x)] \qquad (A.1)$$

If the right side is negative then it is valid always. If the right side is positive – then there exist a lower limit on the price p. For simplicity, neglect term $\gamma^3(x)$ to compare with $2x_0\sigma^2(x)$ and take the conventional power utility $u(c)$ (Cochrane, 2001) as:

$$u(c) = \frac{1}{1-\alpha} c^{1-\alpha} \qquad (A.2)$$

Let us consider the case with negative right side (A.1). Simple but long calculations give:

$$-\beta \frac{u''(c_{t+1;0})}{u''(c_t)} [x_0{}^2 + \sigma^2(x)] < \beta \frac{u'''(c_{t+1;0})}{u''(c_t)} \xi_{max}\, 2x_0\sigma^2(x)$$

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$$\xi_{max}\, 2x_0\sigma^2(x) < -\frac{u'''(c_{t+1;0})}{u''(c_{t+1;0})}\,[{x_0}^2+\sigma^2(x)] \qquad \text{(A.3)}$$

Let us take into account (A.2) and for (A.3) obtain:

$$\frac{u''(c)}{u'''(c)} = -\frac{c}{1+\alpha} \quad ; \quad \xi_{max}\, 2x_0\sigma^2(x) < \frac{e_{t+1}+x_0\xi_{max}}{1+\alpha}\,[{x_0}^2+\sigma^2(x)]$$

$$\xi_{max}x_0\,[(1+2\alpha)\sigma^2(x)-{x_0}^2] < e_{t+1}[{x_0}^2+\sigma^2(x)] \qquad \text{(A.4)}$$

Inequality (A.4) determines that the right side (A.1) is negative in two cases. The left side in (A.4) is negative and

$$(1+2\alpha)\sigma^2(x) < {x_0}^2 \quad ; \quad \frac{1}{3} \le \frac{1}{1+2\alpha} < 1 \qquad \text{(A.5)}$$

Inequality (A.5) describes small payoff volatility $\sigma^2(x)$. In this case the right side of (A.1) is negative for all ξ_{max} and all price p and hence (4.27) that defines max of utility (2.5) is valid. The left side in (A.4) is positive and

$$(1+2\alpha)\sigma^2(x) > {x_0}^2 \quad ; \quad \xi_{max} < \frac{e_{t+1}[{x_0}^2+\sigma^2(x)]}{x_0\,[(1+2\alpha)\sigma^2(x)-{x_0}^2]} \qquad \text{(A.6)}$$

This case describes high payoff volatility and the upper limit on ξ_{max} to utility (2.5). Take the positive right side in (A.1). Then (A.4) is replaced by the opposite inequality

$$\xi_{max}x_0\,[(1+2\alpha)\sigma^2(x)-{x_0}^2] > e_{t+1}[{x_0}^2+\sigma^2(x)] \qquad \text{(A.7)}$$

It is valid for (A.6) only. (A.7) determines a lower limit on ξ_{max} to utility (2.5):

$$\xi_{max} > \frac{e_{t+1}[{x_0}^2+\sigma^2(x)]}{x_0\,[(1+2\alpha)\sigma^2(x)-{x_0}^2]}$$

Appendix B. Approximations of the price characteristic function and probability measure

Taylor series expansions of market price characteristic function result successive approximations of characteristic function. Derivation of approximations is a self-standing research and here we present simple examples of such approximations only. We consider simple approximations of price characteristic function $F_K(t;x)$ and price probability measure $\eta_K(t;p)$ during Δ (5.3) those fit obvious condition. As such we consider approximations $F_K(t;x)$ of the price characteristic function those match (B.1):

$$p(t;n) = \frac{C(t;n)}{U(t;n)} = \frac{d^n}{(i)^n dx^n}F_k(t;x)|_{x=0} = \int dp\,\eta_k(t;p)p^n \;;\; n \le K \qquad \text{(B.1)}$$

Statistical moments determined by $F_K(t;x)$ for $n>K$ will be different from price statistical moments $p(t;n)$ (5.10; 5.12) but first K moments will be equal to $p(t;n)$.

We suggest approximation $F_K(t;x)$ of price characteristic function $F(t;x)$ (5.24) as

$$EF_K(t;x) = exp\left\{\sum_{m=1}^{K}\frac{i^m}{m!}\,a_m\,x^m - b\,x^{2n}\right\} \;;\; K=1,2,\dots;\; K<2n\,;\; b>0 \qquad \text{(B.2)}$$

For each approximation $F_K(t;x)$ terms a_m in (B.2) depend on price statistical moments $p(t;m)$, $m\le K$ and match relations (B.1). The term bx^{2n}, $b>0$, $2n>K$ doesn't impact relations (B.1) but guarantees existence of the price probability measures $\eta_K(t;p)$ as Fourier transforms (5.25). Uncertainty and variability of the coefficient $b>0$ and power $2n>K$ in (B.2) underlines well-known fact that first k statistical moments don't explicitly determine characteristic function and probability measure of a random variable. Relations (B.2) describe the set of characteristic functions $F_K(t;x)$ with different $b>0$ and $2n>K$ and corresponding set of probability measures $\eta_K(t;p)$ those match (B.1; 5.25). For $K=1$ approximate price characteristic function $F_1(t;x)$ and measure $\eta_1(t;p)$ are trivial:

$$F_1(t;x) = exp\{i\,a_1 x\} \;;\; p(t;1) = -i\frac{d}{dx}F_1(t;x)|_{x=0} = a_1 \qquad \text{(B.3)}$$

$$\eta_1(t;p) = \int dx\; A_1(xt;)\,exp -ipx = \delta\big(p - p(t;1)\big) \qquad \text{(B.4)}$$

For $K=2$ approximation $F_2(t;x)$ describes the Gaussian probability measure $\eta_2(t;p)$:

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$$F_2(x;t) = exp\left\{i\,p(t;1)x - \frac{a_2}{2}x^2\right\} \quad \text{(B.5)}$$

It is easy to show that

$$p_2(t;2) = -\frac{d^2}{dx^2}F_2(t;x)|_{x=0} = a_2 + p^2(t;1) = p(t;2)$$

Hence:

$$a_2 = p(t;2) - p^2(t;1) = \sigma^2(t;p) \quad \text{(B.6)}$$

Coefficient a_2 equals price volatility $\sigma^2(t;p)$ (5.27) and Fourier transform (5.25) for $F_2(t;x)$ gives Gaussian price probability measure $\eta_2(t;p)$:

$$\eta_2(pt;) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma(p)} exp\left\{-\frac{(p-p(t;1))^2}{2\sigma^2(t;p)}\right\} \quad \text{(B.7)}$$

For $K=3$ approximation $F_3(t;x)$ has form:

$$F_3(t;x) = exp\left\{i\,p(t;1)x - \frac{\sigma^2(t;p)}{2}x^2 - i\,\frac{a_3}{6}x^3\right\} \quad \text{(B.8)}$$

$$p_3(t;3) = i\frac{d^3}{dx^3}F_3(t;x)|_{x=0} = a_3 + 3p(t;1)\sigma^2(t;p) + p^3(t;1) = p(t;3)$$

$$a_3 = p(t;3) - 3p(t;2)p(t;1) + 2\,p^3(t;1)$$

$$a_3 = E\left[\left(p - p(t;1)\right)^3\right] = Sk(t;p)\sigma^3(t;p) \quad \text{(B.9)}$$

Coefficient a_3 (B.9) depends on price skewness $Sk(t;p)$ that describe asymmetry of the price probability from normal distribution. For $K=4$ approximation $F_4(t;x)$ during Δ (5.3) depends on choice of $b>0$ and degree $2n>4$:

$$F_4(t;x) = exp\left\{i\,p(t;1)x - \frac{\sigma^2(t;p)}{2}x^2 - i\,\frac{a_3}{6}x^3 + \frac{a_4}{24}x^4 - bx^{2n}\right\} \;;\; 2n > 4 \quad \text{(B.10)}$$

Simple, but long calculations give:

$$a_4 = p(t;4) - 4p(t;3)p(t;1) + 12p(t;2)p^2(t;1) - 6p^4(t;1) - 3p^2(t;2)$$

$$a_4 = E\left[\left(p(t_i) - p(t;1)\right)^4\right] - 3E^2\left[\left(p(t_i) - p(t;1)\right)^2\right]$$

Price kurtosis $Ku(p)$ (B.11) describes how the tails of the price probability measure $\eta_K(t;p)$ differ from the tails of a normal distribution.

$$Ku(p)\sigma_p^4(t;p) = E\left[\left(p(t_i) - p(t;1)\right)^4\right] \quad \text{(B.11)}$$

$$a_4 = [Ku(p) - 3]\sigma_p^4(t;p)$$

Even simplest Gaussian approximation $F_2(t;x)$, $\eta_2(t;p)$ (B.5; B.7) uncovers direct dependence of price volatility $\sigma^2(t;p)$ (B.6; 5.27) on 2-d statistical moments of the trade value $C(t;2)$ and volume $U(t;2)$. Thus, prediction of price volatility $\sigma^2(t;p)$ for Gaussian measure $\eta^2(t;p)$ (B.9) should follow non-trivial forecasting of the statistical moments of the market trade value $C(t;2)$ and volume $U(t;2)$.

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