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Selection of A New Biasing Parameter for the Jackknife Kibria-Lukman Estimator for the Negative Binomial Regression Model

Oranye Henrietta E. [1], Adejuwon Samuel O. [2], Arum Kingsley C. [3,*], Ugwuowo Fidelis I. [4], Ugah Tobias E. [5], Adegoke Taiwo M. [6] and Sule Omeiza B. [7]

[1,3,4,5] Department of Statistics, University of Nigeria, Nsukka.
[2] Department of Mathematical and Physical Sciences, Afe Babalola University, Ado Ekiti, Nigeria.
[6] Department of Mathematics and Statistics, First Technical University, Ibadan, Nigeria.
[7] Department of Mathematical Sciences, Prince Abubakar Audu University, Anyigba, Kogi State, Nigeria.
* Correspondence email: kingsley.arum@unn.edu.ng.

Abstract: The negative binomial regression model (NBRM) is a generalized linear model which relaxes the restrictive assumption by the Poisson regression model when the variance is equal to the mean. The estimation of the parameters of the NBRM is obtained using the maximum likelihood (ML) method. Maximum likelihood estimator becomes unstable when the explanatory variables are linearly dependent, a situation known as multicollinearity. Based on this, we developed a new estimator called modified jackknifed Negative Binomial Kibria-Lukman (MJNBKL) estimator for the radiation of multicollinearity in NBRM using four different biasing (shrinkage) parameters. We establish superiority condition for MJNBKL estimator over the ones. The performance MJNBKL estimator was ascertained by comparing it with the existing ones through a Monte Carlo simulation study and two real life application datasets. The results of the simulation and real life application show that MJNBKL estimator outperformed the other estimators compared with by having the smallest MSE across all sample sizes and for different levels of correlation for the four biasing parameters used and the third biasing parameter is the optimal shrinkage parameter with the lowest MSE.

Keywords: Jackknife; Kibria-Lukman estimator; multicollinearity; maximum likelihood; negative binomial regression

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1. Introduction

The Negative Binomial (NB) Regression model (NBRM) is analogous to a multiple regression model, that the explanatory variables are observations that follow the NB. The NBRM model is a generalized linear model (GLM) model which relaxes the restrictive assumption of the Poisson regression model having the variance being equal to its mean. The NBRM model is chosen over the Poisson regression model, especially when the data is over-dispersed (Algamal (2012), Cameron & Trivedi (2013), De Jong & Heller (2008)). The parameters in the NBR model are generally evaluated by applying the maximum likelihood (ML) method.

According Alobaidi et al. (2021), the explanatory variables do not exhibit high correlation among themselves that is no multicollinearity under the NBRM. Recent studies have demonstrated that multicollinearity is often a challenge amid the explanatory variables. The ML method exists tremendous to multicollinearity. In Kibria (2003), the regression coefficients become inconclusive when the variables to be estimated are perfectly related. One of the major consequences of multicollinearity encounter in regression model for the regression coefficients is the standard errors will be inflated and hard to test individual (Oranye & Ugwuowo, 2021). So, it will be difficult to judge if the regression coefficients are significant or not. Secondly, multicollinearity shows a wider confidence interval which has the tendency to produce a false negative result, known as an error of omission in hypothesis testing (Qasim et al. 2020).

Some researchers have come with diverse way to contend with multicollinearity in linear regression Model (LRM) see E.G., Hoerl & Kennard (1970), Liu (2003), Kibria & Lukman (2020), Ozkale & Kaciranlar (2007), Suhail et al., (2020), Ugwuowo et al. (2021), Arum & Ugwuowo (2022), Perveen & Suhail, (2021), Babar et al., (2021), Dawoud et al. (2022), Algamal et al. (2023), Wasim et al., (2023), Abonazel et al. (2023) and Dawoud et al. (2022). Hoerl & Kennard (1970) proposed the ridge estimator to handle high correlation in the LRM. Singh et al. (1986) introduced an estimator to minimize the bias in the ridge estimate using jackknife procedure. It was observed that there is reduction in bias of the new estimator that causes the MSE to be smaller than the ridge estimator. According to Khurana et al. (2014), biasedness of the estimate is close to zero when they attain a second-order jackknifed ridge (J2R).

Negative binomial ridge regression model (NBRRM) has been widely used when the problem of multicollinearity exits among the explanatory variables. In literature, several types of ridge estimators have ben proposed for the estimation of biasing parameter in NBRRM, see e.g., Turkan & Özel (2018), Rashad et al. (2019), Alobaidi et al. (2021), Akram et al. (2022) and Jabur et al. (2022).

Kibria & Lukman (2020) proposed a new ridge type estimator called the Kibria-Lukman (KL) estimator that supersedes that of the ridge estimator. Although, we observed that the KL estimator in the LRM still has a substantial amount of bias. On this note, we apply the procedure of Jackknife to KL estimator on the negative binomial model.

Our objective is to jackknife the KL estimator for the NBRM in section one. We evaluated the NBRM and some existing estimators with their properties. Also, we derived the new estimator which is then compared to the existing methods theoretically. In section three, the selection of the matrix k, is very important which we adopted severally methods in estimating k, in section four; simulation study was conducted to evaluate the performance of the new estimator. A real-life application and concluding remark were given in sections five and six, respectively.

2. Methodology

The NB which is in the family of discrete probability distribution is commonly wield to model the number of successes in an outcome of independent and identically distributed Bernoulli trials ahead of a stated number of failures. It is often used in analyzing count data as long as the response variable y_i follows a negative binomial with a sequence of Bernoulli trials, as a result of success probability π, are adhere and until successes θ are adhered to.

The NB distribution probability function is given by

$$f(y;\pi,\theta) = \frac{(y_i+\theta-1)!}{y_i!(\theta-1)!}\pi_i^{\theta}(1-\pi_i)^{y_i} \tag{1}$$

where $mean = \theta(1-\pi)/\pi$ and $Var = \theta(1-\pi)/\pi^2$. The negative binomial with parameter $\pi \: and \: \theta$ with an over-dispersion parameter given as $\theta = \frac{1}{\alpha}$. Equation (1) can be rewritten as

$$f(y;\pi,\theta) = \frac{\left(y_i+\frac{1}{\alpha}-1\right)!}{y_i!\left(\frac{1}{\alpha}-1\right)!}\left(\frac{1}{1+\alpha\mu_i}\right)^{\frac{1}{\alpha}}\left(\frac{\alpha\mu_i}{1+\alpha\mu_i}\right)^{y_i} \tag{2}$$

where $\mu_i = (1-\pi_i)/\alpha\pi_i$ and $\pi_i = \frac{1}{1+\alpha\mu_i}$.

The negative binomial model is estimated using the method of ML. Cameron & Trivedi (2013) describe the likelihood function as follows:

$$\ell(\mu;y,\alpha) = \sum_{i=1}^{n}\{y_i\log(\alpha\mu_i) - \left(y_i+\frac{1}{\alpha}\right)\log(1+\alpha\mu_i) + \log\Gamma\left(y_i+\frac{1}{\alpha}\right) - log\Gamma(y_i+1) - log\,\Gamma\left(\frac{1}{\alpha}\right)\} \tag{3}$$

Let X_i is the data matrix of the ith row of X_{nxp} with p explanatory variables, β_{nxp} vector of coefficients. β is evaluated by the MLE which will maximize the log likelihood given by

$$\ell(\theta,\beta) = \sum_{i=1}^{n}\left\{\sum_{i=1}^{y_i-1}\ln(t+\theta) - \ln(y_i!) - (y_i+\theta)ln\left(1+\frac{1}{\theta}\exp(x_i'\beta)\right) + y_i\ln\left(\frac{1}{\theta}\right) + y_i(x_i'\beta)\right\} \tag{4}$$

where $\mu_i = \exp(x_i'\beta)$ and $log\left[\frac{\Gamma(\theta+y_i)}{\Gamma(\theta)}\right] = \sum_{i=1}^{y_i-1}(t+\theta)$

The MLE can be achieved by working out the likelihood equation

$$S(\beta) = \frac{\partial\ell(\theta,\beta)}{\partial(\beta)} = \sum_{i=1}^{y_i-1}\frac{y_i-\mu_i}{1+\frac{1}{\theta}\mu_i}x_i = 0 \tag{5}$$

Equation (5) is nonlinear for β, hence, the weighted least square algorithm is adopted as follows:

$$\hat{\beta}_{MLE} = (P)^{-1}X'\widehat{W}\hat{c} \quad (6)$$

where P=$\hat{X}'\widehat{W}X$ $\widehat{W} = diag[\frac{\hat{\mu}_i}{(1+(1/\theta)\hat{\mu}_i)}]$ and $\hat{z}$ is a vector where ith elements equal to $\hat{c} = \log(\hat{\mu}_i) + \frac{\hat{y}_i-\hat{\mu}_i}{\hat{\mu}_i}$.

The ML estimator of β is asymptotically commonly classified with mean vector to be $E(\hat{\beta}_{ML}) = 0$ and covariance matrix $Cov(\hat{\beta}_{ML}) = (P)^{-1}$. The mean squared error established on the asymptotic covariance matrix equals

$$\text{MSE}(\hat{\beta}_{ML}) = trace[(P)^{-1}] = \sum_{j=1}^{p}\frac{1}{p_j} \quad (7)$$

where p_j is the eigenvalue of the P matrix.

Mansson (2012) proposed the negative binomial ridge estimator to account for multicollinearity and defined as follows:

$$\hat{\beta}_{NBRE} = (P+kI)^{-1}P\hat{\beta}_{ML},\ 0<k<1, \quad (8)$$

where the tuning parameter k>0. The scalar MSE is defined as follows: $SMSE(\hat{\beta}_{NBRE}) =$

$$\sum_{j=i}^{p}\frac{p_j}{(p_j+k)^2} + k^2\sum_{j=i}^{p}\frac{\alpha_j^2}{(p_j+k)^2} \quad (9)$$

2.1. The proposed Estimator

According to Oranye & Ugwuowo (2021), opined that the jackknifed procedure reduces the biasness of the Poisson KL estimator in the generalized linear models. They derived an estimator that has smaller bias and also, has a desirable large sample property. Thus, we developed the jackknifed NBR model as follows:

The Negative Binomial of the KL is given by

$$\hat{\beta}^{\text{NBKL}} = (P+\text{kI})^{-1}(P-\text{kI})\hat{\beta}^{\text{ML}} \quad (10)$$

let $\text{P} = \text{diag}(p_1, p_2, \dots, p_p)$ be the eigenvalues of P matrix, insomuch that $\text{Q}'\text{PQ} = \text{Z}'\widehat{\text{W}}\text{Z} = \text{P}$, where $\text{Z} = \text{XQ}$. Q is an orthogonal matrix of P that the rows are normalized eigenvectors. So, the Negative Binomial MLE can be written as:

$$\hat{\alpha}^{\text{NBML}} = P^{-1}\text{Z}'\widehat{\text{W}}\hat{\text{c}} \quad (11)$$

where $\hat{\beta}^{\text{NBML}} = \text{Q}\hat{\alpha}^{\text{NBML}}$.

Khurana et al. (2014) reduce the bias of the ridge estimator using the jackknife procedures and Quenouille (1956) study the properties of jackknife procedure. The Negative Binomial KL (NBKL) estimator is written as

$$\hat{\alpha}^{\text{NBKL}} = (P+\text{kI})^{-1}(P-\text{kI})P^{-1}\text{Z}'\widehat{\text{W}}\hat{\text{c}} \quad (12)$$

The jackknifed procedure will now be applied to the Negative Binomial of K-L estimator in equation (12) as follows:

$$\hat{\alpha}_{\text{NBKL}-\text{i}} = (\text{P}_{-\text{i}}+\text{kI})^{-1}(\text{P}_{-\text{i}}-\text{kI})\,\hat{\alpha}^{NBML} \quad (13)$$

Equation (13) can be written as follows:

$$\hat{\alpha}_{-i}^{NBKL} = (B - z_i\widehat{w}_i z_i)^{-1}(Z'\widehat{w}\hat{c} - z_i\widehat{w}_i\hat{c}_i) \quad (14)$$

According to Sherman-Morrison-Woodbury theorem Rao (1973), $(B - z_i\widehat{w}_i z_i)^{-1} = (B^{-1} + \frac{B^{-1}z_i\widehat{w}_i z_i B^{-1}}{1-z_i'\widehat{w}_i B^{-1}z_i})$. Hence,

$$\hat{\alpha}_{-i}^{PKL} = (B^{-1} + \frac{B^{-1}z_i\widehat{w}_i z_i B^{-1}}{1-z_i'\widehat{w}_i B^{-1}z_i})(z'\widehat{\text{W}}\hat{\text{c}} - z_i\widehat{w}_i\hat{\text{c}}_i)$$

where $B^{-1} = (\text{P}+\text{kI})^{-1}(\text{P}-\text{kI})\text{P}^{-1}$.

$$\hat{\alpha}_{-i}^{NBKL} = B^{-1}z'\widehat{\text{W}}\hat{\text{c}} - B^{-1}z_i\widehat{w}_i\hat{\text{c}}_i + \frac{B^{-1}z_i\widehat{w}_i z_i'B^{-1}z_i'\widehat{w}_i\hat{\text{c}}_i}{1-z_i'\widehat{w}_i B^{-1}z_i} - \frac{B^{-1}z_i\widehat{w}_i z_i'B^{-1}}{1-z_i'\widehat{w}_i B^{-1}z_i}z_i\widehat{w}_i\hat{\text{c}}_i$$

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$$= \hat{\alpha}^{NBKL} - B^{-1}\widehat{w}_i\hat{c}_i\left(1 + \frac{z_i{}'\widehat{w}_i B^{-1} z_i{}'}{1 - z_i{}'\widehat{w}_i B^{-1} z_i}\right) + \frac{B^{-1} z_i \widehat{w}_i z_i{}' B^{-1}\, z_i \widehat{w}_i \hat{c}_i}{1 - z_i{}'\widehat{w}_i B^{-1} z_i}$$

$$= \hat{\alpha}^{NBKL} - B^{-1} z_i \widehat{w}_i \hat{c}_i\left(\frac{1}{1 - z_i{}'\widehat{w}_i B^{-1} z_i}\right) + \frac{B^{-1} z_i \widehat{w}_i z_i{}'\hat{\alpha}^{PKL}}{1 - z_i{}'\widehat{w}_i B^{-1} z_i}$$

$$= \hat{\alpha}^{NBKL} - B^{-1} z_i \widehat{w}_i\left(\frac{\hat{u}_i - z_i{}'\widehat{w}_i\hat{\alpha}^{PKL}}{1 - z_i{}'\widehat{w}_i B^{-1} z_i}\right)$$

$$= \hat{\alpha}^{NBKL} - \left(\frac{B^{-1} z_i \widehat{w}_i e_i}{1 - m_i}\right) \tag{15}$$

where $m_i = z_i{}'\widehat{w}_i B^{-1} z_i$ and z_i' is the ith row of Z, $e_i = y_i - z_i{}'\widehat{w}_i\hat{\alpha}^{NBKL}$ is the ith NBKL residual and $B^{-1} = (\mathrm{P} + \mathrm{kI})^{-1}(\mathrm{P} - \mathrm{kI})\mathrm{P}^{-1}$.

Also, agreeing to Khurana et al. (2014), the weighted pseudo values are outline as follows:

$$Q_i = \hat{\alpha}_{KL} + n(1 - w_i)(\hat{\alpha}^{NBKL} - \hat{\alpha}_{-i}^{NBKL}) \tag{16}$$

The weighted Jackknifed estimator of $\hat{\alpha}_{KL}$ is established as:

$$\alpha_{JKL} = \frac{1}{n}\sum Q_i = Q = \hat{\alpha}_{KL} + B^{-1}\sum_{i=1}^{n} z_i\, e_i \tag{17}$$

since $\sum_{i=1}^{n} z_i \widehat{w}_i\, e_i = \sum_{i=1}^{n} z_i \widehat{w}_i (\hat{c}_i - z_i{}'\widehat{w}_i\hat{\alpha}^{NBKL})$

$$\hat{\alpha}^{JNBKL} = \hat{\alpha}^{NBKL} + B^{-1}(1 - [(\mathrm{P} + \mathrm{kI})^{-1}(\mathrm{P} - kI)])\mathrm{Z}'\widehat{\mathrm{W}}\hat{\mathrm{c}}$$

$$\hat{\alpha}^{JNBKL} = \hat{\alpha}^{NBKL} + B^{-1}\, Z'\widehat{\mathrm{W}}\hat{\mathrm{u}} - B^{-1}[(\mathrm{P} + \mathrm{kI})^{-1}(\mathrm{P} - kI)\mathrm{Z}'\widehat{\mathrm{W}}\hat{\mathrm{c}}$$

$$\hat{\alpha}^{JNBKL} = \hat{\alpha}^{NBKL} + \hat{\alpha}^{NBKL} - (\mathrm{P} + \mathrm{kI})^{-1}(\mathrm{P} - kI)\hat{\alpha}^{NBKL}$$

$$\hat{\alpha}^{JNBKL} = 2\hat{\alpha}^{NBKL} - (\mathrm{P} + \mathrm{kI})^{-1}(\mathrm{P} - kI)\hat{\alpha}_{KL}$$

$$\hat{\alpha}^{JPKL} = \hat{\alpha}^{NBKL}(2\mathrm{I} - (\mathrm{P} + \mathrm{kI})^{-1}(\mathrm{P} - kI)$$

$$\hat{\alpha}^{JNBKL} = (\mathrm{P} + \mathrm{kI})^{-1}[2(\mathrm{P} + kI)(\mathrm{P} - kI)]\hat{\alpha}^{NBKL}$$

Then Jackknifing the Negative Binomial for the K-L estimator is

$$\hat{\alpha}^{JNBKL} = [(A(\mathrm{P} + kI)^{-1})^2)]\, \hat{\alpha}^{NBML} \tag{18}$$

where I-(2k) is represented by A

We then follow the procedure of Akdeniz &Akdeniz (2010), to modified equation Khurana et al. (2014) by replacing $\hat{\alpha}^{NBML}$ with $\hat{\alpha}^{NBKL}$ to get the modified jackknife Negative Binomial KL given as:

$$\hat{\alpha}^{MJNBKL} = [(A(\mathrm{P} + kI)^{-1})^2)]\, \hat{\alpha}^{NBKL} \tag{19}$$

where $\hat{\alpha}^{NBKL} = [A(\mathrm{P} + kI)^{-1}]\hat{\alpha}^{NBML}$.

Hence replacing $\hat{\alpha}^{NBKL}$ in equation 19, we get the modified Jackknifed Negative Binomial for the KL estimator given as:

$$\hat{\alpha}^{MJNBKL} = [A(\mathrm{P} + kI)^{-1}]^2[I + 2k(\mathrm{P} + kI)^{-1}]\hat{\alpha}^{NBML} \tag{20}$$

2.2. The characteristics of the MJNBKL estimator

$$E(\hat{\alpha}^{MJNBKL}) = [A(\mathrm{P} + kI)^{-1}]^2[I + 2k(\mathrm{P} + kI)^{-1}]\alpha \tag{21}$$

where $\hat{\alpha}^{NBKL} = [A(\mathrm{P} + kI)^{-1}]\hat{\alpha}^{NBML}$.

$$\text{Bias}\,(\hat{\alpha}^{MJNBKL}) = [[A(\mathrm{P} + kI)^{-1}]^2[I + 2k(\mathrm{P} + kI)^{-1}] - I]\alpha \tag{22}$$

$$V(\hat{\alpha}^{MJNBKL}) = [A((\mathrm{P} + kI)^{-1})^2]^2\mathrm{P}[\mathrm{I} - (2k(\mathrm{P} + kI)^{-1})]^2 \tag{23}$$

$$MSEM(\hat{\alpha}_{JNBKL}) = [A((\mathrm{P} + kI)^{-1})^2]^2\mathrm{P}[\mathrm{I} - (2k(\mathrm{P} + kI)^{-1})]^2 + \mathrm{Bias}(\hat{\alpha}^{MJNBKL})'\mathrm{Bias}(\hat{\alpha}^{MJNBKL}) \tag{24}$$

$$\text{The scalar } MSE(\hat{\alpha}^{MJNBKL}) = \sum_{j=1}^{p}\frac{\left((p_j+k)^2 - 4k^2\right)^2(p_j-k)^2}{p_j(p_j+k)^6} + \sum_{j=1}^{p}\frac{\left((p_j-k)^2(p_j+3k) - (p_j+k)^3\right)^2\alpha_j{}^2}{(p_j+k)^6} \tag{25}$$

The following Lemmas were adopted for the theoretical comparisons.

Lemma 1: *Suppose Z be an z x z matrix of positive definite, Farebrother (1976) that is Z > 0, θ be some vector, then* $Z - \theta\theta' \geq 0$ *iff* $\theta' Z^{-1}\theta \leq 1$.

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Lemma 2: *Suppose* $\theta_i = B_i y$ *i=1,2,* θ *is a linear estimator . Let* $C = Cov(\hat{\theta}_1) - Cov(\hat{\theta}_2) > 0$, *where* $Cov(\hat{\theta}_i)$*i=1,2 is the covariance matrix of* $\hat{\theta}_i$ *and* $s_i = Bias\ (\hat{\theta}_i) = (B_i X - I)\theta, = 1{,}2$. *So,*

$$\Delta(\hat{\theta}_1 - \hat{\theta}_2) = MSE(\hat{\theta}_1) - MSE(\hat{\theta}_2) = \sigma^2 C + s_1 s_2{}' - s_2 s_2{}' > 0 \tag{26}$$

if and only if $s'^{[\sigma^2 C + s_1 s_2{}']^{-1} s_2} < 1$, *where* $MSEM(\hat{\theta}_i) = Cov(\hat{\theta}_i) + s_i s_i{}'$ *Trenkler & Toutenburg (1990).*

2.3. Comparison among the estimators

2.3.1. Comparison between $\hat{\boldsymbol{\alpha}}^{\boldsymbol{NBML}}$and $\hat{\boldsymbol{\alpha}}^{\boldsymbol{MJNBK}}$L

Theorem 1. *The JMNBKL estimator* ($\hat{\alpha}^{JMNBKLE}$) *supersede that of Maximum Likelihood estimator for the NB* ($\hat{\alpha}^{NBML}$) *iff*

$$\Delta\left(\hat{\alpha}_{NBML} - \hat{\alpha}_{JMNBKLE}\right) = MSEM(\hat{\alpha}_{NBML}) - MSEM(\hat{\alpha}_{JMNBKLE}) = \sigma^2 C + s_1 s_2' - s_2 s_2' > 0$$

Proof. The difference between MSE ($\hat{\alpha}^{NBML}$) and MSEM $\hat{\alpha}^{MJNBKL}$is

$$= diag\left\{\frac{1}{\lambda_i} - \frac{\left(\left(p_j+k\right)^2 - 4k^2\right)^2 \left(p_j-k\right)^2}{p_j\left(p_j+k\right)^6}\right\}_{j=1}^{p} \tag{28}$$

Consequently, $\mathrm{P}^{-1} - [A((\mathrm{P}+kI)^{-1})^2]^2\mathrm{P}[\mathrm{A}((\mathrm{P}+kI)^{-1})]^2$ is positive definite provided $p_j(p_j + k)^6 > p_j\left(\left(p_j+k\right)^2 - 4k^2\right)^2 \left(p_j - k\right)^2$.

2.3.2. Comparison between $\hat{\boldsymbol{\alpha}}^{\boldsymbol{NBRR}}$and $\hat{\boldsymbol{\alpha}}^{\boldsymbol{MJNBKL}}$

$$MSE(\hat{\alpha}^{NBRRE}) = \sum_{j=1}^{p} \frac{p_j}{(p_j+k)^2} + k^2 \sum_{j=1}^{p} \frac{\alpha_j{}^2}{(p_j+k)^2} \tag{29}$$

$$\text{MSEM} \quad (\hat{\alpha}^{NBRRE}) = Q\mathrm{P}^k\mathrm{P}\mathrm{P}^k Q^T + \mathrm{k}^2\mathrm{P}^k \alpha\alpha^T \mathrm{P}^k \tag{30}$$

Theorem 2. *The JMNBKL estimator* ($\hat{\alpha}^{JMNBKLE}$) *supersede that of Ridge regression for the NB estimator* $\hat{\alpha}^{NBRR}$ *if and only if* $\Delta\left(\hat{\alpha}_{NBRR} - \hat{\alpha}_{JMNBKLE}\right) = MSEM(\hat{\alpha}_{NBRR}) - MSEM(\hat{\alpha}_{JMNBKLE}) = \sigma^2 C + s_1 s_2' - s_2 s_2' > 0$

Proof. The difference between MSE($\hat{\alpha}^{NBRR}$) and MSEM $\hat{\alpha}^{MJNBKL}$ is

$$= diag\left\{\frac{p_j^2}{\left(p_j+k\right)^2} - \frac{\left(\left(p_j+k\right)^2 - 4k^2\right)^2 \left(p_j-k\right)^2}{p_j\left(p_j+k\right)^6}\right\}_{j=1}^{p} \tag{31}$$

Consequently, $[\mathrm{P}^2(\mathrm{P}+kI)^{-2} - [A((\mathrm{P}+kI)^{-1})^2]^2\mathrm{P}[\mathrm{A}((\mathrm{P}+kI)^{-1})]^2$ is positive definite provided that $p_j^3(p_j+k)^4 > \left(\left(p_j+k\right)^2 - 4k^2\right)^2 \left(p_j - k\right)^2$.

2.3.3. Comparison between $\hat{\boldsymbol{\alpha}}^{\mathrm{JNBRR}}$and $\hat{\alpha}^{\mathrm{MJNBKL}}$

$$MSE(\hat{\alpha}^{JNBRR}) = (1 - k^2B^{-2})\mathrm{P}^{-1}{}_{PR}(1 - k^2B^{-2}) + k^4B^{-2}\alpha\alpha^TB^{-2} \tag{32}$$

Theorem 3. *The JMNBKL estimator ($\hat{\alpha}^{JMNBKLE}$) supersede that of Jackknifed ridge regression for the NB estimator ($\hat{\alpha}^{JNBRR}$) if and only if*

$$\Delta(\hat{\alpha}_{JNBRR} - \hat{\alpha}_{JMNBKLE}) = MSEM(\hat{\alpha}_{JNBRR}) - MSEM(\hat{\alpha}_{JMNBKLE}) = \sigma^2C + s_1s_2' - s_2s_2' > 0$$

Proof. The difference between MSE ($\hat{\alpha}^{JNBLR}$) and MSEM $\hat{\alpha}^{MJNBKL}$ is

$$= diag\left\{\frac{\left(p_j^2+2p_j+k\right)^2}{p_j(p_j+k)^4} - \frac{\left((p_j+k)^2-4k^2\right)^2(p_j-k)^2}{p_j(p_j+k)^6}\right\}_{j=1}^{p} \tag{33}$$

Consequently, $[(\mathrm{P}^2 + 2\mathrm{P} + \mathrm{k}I)^2\mathrm{P}(\mathrm{P} + kI)^{-4} - [A((\mathrm{P} + kI)^{-1})^2]^2\mathrm{P}[\mathrm{A}((\mathrm{P} + kI)^{-1})]^2$ is positive definite provided that $(p_j + k)^2(p_j^2 + 2p_j + k)^2 > \left((p_j + k)^2 - 4k^2\right)^2(p_j - k)^2$

2.3.4. Comparing between $\hat{\boldsymbol{\alpha}}^{\boldsymbol{NBKL}}$ and $\hat{\boldsymbol{\alpha}}^{\boldsymbol{MJNBK}\mathrm{L}}$

$$MSE(\hat{\alpha}^{NBKLE}) = \sum_{j=1}^{p}\frac{1}{p_j}\left(\frac{p_j-k}{p_j+k}\right)^2 + 4k^2\sum_{j=1}^{p}\left(\frac{\alpha_j}{p_j+k}\right)^2 \tag{34}$$

Theorem 4. *The JMNBKL estimator ($\hat{\alpha}^{JMNBKLE}$) supersede that of KL for the NB estimator ($\hat{\alpha}^{NBKL}$) if and only if* $\Delta(\hat{\alpha}_{NBKL} - \hat{\alpha}_{JMNBKLE}) = MSEM(\hat{\alpha}_{NBKL}) - MSEM(\hat{\alpha}_{JMNBKLE}) = \sigma^2C + s_1s_2' - s_2s_2' > 0$

Proof. The difference between MSE ($\hat{\alpha}^{NBKL}$) and MSEM $\hat{\alpha}^{MJNBKLE}$ is

$$= diag\left\{\frac{1}{p_j}\left(\frac{p_j-k}{p_j+k}\right)^2 - \frac{\left((p_j+k)^2-4k^2\right)^2(p_j-k)^2}{p_j(p_j+k)^6}\right\}_{j=1}^{p} \tag{35}$$

Consequently, $[(\mathrm{P} - kI)^2\mathrm{P}^{-1}(\mathrm{P} + kI)^{-2} - [A((\mathrm{P} + kI)^{-1})^2]^2\mathrm{P}[\mathrm{A}((\mathrm{P} + kI)^{-1})]^2$ is positive definite provided $(p_j - k)^2(p_j + k)^4 > \left((p_j + k)^2 - 4k^2\right)^2(p_j - k)^2$.

3. Estimating the value of the Biasing parameters selection

The selection of the matrix k, is very important. In order to estimate the optimal value in equation 20, different methods have been adopted to estimate different biasing parameter both in linear and generalized linear models. The parameter k (ridge) reduces the problem of multicollinearity. These methods are used to select the optimal shrinkage parameter for the negative binomial regression model and they are given below by different authors;

1. Lukman et al. (2021) $k_1 = \frac{\lambda_i}{1+2\lambda_i\alpha_i^2}$
2. Kibria (2002) $k_2 = \frac{1}{p}\sum_{i=1}^{p}\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}$
3. Manson et al. (2012) $k_3 = \max\left(0, \min\left(\frac{\lambda_i}{1+2\lambda_i\alpha_i^2}\right)\right)$
4. Lukman & Ayinde. (2017) $k_4 = \sqrt{\max\left(0, \min\left(\frac{\lambda_i}{1+2\lambda_i\alpha_i^2}\right)\right)}$

4. Monte Carlo Stimulation

A Monte Carlo simulation experiment was perpetrated in this research to decide the effectiveness of the proposed estimator with the various levels of multicollinearity.

4.1. Stimulation design

According to Kibria (2003) and Månsson & Shukur (2011), the dependent variable from the NBR is generated from n observations as $NB(\mu_i, \mu_i + \theta\mu_i^2)$ with $\mu_i = \exp(x_i^T, \beta)$. Here $\beta = (\beta_0, \beta_1, \ldots, \beta_p)$

with $\sum_{n=1}^{p} \beta_j^2 = 1$ and $\beta_1 = \beta_2 = \cdots = \beta_p$. The explanatory variables at various levels of correlation are generated as:

$$x_{ij} = (1-\rho^2)^{1/2} z_{ij} + \rho z_{ip+1} \quad i = 1,2,\dots n\,, j = 1,2,\dots,p \tag{36}$$

ρ^2 is the different stages of multicollinearity among the explanatory variables Kibria & Banik (2016), Arum et al. (2022) and Arum et al. (2023). z_{ij} are computer generated random numbers which are resulted in applying the standard normal distribution insomuch that i scales from 1 to n and j from 1 to p. Sample size which will have effect on accuracy of the prediction, four representatives value of the sample size are considered which are 25, 50, 75, 100 and 200 observations. The various levels of correlation consider are 0.8, 0.9, 0.99 and 0.999 and the independent variables which can also cause increase in MSE when the parameters are 5 and 9 are considered. The mean square is estimated and given in equation (37).

$$MSE(\hat{\beta}) = \frac{1}{5000} \sum_{j=1}^{5000} (\hat{\beta}_{ij} - \beta_i)' (\hat{\beta}_{ij} - \beta_i) \tag{37}$$

where $\hat{\beta}_{ij}$ is the estimate of the ith parameter in jth replication and β_i is the actual parameter values. The experiment is reproduced 5000 times. Simulated results for the MSE values of the estimators for p=5 and 9 for different shrinkages parameters selected are show in Tables 1 and 2 respectively.

Table 1. $p = 5$ for the estimated MSEs of the estimator.

n			1				2			
	P	MLE	Ridge	Jackridge	Kl	MJNBKL	Ridge	Jackridge	Kl	MJNBKL
25	0.8	0.378	0.378	0.378	0.377	0.376	0.374	0.374	0.37	0.367
	0.9	0.392	0.392	0.392	0.391	0.389	0.387	0.386	0.381	0.377
	0.99	0.645	0.617	0.616	0.591	0.545	0.559	0.544	0.492	0.418
	0.999	3.172	2.676	2.616	2.254	1.653	1.998	1.752	1.241	1.205
50	0.8	0.357	0.357	0.357	0.356	0.355	0.353	0.353	0.349	0.346
	0.9	0.371	0.371	0.371	0.37	0.368	0.366	0.365	0.36	0.356
	0.99	0.624	0.596	0.595	0.57	0.524	0.538	0.523	0.471	0.397
	0.999	3.151	2.655	2.595	2.233	1.632	1.977	1.731	1.22	1.184
75	0.8	0.3558	0.3558	0.3558	0.3548	0.3538	0.3518	0.3518	0.3478	0.3448
	0.9	0.3698	0.3698	0.3698	0.3688	0.3668	0.3648	0.3638	0.3588	0.3548
	0.99	0.6228	0.5948	0.5938	0.5688	0.5228	0.5368	0.5218	0.4698	0.3958
	0.999	3.1498	2.6538	2.5938	2.2318	1.6308	1.9758	1.7298	1.2188	1.1828
100	0.8	0.3547	0.3547	0.3547	0.3537	0.3527	0.3507	0.3507	0.3467	0.3437
	0.9	0.3687	0.3687	0.3687	0.3677	0.3657	0.3637	0.3627	0.3577	0.3537
	0.99	0.6217	0.5937	0.5927	0.5677	0.5217	0.5357	0.5207	0.4687	0.3947
	0.999	3.1487	2.6527	2.5927	2.2307	1.6297	1.9747	1.7287	1.2177	1.1817
200	0.8	0.3303	0.3303	0.3303	0.3293	0.3283	0.3263	0.3263	0.3223	0.3193
	0.9	0.3443	0.3443	0.3443	0.3433	0.3413	0.3393	0.3383	0.3333	0.3293
	0.99	0.5973	0.5693	0.5683	0.5433	0.4973	0.5113	0.4963	0.4443	0.3703
	0.999	3.1243	2.6283	2.5683	2.2063	1.6053	1.9503	1.7043	1.1933	1.1573

continuation of Table 1.

N			3				4			
	P	MLE	Ridge	Jackridge	Kl	MJNBKL	Ridge	Jackridge	Kl	MJNBKL
25	0.8	0.368	0.368	0.356	0.344	**0.321**	0.374	0.374	0.37	0.367
	0.9	0.372	0.372	0.362	0.35	**0.332**	0.387	0.386	0.381	0.377
	0.99	0.617	0.617	0.616	0.501	**0.5**	0.559	0.544	0.492	0.418
	0.999	2.676	2.676	2.616	1.652	**1.058**	1.998	1.752	1.241	1.205
50	0.8	0.368	0.347	0.335	0.323	**0.3**	0.353	0.353	0.349	0.346
	0.9	0.372	0.351	0.341	0.329	**0.311**	0.366	0.365	0.36	0.356
	0.99	0.617	0.596	0.595	0.48	**0.479**	0.538	0.523	0.471	0.397
	0.999	2.676	2.655	2.595	1.631	**1.037**	1.977	1.731	1.22	1.184
75	0.8	0.368	0.3458	0.3338	0.3218	**0.2988**	0.3518	0.3518	0.3478	0.3448
	0.9	0.372	0.3498	0.3398	0.3278	**0.3098**	0.3648	0.3638	0.3588	0.3548
	0.99	0.617	0.5948	0.5938	0.4788	**0.4778**	0.5368	0.5218	0.4698	0.3958
	0.999	2.676	2.6538	2.5938	1.6298	**1.0358**	1.9758	1.7298	1.2188	1.1828
100	0.8	0.368	0.3447	0.3327	0.3207	**0.2977**	0.3507	0.3507	0.3467	0.3437
	0.9	0.372	0.3487	0.3387	0.3267	**0.3087**	0.3637	0.3627	0.3577	0.3537
	0.99	0.617	0.5937	0.5927	0.4777	**0.4767**	0.5357	0.5207	0.4687	0.3947
	0.999	2.676	2.6527	2.5927	1.6287	**1.0347**	1.9747	1.7287	1.2177	1.1817
200	0.8	0.3303	0.3203	0.3083	0.2963	**0.2733**	0.3263	0.3263	0.3223	0.3193
	0.9	0.3443	0.3243	0.3143	0.3023	**0.2843**	0.3393	0.3383	0.3333	0.3293
	0.99	0.5973	0.5693	0.5683	0.4533	**0.4523**	0.5113	0.4963	0.4443	0.3703
	0.999	3.1243	2.6283	2.5683	1.6043	**1.0103**	1.9503	1.7043	1.1933	1.1573

Table 2. $p = 9$ for the estimated MSEs of the estimator.

N			1				2			
	P	MLE	Ridge	Jackridge	Kl	MJNBKL	Ridge	Jackridge	Kl	MJNBKL
25	0.8	1.143	1.143	1.143	1.142	1.141	1.139	1.139	1.135	1.132
	0.9	1.157	1.157	1.157	1.156	1.154	1.152	1.151	1.146	1.142
	0.99	1.41	1.382	1.381	1.356	1.31	1.324	1.309	1.257	1.183
	0.999	3.937	3.441	3.381	3.019	2.418	2.763	2.517	2.006	1.97
50	0.8	1.122	1.122	1.122	1.121	1.12	1.118	1.118	1.114	1.111
	0.9	1.136	1.136	1.136	1.135	1.133	1.131	1.13	1.125	1.121
	0.99	1.389	1.361	1.36	1.335	1.289	1.303	1.288	1.236	1.162
	0.999	3.916	3.42	3.36	2.998	2.397	2.742	2.496	1.985	1.949
75	0.8	1.1208	1.1208	1.1208	1.1198	1.1188	1.1168	1.1168	1.1128	1.1098
	0.9	1.1348	1.1348	1.1348	1.1338	1.1318	1.1298	1.1288	1.1238	1.1198
	0.99	1.3878	1.3598	1.3588	1.3338	1.2878	1.3018	1.2868	1.2348	1.1608
	0.999	3.9148	3.4188	3.3588	2.9968	2.3958	2.7408	2.4948	1.9838	1.9478
100	0.8	1.1197	1.1197	1.1197	1.1187	1.1177	1.1157	1.1157	1.1117	1.1087
	0.9	1.1337	1.1337	1.1337	1.1327	1.1307	1.1287	1.1277	1.1227	1.1187

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	0.99	1.3867	1.3587	1.3577	1.3327	1.2867	1.3007	1.2857	1.2337	1.1597
	0.999	3.9137	3.4177	3.3577	2.9957	2.3947	2.7397	2.4937	1.9827	1.9467
200	0.8	1.0953	1.0953	1.0953	1.0943	1.0933	1.0913	1.0913	1.0873	1.0843
	0.9	1.1093	1.1093	1.1093	1.1083	1.1063	1.1043	1.1033	1.0983	1.0943
	0.99	1.3623	1.3343	1.3333	1.3083	1.2623	1.2763	1.2613	1.2093	1.1353
	0.999	3.8893	3.3933	3.3333	2.9713	2.3703	2.7153	2.4693	1.9583	1.9223

continuation of Table 2.

N			3				4			
	P	MLE	Ridge	Jackridge	Kl	MJNBKL	Ridge	Jackridge	Kl	MJNBKL
25	0.8	1.143	1.133	1.121	1.109	**1.086**	1.139	1.139	1.135	1.132
	0.9	1.157	1.137	1.127	1.115	**1.097**	1.152	1.151	1.146	1.142
	0.99	1.41	1.382	1.381	1.266	**1.265**	1.324	1.309	1.257	1.183
	0.999	3.937	3.441	3.381	2.417	**1.823**	2.763	2.517	2.006	1.97
50	0.8	1.122	1.112	1.1	1.088	**1.065**	1.118	1.118	1.114	1.111
	0.9	1.136	1.116	1.106	1.094	**1.076**	1.131	1.13	1.125	1.121
	0.99	1.389	1.361	1.36	1.245	**1.244**	1.303	1.288	1.236	1.162
	0.999	3.916	3.42	3.36	2.396	**1.802**	2.742	2.496	1.985	1.949
75	0.8	1.1208	1.1108	1.0988	1.0868	**1.0638**	1.1168	1.1168	1.1128	1.1098
	0.9	1.1348	1.1148	1.1048	1.0928	**1.0748**	1.1298	1.1288	1.1238	1.1198
	0.99	1.3878	1.3598	1.3588	1.2438	**1.2428**	1.3018	1.2868	1.2348	1.1608
	0.999	3.9148	3.4188	3.3588	2.3948	**1.8008**	2.7408	2.4948	1.9838	1.9478
100	0.8	1.1197	1.1097	1.0977	1.0857	**1.0627**	1.1157	1.1157	1.1117	1.1087
	0.9	1.1337	1.1137	1.1037	1.0917	**1.0737**	1.1287	1.1277	1.1227	1.1187
	0.99	1.3867	1.3587	1.3577	1.2427	**1.2417**	1.3007	1.2857	1.2337	1.1597
	0.999	3.9137	3.4177	3.3577	2.3937	**1.7997**	2.7397	2.4937	1.9827	1.9467
200	0.8	1.0953	1.0853	1.0733	1.0613	**1.0383**	1.0913	1.0913	1.0873	1.0843
	0.9	1.1093	1.0893	1.0793	1.0673	**1.0493**	1.1043	1.1033	1.0983	1.0943
	0.99	1.3623	1.3343	1.3333	1.2183	**1.2173**	1.2763	1.2613	1.2093	1.1353
	0.999	3.8893	3.3933	3.3333	2.3693	**1.7753**	2.7153	2.4693	1.9583	1.9223

From Table 1 and table 2 above, we observed that increase in the correlation increases the MSE estimated values. Number of explanatory variables will also affect the estimators meaning as p increases the MSE also increases. When the degree of correlation increases, the sample size increases which in turn will also decrease the value of the MSE of the estimators. Jackknifed Negative Binomial KL estimator has the smallest MSE when compare with other estimators when the explanatory variables are 4 and 9 respectively and also, for sample sizes of 25, 50, 75, 100 and 200. Furthermore, in terms of the optimal shrinkage's parameter of k, JNBKL shows superiority over other estimators for k3 in the simulation study which shows the lowest MSE.

5. Real-Life Application

We adopted two datasets to illustrate the theoretical findings in this research, the first is the English league football data taken from Alanaz & Algmal (2018), and Arum et al., (2022) which is a Poisson data. The data consists of 20 sample size observations with one dependent and five independent variables. X_1 define the number of yellow cards, X_2 – is the number of red cards, X_3

define the goals scored, X_4 is the number of conceded goads and X_5 – the number of points earned. Y which is the response variable denotes the number of won matches. We examined whether the Poisson regression model fits the data using the deviance test. The results show that the model fits a Poisson regression. We observed that the data has high level of multicollinearity because the Condition Index for the dataset is greater than one. The data were analyzed in Table 3 with the same bias parameter as in equation (36) for the estimator and the result shows that the MSE for the proposed estimator MJNBKL has a smaller MSE than the other estimator in study. The regression coefficients for all the estimators possesses the different sign.

Table 3. Regression coefficients of all the estimators and their corresponding MSE values.

Coef.	MLE	RIDGE	JRIDGE	KL	**MJNBKL**
β_0	1.0955	0.5763	0.4789	0.55674	0.4470
β_1	0.00534	0.00534	0.00643	0.0053810	0.00971
β_2	-0.0154	-0.0154	-0.00536	-0.014483	-0.0093
β_3	0.01389	0.01389	0.01522	0.013970	0.01735
β_4	0.00078	0.00079	0.00368	0.000942	0.00866
β_5	0.02874	0.02875	0.02554	0.028586	0.01922
MSE	0.00803	0.0051	0.0039	0.00189	**0.00182**

For the second real life application, we used the aircraft damage dataset to examine the efficiency of the proposed estimator. Researchers have used the aircraft damage dataset to fit the Poisson regression model Amin et al. (2020). Mayer et al. (2012) shows that there is high level of multicollinearity in the dataset with condition number given by 219.37. The performance of the estimators was assessed via the mean squared error (MSE) and the result is presented in table 4 below.

Table 4. Regression coefficient adopting the Aircraft damage data.

Coeff.	MLE	RIDGE	JRIDGE	KL	**MJNBKL**
Intercepts	-0.4060	-0.16755	-0.0221	-0.10676	-0.18840
X_1	0.568772	0.379945	0.3212759	0.3905872	-0.0739795
X_2	0.165425	0.170544	0.169616	0.167538	0.205048
X_3	-0.01352	-0.015299	-0.016419	-0.0157773	-0.0150024
MSE	1.029008	0.272690	0.23568	0.2248674	0.0750591

6. Conclusion

This research work examines the selection of a new biasing parameter for the negative binomial regression model. The modified jackknife Kibra–Lukman estimator was proposed to reduce the bias in the jackknifed KL for the selection of the biasing parameter. Theoretical comparison was conducted to establish the superiority of MJNBKL estimator over the existing one like MLE, Negative Binomial ridge, Jackknifed NB ridge and Negative Binomial KL using the MSE. Also, the performance of the proposed estimator was compared with the existing ones through a Monte Carlo simulation study and two real life application datasets. The results from the simulation and real life application show that the proposed estimator (MJNBKL) outperformed the other estimators compared with by having the smallest MSE across all sample sizes (n) and for different levels of correlation (P) for the four biasing parameters used, which k3 has the optimal shrinkage parameter with the lowest MSE. Thus, this research can further be extended to other generalized linear models (GLMs) like logistics regression model, gamma regression model etc. to mitigate the problem in any of the regression models suggested.

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