Concept Paper

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Meaningfully Averaging Unbounded Functions

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Abstract: In this paper, we want to meaningfully average an "infinite collection of objects covering an infinite expanse of space". We illustrate this quote with an explicit n-dimensional function, where the graph of the function is dense in $\mathbb{R}^{n+1}$. The problem is no meaningful expected value of the function (e.g., w.r.t the Lebesgue or Hausdorff measure) has a finite value. In fact, "almost no" Borel measurable functions have a finite and meaningful average. In rigorous terms, suppose for $n \in \mathbb{N}$, set $A \subseteq \mathbb{R}^n$ and function $f : A \to \mathbb{R}$. If set A is Borel, B^* is the set of Borel measurable functions in $\mathbb{R}^A$ for all $A \subseteq \mathbb{R}^n$, and B^{**} is the set of all $f \in B^*$ with an finite expected value—w.r.t the Hausdorff measure—then B^{**} is a shy subset of B^*. To fix this issue, we wish to find an unique and "natural" extension of the expected value—w.r.t the Hausdorff measure—on bounded functions to unbounded/bounded f, which takes finite values only, so B^{**} is a non-shy subset of B^*. Note, we haven't found evidence suggesting mathematicians thought of this problem; however, it's assumed, in general, there's no meaningful way of averaging functions which cover an infinite expanse of space. Regardless, we'll attempt to solve the problem by defining a choice function—this shall choose a unique set of equivalent sequences of sets (i.e. $(F_k^{\star\star\star})_{k\in\mathbb{N}}$), where the set-theoretic limit of $F_k^{\star\star\star}$ is the graph of f; the measure H^h is the h-Hausdorff measure, where for each $k \in \mathbb{N}$, $0 < H^h(F_k^{\star\star\star}) < +\infty$; and $(f_k^{\star})_{k\in\mathbb{N}}$ is a sequence of functions, where: $\{(x_1, \cdots, x_n, f_k^{\star}(x_1, \cdots, x_n)) : (x_1, \cdots, x_n) \in \text{dom}(F_k^{\star\star\star})\} = F_k^{\star\star\star}$ .Thus, if $(F_k^{\star\star\star})$ converges to the graph of f at a rate linear or super-linear to the rate non-equivalent sequences of sets converge, and the extended expected value of f *or* $\mathbb{E}^{**}[f, F_k^{\star\star\star}]$ in the eq. below: $\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(k \in \mathbb{N})\left(k \geq N \Rightarrow \left|\frac{1}{H^h\left(\text{dom}(F_k^{\star\star\star})\right)}\int_{\text{dom}(F_k^{\star\star\star})} f_k^{\star}\, dH^h - \mathbb{E}^{**}[f, F_k^{\star\star\star}]\right| < \epsilon\right)$ exists, then $\mathbb{E}^{**}[f, F_k^{\star\star\star}]$ is an unique and "natural" extension of the original expected value on bounded f, so B^{**}—the set of all $f \in B^*$ with a finite $\mathbb{E}^{**}[f, F_k^{\star\star\star}]$—is a non-shy set in B^*. Note we guessed the choice function using computer programming: we redefine linear and super-linear convergence in terms of partitions of $F_k^{\star\star\star}$ with equal h-Hausdorff measure, a sample point from each partition, pathways between points in the sample, length of segments in the pathway (except for the lengths that are outliers), multiplying remaining lengths in the pathway by constant so they add up to one (i.e, a probability distribution), taking the entropy of the distribution, maximizing the entropy w.r.t all pathways, and creating limits—w.r.t the constant h-Hausdorff measure of the partitions and index of the sequences of sets—which redefines linear and super-linear convergence. Despite this, we don't know if the choice function solves the problem. (Infact, we're unable to prove most of the concepts in the paper: we require assistance for proving certain statements.) Even then, we'll visualize the paper using examples in this paper and examples in sec. 3 & 4 of "Mean of Unbounded Sets Using Conditional Expectation" [9]. The biggest use of this research is the extension of the expected value is unique and finite for a "non-negligible" amount of Borel measurable functions: this is easier to use in application when finding the "average" of functions covering an infinite expanse of space.

Keywords: expected value; Hausdorff measure; (Exact) dimension function; measurable functions; Function Space; Prevalent and Shy sets; entropy; choice function

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0. Introduction

According to an article in Quanta Magazine [3] Wood writes, "No known mathematical procedure can meaningfully average an infinite number of objects covering an infinite expanse of space in general. The path integral is more of a physics philosophy than an exact mathematical recipe." Note the quote can be illustrated by an explicit n-dimensional function where the graph of the function is dense in $\mathbb{R}^{n+1}$. Further note, a meaningful average for the function should be finite for the sake of application. The cited paper [5] presents a constructive approach to such a function using filters over families of

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finite set; however, the average in the approach is not unique. The method of the paper determines the average value for functions with a range that lies in any algebraic structure for which the finite averages make sense. In this paper, we will explore a more constructive approach where the average is unique, finite, and "meaningful" (§2.3 & §2.4) for such functions and also for "a sizable portion" of all functions in (i.e., a non-shy subset of [12]) the set of all Borel measurable functions. The reason we want this is explained in the next paragraph. (Note, the functions must be Borel measurable for application purposes).

Suppose for $n \in \mathbb{N}$, set $A \subseteq \mathbb{R}^n$ and the function $f : A \to \mathbb{R}$. If A is Borel, note the expected value of f, w.r.t the d-dimensional Hausdorff measure (i.e., d is the Hausdorff dimension of A), is undefined when the Hausdorff-dimension (of A)-dimensional Hausdorff measure (of also A) is either $+\infty$, zero or the function f is unbounded (def. 4, 5 & 8). Infact, if B^* is the set of all Borel measurable functions in $\mathbb{R}^A$ (for all the sets $A \subseteq \mathbb{R}^n$—i.e, def. 9) and B^{**} is the set of all $f \in B^*$ with a finite expected value, then B^{**} is a shy subset of B^*. This means "almost no" Borel measurable functions have a finite expected value. Specifically, within the definition of B^{**}—we want to manipulate the expected value of f to be finite—such that B^{**} is a non-shy (i.e., prevelant or neither prevelant nor shy [12]) subset of B^*.

This can be done by taking the expected value of a sequence of bounded functions which converge to bounded/unbounded f. A sequence of bounded functions is chosen using a "choice function", where criteria are added to determine the choice function. Note, we want the expected value from the chosen sequence of bounded functions to be unique and "natural" extension of the original expected value w.r.t Hausdorff measure on bounded f, taking finite values only.

We do this by defining a sequence of sets called *$\star$-sequence of sets* (def. 12), where the $\star$-sequences of sets converge to the graph of f rather than A. If not, the *generalized expected value* of f w.r.t to a $\star$-sequence (def. 13) cannot be finite for a non-shy subset of Borel measurable functions. Moreover, since there are graphs of functions with multiple $\star$-sequences of sets, s.t. the generalized expected values of f w.r.t each $\star$-sequence are different and non-unique (depending on the starred-sequence chosen)—we must have a choice function which chooses a unique set of equivalent $\star$-sequences with the same, unique expected value.

Therefore, when defining the choice function, we ask a question in §2.4 where with previous sections; we define equivalent & non-equivalent $\star$-sequences of sets for §2.1, and "natural" extensions of expected values for §2.3. We attempt to answer the question in §2.4 by redefining linear/super-linear convergence (def. 22) in terms of entropy, samples and "pathways" where the samples are derived by taking a point from each partition of a $\star$-sequence of sets, such the partitions have equal Hausdorff measure—§3. Since all samples have finite points; we take a "pathway" of line segments between the nearest point to each start-point of all segments in the pathway (i.e., the pathway should intersect every point once), where in def. 26 we *exclude* segments with lengths which are outliers [8]. The procedure is similar to the ones used in computers to graph functions [6]. We also take the length of each of the line segments in the "pathway", multiplying all lengths by a constant so they add up to one (i.e. a discrete probability distribution). We take the supremum of the Entropy of the distribution [10] w.r.t all "pathways" to redefine def. 22 as def. 27, where the redefined definition is used to create a choice function in §4.1.

1. Preliminary Definitons/Motivation

Other than integration with filters [5], there are few other constructive approaches to finding a unique and "natural" extension of the average that takes a finite value for additional functions. Before beginning, consider the following mathematical definitions:

1.1. Preliminary Definitions

Let X be a completely metrizable topological vector space.

Definition 1 (**Prevalent Subset of** X)**.** *A Borel set $E \subset X$ is said to be **prevalent** if there exists a Borel measure μ on X such that:*

(1) $0 < \mu(C) < \infty$ *for some compact subset C of X, and*
(2) *the set $E + x$ has full μ-measure (that is, the complement of $E + x$ has measure zero) for all $x \in X$.*

More generally, a subset F of X is prevalent if F contains a prevalent Borel Set. Also note:

Definition 2 (**Shy Subset of** X)**.** *The complement of a prevalent set is called a shy set.*

such that we define:

Definition 3 (**Non-Shy Subset of** X)**.** *A subset of X that is prevalent or neither prevalent nor shy.*

Furthermore, suppose we define:

Definition 4 (**Hausdorff Measure**)**.** *Let (V, d) be a metric space, $\alpha \in [0, \infty)$. For every $C \in V$, define the diameter of C as:*

$$diam(C) := \sup\{d(x, y) : x, y \in C\}, \quad diam(\emptyset) := 0$$

We define:

$$H_{\delta}^{\alpha}(E) = \inf\left\{\sum_{i=1}^{\infty}(diam(C_i))^{\alpha} : diam(C_i) \leq \delta, E \subseteq \bigcup_{i=1}^{\infty} C_i\right\}. \tag{1.1.1}$$

The Hausdorff Outer Measure is defined by

$$H^{\alpha}(E) = \sup_{\delta > 0} H_{\delta}^{\alpha}(E) = \lim_{\delta \to 0} H_{\delta}^{\alpha}(E)$$

If $i \in \mathbb{N}$ and $\delta \in \mathbb{R}$ such that $\delta > 0$, where the Euler's Gamma function is Γ and constant $\mathcal{N}_{\alpha}$ is:

$$\mathcal{N}_{\alpha} = \frac{\pi^{\alpha/2}}{2\Gamma\left(\frac{\alpha}{2} + 1\right)} \tag{1.1.2}$$

when $\alpha \in \mathbb{N}$ and E is a Borel set we have that

$$L^{\alpha}(E) = \frac{1}{2}\mathcal{N}_{\alpha} H^{\alpha}(E) \tag{1.1.3}$$

such that $H^{\alpha}(E)$ is related to the α-dimensional Lebesgue Measure.

Definition 5 (**Hausdorff Dimension**)**.** *The Hausdorff Dimension of E is defined by $dim_H(E)$ where:*

$$H^{\alpha}(E) = \begin{cases} \infty & \text{if } 0 \leq \alpha < dim_H(E) \\ 0 & \text{if } dim_H(E) < \alpha < \infty \end{cases} \tag{1.1.4}$$

Therefore, we can use definitions 1, 2, 4 to prove or disprove:

Theorem 6. *The set of Borel measurable, unbounded functions forms a prevalent subset of the set of all Borel measurable functions.*

Note 7 (**Notes on Theorem 6**)**.** *By measurable function, we mean the pre-image of any subset of $\mathbb{R}$ (under a measurable function) is in the Borel sigma algebra. (Note function f on set A is unbounded when there is no $I \geq 0$ such that for all $x \in A$):*

$$|f(x)| \leq I$$

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however, we're unsure if theorem 6 is correct. Despite this, we could prove or disprove theorem 6 using the paper on prevalence in [12].

If the theorem is true, it is easier to show a shy subset of Borel measurable, unbounded function have finite expected values (see the next definition). Hence, a shy subset of all Borel measurable function, including bounded Borel functions, have finite expected values.

We, therefore, define the expected value w.r.t the Hausdorff measure to be the following:

Definition 8 (Expected Value of f). *If $n \in \mathbb{N}$, where set $A \subseteq \mathbb{R}^n$, the expected value of function $f : A \to \mathbb{R}$ (using def. 4 and 5) is*

$$\mathbb{E}[f] = \frac{1}{H^{\dim_H(A)}(A)} \int_A f \, dH^{\dim_H(A)}$$

where we can see there are cases where $\mathbb{E}[f]$ is undefined or infinite (e.g. $H^{\dim_H(A)}(A)$ is zero, $+\infty$ or f is unbounded). In this case, if topological vector space X is $\mathbb{R}^A$ (see §1.1) where we define B^* such that:

Definition 9 (The set of all measurable functions). *B^* is the set of all Borel measurable functions in $\mathbb{R}^A$ for all the sets $A \subseteq \mathbb{R}^n$. This can also be described as:*

$$\bigcup_{\mathscr{A} \in 2^{\mathbb{R}^n}} \mathbb{R}^{\mathscr{A}}$$

Thus, we must prove:

Theorem 10. *If set $B^{**} \subseteq B^*$ is the set of all $f \in B^*$ (def. 9) with a finite $\mathbb{E}[f]$, then B^{**} is a shy subset of B^*.*

Note 11 (Note on Theorem 10). *We're not sure how to prove theorem 10; however, we refer to an answer from @Mathe at the last page of this citation [11],*

"We can follow the argument presented in example 3.6 of [12]:

Because a function can always be represented as $f = f^+ - f^-$ we only consider whether positive functions have a mean value. We consider the case of a set A with finite positive measure. In this context having a mean means having a finite integral, and not being integrable means having an infinite integral.

Take $X := L^0(A)$ (measurable functions over A) let P denote the one-dimensional subspace of $L^0(A)$ consisting of constant functions (assuming the Hausdorff measure on A) and let $F := L^0(A) \setminus L^1(A)$ (measurable functions over A with no finite integral)

If λ_P denotes the Lebesgue measure over P, for any fixed $f \in F$

$$\lambda_P\left(\left\{\beta \in \mathbb{R} : \int_A (f + \beta)\mu < \infty\right\}\right) = 0$$

Meaning P is a 1-dimensional probe of F, so F is a 1-prevalent set. (In other terms, the set of measurable functions over A with no finite integral or mean, forms a prevalent subset of the set of all measurable functions in $\mathbb{R}^A$. Therefore, using def. 2, the set of measurable functions with a finite integral or mean forms a shy subset of all Borel measurable functions in $\mathbb{R}^A$.)

1.2. Extended Expected Values

Here are four ways to extend $H^{\dim_H(A)}(A)$ (def. 4 & 5) in $\mathbb{E}[f]$ (def. 8), so when f is bounded, if B^{***} is the set of all $f \in B^*$ where extended expected values are finite, then set $B^{***} \supset B^{**}$ (i.e., thm. 10):

(1) Defining a **(exact) dimension function**; i.e., $h : [0, +\infty) \to [0, +\infty]$, that's monotonically increasing, strictly positive and right continuous, such that when D denotes the diameter of a ball in a

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covering for the definition of the Hausdorff Measure, we replace $D^{\dim_{\mathrm{H}}(A)}$ with $h(D)$ so $H^h(A)$: the h-**Hausdorff measure**, is positive and finite. This leads to the extended expected value $\mathbb{E}^{\star}[f]$, where:

$$\mathbb{E}^{\star}[f] = \frac{1}{H^h(A)} \int_A f \, dH^h$$

Note, however, not all A has dimension function h which leads to:

(2) If A is fractal but has no gauge function, we could use this paper [1] which is an extension of the Lebesgue density theorem and this paper [2] which is an extension of the Hausdorff measure using Hyperbolic Cantor sets. Note, however, when A is non-fractal (e.g. countably infinite) or f is unbounded, there is a possibility that the expected value is infinite or undefined. Hence,

(3) In the case f is unbounded and fractal, we could use [4, p.19-47], which applies a Henstock-Kurzweil type integral (i.e., μ-HK integral) on a measure Metric Space. This coincides with unbounded functions with finite improper Riemman integrals, including bounded functions with finite Lebesgue integrals, bounded function with finite integrals w.r.t the Hausdorff measure, or function with finite Henstock-Kurzweil integrals.

1.3. Examples

If $n \in \mathbb{N}$, set $A \subseteq \mathbb{R}^n$ and function $f : A \to \mathbb{R}$, we want to apply the definitions of the next section for the following examples:

(a) The most important example is any explicit, bijective $f : \mathbb{R}^n \to \mathbb{R}$ where the graph of f is dense in $\mathbb{R}^{n+1}$. (This is a nice example of "infinite number of objects covering an infinite expanse of space" described by Wood [3].) Note $\mathbb{E}[f]$ is undefined, since the Hausdorff/Lebesgue measure of the graph of f in each hyperrectangle (i.e., the n-dimensional analogue of rectangle) of $\mathbb{R}^{n+1}$ is zero [7]. In other words, using

$$\mathbb{E}[f] = \frac{1}{H^{\dim_{\mathrm{H}}(A)}(A)} \int_A f \, dH^{\dim_{\mathrm{H}}(A)}$$

$\mathbb{E}[f]$ is undefined because of division by zero (i.e, $1/(H^{\dim_{\mathrm{H}}(A)}(A)) = 0$).

Further, we assume using §1.2, crit. (1), there is no (exact) dimension function of A where $H^h(A)$ is positive & finite, since A is unbounded. Furthermore, the graph of f might be "too chaotic" for extensions of the Lebesgue Density Theorem [1], the Hausdorff measure using Hyperbolic Cantor Sets [2], or the Henstock-Kurzweil integral on the Metric Space [4, p.19-47].

(b) A simpler, more explicit example is $A = \mathbb{Q}$, gcd is the greatest common divisor, and $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ where:

$$f(x) = \begin{cases} f_1(x) & x \in A_1 := \{r/q : r \in \text{odd } \mathbb{Z}, q \in \text{even } \mathbb{Z}, q \neq 0, \gcd(r,q) = 1\} \\ f_2(x) & x \in A_2 := \{r_1/(q_1) : r_1 \in \mathbb{Z}, q_1 \in \text{odd } \mathbb{Z}, \gcd(r_1,q_1) = 1\} \end{cases} \tag{1.3.1}$$

For instance, point $(1/4, f_1(1/4))$ is a point in the graph of f (since $1/4 \in \mathbb{Q}$ and $1/4 \in A_1$, making $f(1/4) = f_1(1/4)$). Also, point $(1/3, f_2(1/3))$ is a point in the graph of f (since $1/3 \in \mathbb{Q}$ and $1/3 \in A_2$, making $f(1/3) = f_2(1/3)$); however, point $(\sqrt{2}, 1)$ is not in the graph of f (since $\sqrt{2} \notin \mathbb{Q}$).

Note the function in eq. 1.3.1 is bounded; however, the expected value & extensions are undefined. (Using def. 8, we know $\dim_{\mathrm{H}}(A) = 0$ but $H^{\dim_H(A)}(A) = +\infty$, which makes $\mathbb{E}[f]$:

$$\mathbb{E}[f] = \frac{1}{H^{\dim_{\mathrm{H}}(A)}(A)} \int_A f \, dH^{\dim_{\mathrm{H}}(A)}$$

undefined by division of $+\infty$.) Further, we assume using §1.2, crit. (1), there is no (exact) dimension function of A where $H^h(A)$ is finite. Worse, A isn't "fractal" enough for extensions of the Lebesgue Density Theorem [1], the Hausdorff measure using Hyperbolic Cantor Sets [2], or the Henstock-Kurzweil integral on the Metric Space [4, p.19-47].

(c) An extremely simple example is $A = \mathbb{R} \setminus \{0\}$ and $f(x) = 1/x$. This function is unbounded and has an undefined expected value, even with the improper Riemann integral since:

$$\lim_{(x_1,x_2,x_3,x_4)\to(-\infty,0^-,0^+,+\infty)} \frac{1}{(x_4-x_3)+(x_2-x_1)}\left(\int_{x_1}^{x_2}\frac{1}{x}\,dx+\int_{x_3}^{x_4}\frac{1}{x}\,dx\right) = \quad (1.3.2)$$

$$\lim_{(x_1,x_2,x_3,x_4)\to(-\infty,0^-,0^+,+\infty)} \frac{1}{(x_4-x_3)+(x_2-x_1)}\left(\ln(|x|)+C\Big|_{x_1}^{x_2}+\ln(|x|)+C\Big|_{x_3}^{x_4}\right) = \quad (1.3.3)$$

$$\lim_{(x_1,x_2,x_3,x_4)\to(-\infty,0^-,0^+,+\infty)} \frac{1}{(x_4-x_3)+(x_2-x_1)}(\ln(|x_2|)-\ln(|x_1|)+\ln(|x_4|)-\ln(|x_3|)) \quad (1.3.4)$$

is $+\infty$ (when $x_2 = 1/x_1$, $x_3 = 1/x_4$, and $x_1 = \exp(x_4^2)$) or $-\infty$ (when $x_2 = 1/x_1$, $x_3 = 1/x_4$, and $x_4 = -\exp(x_1^2)$), making the expected value undefined.

2. Attempt to Answer Thesis

Suppose for $n \in \mathbb{N}$, set $A \subseteq \mathbb{R}^n$ and function $f : A \to \mathbb{R}$. Moreover, H^h is the h-Hausdorff measure (§1.2, crit. (1)) where h is the dimension function, and B^* is the set of all Borel measurable functions in $\mathbb{R}^A$.

(Note for the definitions below, I prefer the generalized extensions of $\mathbb{E}[f]$ (def. 8) in §1.2, crit. (2) & (3). Unfortunately, I'm unsure how to describe most of these generalized measures. If possible, replace the h-Hausdorff measure with §1.2, crit. (2) or crit. (3))

Definition 12 **($\star$-Sequence of Sets).** *When we define a sequence of sets $(F_r^\star)_{r\in\mathbb{N}}$, where h is the dimension function (§1.2, crit. (1)), then if:*

(a) *The set theoretic limit of $(F_r^\star)_{r\in\mathbb{N}}$ is the graph of f (i.e., $(F_r^\star)_{r\in\mathbb{N}}$ **converges** to the graph of f) such that*

$$\limsup_{r\to\infty} F_r^\star = \bigcap_{r\geq 1}\bigcup_{q\geq r} F_q^\star$$
$$\liminf_{r\to\infty} F_r^\star = \bigcup_{r\geq 1}\bigcap_{q\geq r} F_q^\star$$

with the graph of f being:

$$\{(x_1,\cdots,x_n,f(x_1,\cdots,x_n)) : (x_1,\cdots,x_n)\in A\}$$

such that the set-theoretic limit of $(F_r^\star)_{r\in\mathbb{N}}$ should be:

$$\limsup_{r\to\infty} F_r^\star = \liminf_{r\to\infty} F_r^\star = \{(x_1,\cdots,x_n,f(x_1,\cdots,x_n)) : (x_1,\cdots,x_n)\in A\}$$

(b) *For all $r \in \mathbb{N}$, where H^h is the h-Hausdorff measure (§1.2, crit. (1)):*

$$0 < H^h(F_r^\star) < +\infty$$

(c) *we define sequence of functions $(f_r^\star)_{r\in\mathbb{N}}$ where $f_r^\star : \mathrm{dom}(F_r^\star) \to \mathrm{range}(F_r^\star)$ such that:*

$$\{(x_1,\cdots,x_n,f_r^\star(x_1,\cdots,x_n)) : (x_1,\cdots,x_n)\in \mathrm{dom}(F_r^\star)\} = F_r^\star$$

*we have $(F_r^\star)$ is a **$\star$-sequence of sets** or **starred-sequence of sets**.*

Example 12. *One $\star$-sequence of sets of $f(x) = 1/x$ on $\mathbb{R}\setminus\{0\}$ (§1.3, crit. (c)) is:*

$$(F_r^\star)_{r\in\mathbb{N}} = (\{(x,1/x) : x\in[-r,-1/r]\cup[1/r,r]\})_{r\in\mathbb{N}}$$

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Example 12. *Another example of a $\star$-sequence of sets of $f:\mathbb{Q}\to\mathbb{R}$ where:*

$$f(x)=\begin{cases}1 & x\in A_1:=\{r/q: r\in \textit{odd}\ \mathbb{Z}, q\in \textit{even}\ \mathbb{N}, q\neq 0, \gcd(r,q)=1\}\\ 0 & x\in A_2:=\{r_1/(q_1): r_1\in\mathbb{Z}, q_1\in \textit{odd}\ \mathbb{N}, \gcd(r_1,q_1)=1\}\end{cases} \tag{2.0.1}$$

using (§1.3, crit. (b)) is the following:

$$(F_r^\star)_{r\in\mathbb{N}}=((x,f(x)): x\in\{c/(r!): -r\cdot r!\le c\le r\cdot r!\})_{r\in\mathbb{N}} \tag{2.0.2}$$

another example is:

$$(F_r^\star)_{r\in\mathbb{N}}=((x,f(x)): x\in\{c/d: d\le r,\ -d\cdot r\le c\le d\cdot r\})_{r\in\mathbb{N}} \tag{2.0.3}$$

Note this leads to a new extension of the expected value where when set $B^{***}\subseteq B^*$ (def. 9) is the set of all $f\in B^*$, there exists *at least* one starred-sequence of sets (of the graph of f) s.t. the extended expected value of f is finite, B^{***} is a non-shy subset of B^*.

Definition 13 (Generalized Expected Value). *If $(F_r^\star)_{r\in\mathbb{N}}$ is a $\star$-sequence of sets (def. 12), the generalized expected value of f w.r.t $(F_r^\star)_{r\in\mathbb{N}}$ is $\mathbb{E}^{**}[f,F_r^\star]$ (when it exists) where:*

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(r\in\mathbb{N})\left(r\ge N\Rightarrow\left|\frac{1}{H^h(dom(F_r^\star))}\int_{dom(F_r^\star)}f_r^\star\,dH^h-\mathbb{E}^{**}[f,F_r^\star]\right|<\epsilon\right) \tag{2.0.4}$$

Example 13. *Using example 12, we find that when $(F_r^\star)_{r\in\mathbb{N}}=(\{(x,1/x): x\in[-r,-1/r]\cup[1/r,r]\})_{r\in\mathbb{N}}$:*

(a) $dom(F_r^\star)=([-r,-1/r]\cup[1/r,r])_{r\in\mathbb{N}}$
(b) $f_r(x)=1/x$ *for* $x\in[-r,-1/r]\cup[1/r,r]$

and the generalized expected value is:

$$\lim_{(x_1,x_2,x_3,x_4)\to(-\infty,0^-,0^+,+\infty)}\frac{1}{(x_4-x_3)+(x_2-x_1)}\left(\int_{x_1}^{x_2}\frac{1}{x}\,dx+\int_{x_3}^{x_4}\frac{1}{x}\,dx\right)= \tag{2.0.5}$$

$$\lim_{r\to\infty}\frac{1}{(r-1/r)+(-1/r-(-r))}\left(\int_{-r}^{-1/r}\frac{1}{x}\,dx+\int_{1/r}^{r}\frac{1}{x}\,dx\right)= \tag{2.0.6}$$

$$\lim_{r\to\infty}\frac{1}{(r-1/r)+(-1/r+r)}\left(\ln(|x|)+C\Big|_{-r}^{-1/r}+\ln(|x|)+C\Big|_{1/r}^{r}\right)= \tag{2.0.7}$$

$$\lim_{r\to\infty}\frac{1}{(r-1/r)+(-1/r+r)}(\ln(|-r|)-\ln(|-1/r|)+\ln(|r|)-\ln(|1/r|))= \tag{2.0.8}$$

$$\lim_{r\to\infty}\frac{1}{2r-2/r}\cdot 4\ln(r)= \tag{2.0.9}$$

$$0 \tag{2.0.10}$$

We can see from example 1.3 crit. (c), the average was once undefined but now we've "chosen" a $\star$-sequence which gives a finite expected value.

2.1. Equivalent and Non-Equivalent $\star$-Sequences of Sets

Suppose we define the following:

Definition 14 (Set V′). *Set V' is the set of all f, where the generalized expected value—w.r.t at least one starred sequence—exists.*

The following are definitions of equivelant and non-equivelant starred-sequences of sets:

Definition 15 (Non-Equivalent Starred-Sequences of Sets). *All starred-sequences of sets (in a set of $\star$-sequences of sets) are non-equivalent, if there exists an $f\in V'$ (def. 14), where the generalized expected values*

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of f (def. 13) w.r.t each starred-sequence of sets has two or more different values (e.g., defined and undefined values are different). See Figure 1.

Figure 1. Below $F_r^{\star}$, $F_k^{\star\star}$, $F_z^{\star\star\star}$ are *non-equivalent* starred sequences of sets, where V' is all circles and E^{**} is the generalized expected value of f w.r.t either $\star$-sequence of sets (def. 12)

Definition 16 (Equivalent Starred-Sequences of Sets). *All starred-sequences of sets (in the set of $\star$-sequences of sets) are equivalent, if we get for all $f \in V'$ (def. 14); the generalized expected value of f (def. 13) w.r.t each starred-sequence of sets has the same value. See Figure 2.*

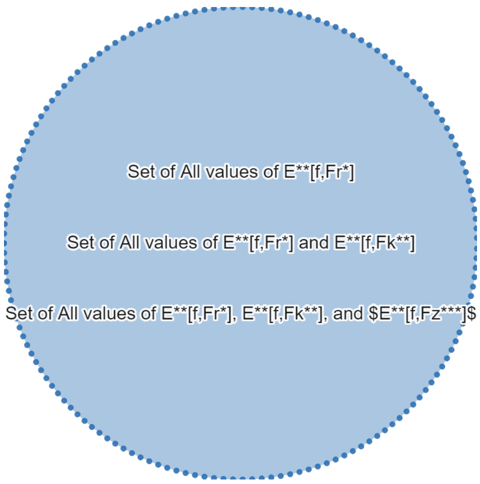

Figure 2. Below $F_r^{\star}$, $F_k^{\star\star}$, $F_z^{\star\star\star}$ are equivalent starred sequences of sets, where V' is the entire circle and E^{**} is the generalized expected value of f w.r.t either $\star$-sequence of sets (def. 12)

However, proving that two or more starred-sequences of sets are non-equivalent or equivalent (using def. 16 or 15) is tedious, since we constantly compute def. 13. Therefore, we ask the following:

2.1.1. Question 1

Is there are a simpler definition of equivalent and non-equivalent $\star$-sequences of sets?

2.1.2. Possible Answer

For the sake of brevity, suppose starred-sequences $F_{r_1}^{(1)} = F_{r_1}^{\star}$ (i.e., def. 12), such that $F_{r_2}^{(2)} = F_{r_2}^{\star\star}$, $F_{r_3}^{(3)} = F_{r_3}^{\star\star\star}$, and $F_{r_s}^{(s)} = F_{r_s}^{\star\text{-s times}}$

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Definition 17 (**Equivalent Starred-Sequences of Sets [Revisited]**). *Starred-sequence of sets* $(F^{\star}_{r_1})_{r_1\in\mathbb{N}}$ *and* $(F^{\star\star}_{r_2})_{r_2\in\mathbb{N}}$ *are equivalent, if there exists a* $N'\in\mathbb{N}$*, where for all* $r_1\geq N'$*, there exists a* $r_2\in\mathbb{N}$*, where if* h_1 *is the (exact) dimension function (§1.2, crit. (1)) of* $F^{\star}_{r_1}$ *and* H^{h_1} *is the* h_1*-Hausdorff measure:*

$$H^{h_1}(F^{\star}_{r_1}\Delta F^{\star\star}_{r_2})=0$$

and also for all $r_2\geq N'$*, there exists a* $r_1\in\mathbb{N}$*, where if* h_2 *is the (exact) dimension function of* $F^{\star\star}_{r_2}$ *and* H^{h_2} *is the* h_2*-Hausdorff measure (§1.2, crit. (1)) then:*

$$H^{h_2}(F^{\star}_{r_1}\Delta F^{\star\star}_{r_2})=0$$

Note we denote these equivalent starred-sequence of sets as

$$(F^{\star}_{r_1})_{r_1\in\mathbb{N}}\sim(F^{\star\star}_{r_2})_{r_2\in\mathbb{N}}$$

Definition 18 (**Multiple Equivalent Starred-Sequences of Sets [Revisited]**). *All starred-sequences of sets in:*

$$\left\{(F^{\star}_{r_1})_{r_1\in\mathbb{N}},(F^{\star\star}_{r_2})_{r_1\in\mathbb{N}},\cdots,(F^{(j)}_{r_j})_{r_1\in\mathbb{N}}\right\}$$

are equivalent, if for all $k,v\in\{1,\cdots,j\}$ *where* $k\neq v$*,* $(F^{(k)}_{r_k})_{r_k\in\mathbb{N}}$ *and* $(F^{(v)}_{r_v})_{r_v\in\mathbb{N}}$ *are equivelant (def. 17). We also state the former as:*

$$(F^{(k)}_{r_k})_{r_k\in\mathbb{N}}\sim(F^{(v)}_{r_v})_{r_v\in\mathbb{N}}$$

Theorem 19. *If starred-sequences of sets in:*

$$\left\{(F^{\star}_{r_1})_{r_1\in\mathbb{N}},(F^{\star\star}_{r_2})_{r_1\in\mathbb{N}},\cdots,(F^{(j)}_{r_j})_{r_1\in\mathbb{N}}\right\}$$

are equivalent (def. 18), then for all $k,v\in\{1,\cdots,j\}$ *where* $k\neq v$*, the generalized means of* f *w.r.t the* $\star$*-sequences (def. 13) have the same mean value. In other words:*

$$\mathbb{E}^{**}[f,F^{(k)}_{r_k}]=\mathbb{E}^{**}[f,F^{(v)}_{r_v}]$$

Note this is similar to def. 16.

Definition 20 (**Non-Equivalent Starred-Sequences of Sets [Revisited]**). *All starred-sequences of sets in*

$$\left\{(F^{\star}_{r_1})_{r_1\in\mathbb{N}},(F^{\star\star}_{r_2})_{r_1\in\mathbb{N}},\cdots,(F^{(j)}_{r_j})_{r_1\in\mathbb{N}}\right\}$$

are non-equivalent, if def. 18 is false, meaning for all $k,v\in\{1,\cdots,j\}$ *where* $k\neq v$*, there exists an* $N'\in\mathbb{N}$*, where for all* $k\geq N'$ *there is either a* $v\in\mathbb{N}$*, where if* h_k *is the (exact) dimension function (§1.2, crit. (1)) of* $F^{(k)}_{r_k}$*, and* H^{h_k} *is the* h_k*-Hausdorff measure:*

$$H^{h_k}(F^{(k)}_{r_k}\Delta F^{(v)}_{r_v})\neq 0$$

for all $v\geq N'$ *there exists a* $k\in\mathbb{N}$*, where if* h_v *is the (exact) dimension function of* $F^{(v)}_{r_v}$*, and* H^{h_v} *is the* h_v*-Hausdorff measure (§1.2, crit. (1)) then:*

$$H^{h_v}(F^{(k)}_{r_k}\Delta F^{(v)}_{r_v})\neq 0$$

2.2. Motivation For Sec. 2.4

If the set $B^{***}\subseteq B^{*}$ (def. 9) is the set of all $f\in B^{*}$ where we choose a $\star$-sequence of sets of the graph of f (def. 12)—where the generalized expected value of f, w.r.t the chosen starred-sequence, is finite—then B^{***} is a non-shy subset of B^{*}. However, consider the following problem:

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Theorem 21. *If set $B^{***} \subseteq B^*$, is the set of all $f \in B^*$ where the generalized expected values of f w.r.t two or more non-equivalent $\star$-sequences of sets (def. 20) have different values, then B^{***} is a non-shy subset of B^* (def. 9).*

This means "almost all" measurable functions have *several* generalized expected values depending on the starred-sequence chosen. Therefore, we need to choose a unique $\star$-sequence of sets where the new extended expected value is an "meaningful" extension of $\mathbb{E}[f]$ (def. 8).

2.3. Essential Definitions for a "Meaningful" Expected Value

Suppose $(F_r^\star)_{r\in\mathbb{N}}$ and $(F_j^{\star\star})_{j\in\mathbb{N}}$ are non-equivelant starred-sequences of sets (def. 12 & 20): we have the following is essential for a "natural" extension of the expected value.

Definition 22 (Linear & Super-linear Convergence of a $\star$-Sequence of Sets To That Of Another $\star$-Sequence of Sets). *If we define function $S : \mathbb{R} \to \mathbb{R}$, where $r \in \mathbb{N}$ and for any linear $j_1 : \mathbb{N} \to \mathbb{N}$, where $j = j_1(r)$, $\mathcal{O}$ is the Big-O notation, and:*

$$H^h(F_r^\star) = \mathcal{O}(S(H^h(F_j^{\star\star})))$$

where if the following is true:

$$0 < \lim_{x\to\infty} S(x)/x$$

then $(F_r^\star)_{r\in\mathbb{N}}$ converges to the graph of f: i.e.,

$$\{(x_1, \cdots, x_n, f(x_1, \cdots, x_n)) : (x_1, \cdots, x_n) \in A\}$$

*at a **linear** or **super-linear** rate compared to that of $(F_j^{\star\star})_{j\in\mathbb{N}}$.*

Now we may combine the previous definitions into a main question with an answer that solves the thesis [1].

2.4. Main Question

Does there exist a choice function that chooses a unique set (of equivalent $\star$-sequences of sets—def. 18) such that:

(a) The chosen starred-sequences of sets converge to $\{(x_1, \cdots, x_n, f(x_1, \cdots, x_n)) : (x_1, \cdots, x_n) \in A\}$ at a rate *linear* or *super-linear* (def. 22) to the rate non-equivalent $\star$-sequences of sets (def. 20) converge to $\{(x_1, \cdots, x_n, f(x_1, \cdots, x_n)) : (x_1, \cdots, x_n) \in A\}$
(b) The *generalized expected value* (def. 13) of f w.r.t the chosen (and equivalent) starred-sequences of sets (def. 18) is finite.
(c) When set $Q \subseteq B^*$ (def. 9) is the set of all $f \in B^*$ such that the choice function chooses a unique set of equivalent $\star$-sequences of sets satisfying (1) and (2), then Q is a non-shy subset (def. 5) of B^* (i.e., def. 9).
(d) Out of all the choice functions which satisfy (1), (2) and (3), we choose the one with the *simplest form*, meaning for each choice function fully expanded, we take the one with the fewest variables/numbers (excluding those with quantifiers)?

[1] If the set $B^{**} \subseteq B^*$ (def. 9) is the set of all $f \in B^*$, with an unique and "meaningful" extension of the expected value—w.r.t the Hausdorff measure—on bounded functions to bounded/unbounded f taking finite values, then B^{**} should be non-shy subset of B^*

Note 23 (Notes On Question). *Note, the unique set of equivalent and chosen starred-sequences of sets is defined using notation* $\sim (F_k^{\star\star\star})_{k\in\mathbb{N}}$*, where* $(F_k^{\star\star\star})_{k\in\mathbb{N}}$ *is a starred-sequence in* $\sim (F_k^{\star\star\star})_{k\in\mathbb{N}}$*. Therefore, after we define the choice function, the answer should be* $\mathbb{E}^{**}[f, F_k^{\star\star\star}]$*—using def. 13 (when it exists):*

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(k \in \mathbb{N})\left(k \geq N \Rightarrow \left|\frac{1}{H^h(dom(F_k^{\star\star\star}))}\int_{dom(F_k^{\star\star\star})} f_k^{\star}\, dH^h - \mathbb{E}^{**}[f, F_k^{\star\star\star}]\right| < \epsilon\right) \tag{2.4.1}$$

Also, consider three things:

(a) *If the solution to the main question is extraneous, what other criteria can be included to get a unique choice function? (Note if the solution is always extraneous, we want to replace "equivelant starred-sequences of sets" with the following: "the set of all ⋆-sequences of sets, where the generalized expected values of f w.r.t each starred-sequence is the same".)*
(b) *The h-Hausdorff measure (§1.2, crit. (1)) isn't the most generalized measure in §1.2. How do we use the other measures in §1.2 to answer the thesis[2] of this paper?*
(c) *How do we change the definitions and main question in §2 of this paper to counter Wood's statement [3] which states, "No known mathematical procedure can meaningfully average an infinite number of objects covering an infinite expanse of space in general" by finding an exact mathematical recipe that does otherwise.*

3. Preliminaries to Solve Main Question of Section 2.4 (In Current Form)

Suppose h is the dimension function, H^h is the h-Hausdorff measure (§1.2, crit. (1)), and $(F_r^\star)_{r\in\mathbb{N}}$ is the starred-sequence of sets (def. 12). We will use an alternative approach to definition 22 so we can define a choice function which solves the main question. Read from the second sentence of the last paragraph of the intro of §0 for a summary. Also, refer to sec. 3 and 4 of [9] for examples: (the cited paper uses sets instead of the graphs of functions).

While reading, keep in mind the following questions:

(a) How do we use mathematica code to illustrate §3 and 4?
(b) Is there a more efficient solution to §2.4?
(c) If §2.4 should be changed, (see note 23) what else should be §2.4? What is the most efficient solution to the improved version of §2.4?

3.1. Preliminary Definitions

Definition 24 (Uniform ε coverings of each term of a ⋆-sequence of sets). *We define uniform ε coverings of each term of* $(F_r^\star)_{r\in\mathbb{N}}$ *as a group of pair-wise disjoint sets which cover* $F_r^\star$ *(for some* $r \in \mathbb{N}$*), such when taking dimension function h of* $F_r^\star$*, we want* H^h *of each pair-wise disjoint set to have the same value* $\varepsilon \in range(H^h)$*, where* $\varepsilon > 0$ *and the total sum of* H^h *of the coverings is minimized. In shorter notation, if*

- *The element* $t \in \mathbb{N}$
- *The set* $T \supset \mathbb{N}$ *is arbitrary and uncountable.*

and set Ω *is defined as:*

$$\Omega = \begin{cases} \{1, \cdots, t\} & \text{if there are } t \text{ ways of writing uniform } \varepsilon \text{ coverings of } F_r^\star \\ \mathbb{N} & \text{if there are countably infinite ways of writing uniform } \varepsilon \text{ coverings of } F_r^\star \\ T & \text{if there are uncountable ways of writing uniform } \varepsilon \text{ coverings of } F_r^\star \end{cases} \tag{3.1.1}$$

then for every $\omega \in \Omega$*, the set of uniform ε coverings is defined using* $\mathscr{U}(\varepsilon, F_r^\star, \omega)$ *where ω "enumerates" all possible uniform ε coverings of* $F_r^\star$ *for every* $r \in \mathbb{N}$*.*

---

[2] If the set $B^{**} \subseteq B^*$ (def. 9) is the set of all $f \in B^*$, with an unique and "meaningful" extension of the expected value—w.r.t the Hausdorff measure—on bounded functions to bounded/unbounded f taking finite values, then B^{**} should be non-shy subset of B^*

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Definition 25 (Sample of the uniform ε coverings of each term of a $\star$-sequence of sets). *The sample of uniform ε coverings of each term of $(F_r^\star)_{r\in\mathbb{N}}$ is the set of points where for every $\varepsilon \in \text{range}(H^h)$ and $r \in \mathbb{N}$, we take a point from each pair-wise disjoint set in the uniform ε coverings of $F_r^\star$ (def. 24). In shorter notation, if*

- *The element $k \in \mathbb{N}$*
- *The set $\mathcal{K} \supset \mathbb{N}$ is arbitrary and uncountable.*

and set Ψ_ω is defined as:

$$\Psi_\omega = \begin{cases} \{1,\cdots,k\} & \text{if there are } k \text{ ways of writing the sample of uniform } \varepsilon \text{ coverings of } F_r^\star \\ \mathbb{N} & \text{if there are countably infinite ways of writing the sample of uniform } \varepsilon \text{ coverings of } F_r^\star \\ \mathcal{K} & \text{if there are uncountable ways of writing the sample of uniform } \varepsilon \text{ coverings of } F_r^\star \end{cases} \tag{3.1.2}$$

then for every $\psi \in \Psi_\omega$, the set of all samples of the set of uniform ε coverings is defined using $\mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi)$, such that ψ "enumerates" all possible samples of $\mathscr{U}(\varepsilon, F_r^\star, \omega)$.

Definition 26 (Entropy on the sample of uniform coverings of each term of $\star$-sequence of sets). *Since there are finitely many points in the sample of the uniform ε coverings of each term of $(F_r^\star)_{r\in\mathbb{N}}$ (def. 25), we:*

(a) *Take a "pathway" of line segments between all points in each sample (def. 25), such that if we define the following:*

  (i) *$\lceil\cdot\rceil$ is the ceiling function*
  (ii) *$d(Q,R)$ is the Euclidean-distance between points $Q \in \mathbb{R}^n$ and $R \in \mathbb{R}^n$*
  (iii) *The sequence:*

$$\{x_{i-1}\}_{i=1}^{\lceil H^h(F_r^\star)/\varepsilon\rceil - 1}$$

  contains all points in the "original" sample $\mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi)$ where we define a "pathway" for which we:

    (A) *Choose a point $x_0 \in \mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi)$*
    (B) *Take a point from $\mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi)$ (excluding x_0) with smallest euclidean distance from point $x_0 \in \mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi)$. We denote this point x_1 where we take $d(x_0, x_1)$. (If more than one point has the smallest Euclidean distance from x_0, we take either point).*
    (C) *Take a point in $\mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi)$ (excluding x_0 and x_1) with smallest euclidean distance from x_1. We denote this point x_2, where we take $d(x_1, x_2)$. (If more than one point has the smallest Euclidean distance from x_1, we take either point).*
    (D) *Take a point in $\mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi)$ (excluding x_0, x_1, and x_2) with smallest euclidean distance from x_2. We denote this point x_3 then take $d(x_2, x_3)$. (If more than one point has the smallest Euclidean distance from x_2, we take either point).*
    (E) *Repeat the process excluding points x_0, x_1, x_2, x_3, etc. until all points in the sample are "denoted". (This should occur $\lceil H^h(F_r^\star)/\varepsilon\rceil - 1$ times.)*

  (iv) *$\mathbf{V}$ is a subset of $\left\{i \in \mathbb{N} : 1 \le i \le \lceil H^h(F_r^*)/\varepsilon\rceil - 1\right\}$ with the largest cardinality, where we take the subset of i-values where x_i has the r_i-th smallest Euclidean distance from x_{i-1} (compared to every point in $\mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi) \setminus \{x_{i-1}\}$) such that r_i is not an outlier [8] of*

$$\left\{r_t : t \in \mathbb{N}, 1 \le t \le \lceil H^h(F_r^\star)/\varepsilon\rceil - 1\right\}$$

  In other words:

    i. *For all $w \in \mathbf{V}$, we want $\mathbf{V}$ to be the largest subset of $\left\{i \in \mathbb{N} : 1 \le i \le \lceil H^h(F_r^*)/\varepsilon\rceil - 1\right\}$ for which w-values are all i-values satisfying def. 26, criteria (iv).*

  (v) *Combining everything in def. 26, crit. (a), we ultimately want all lengths between every point in the "pathway" (def. 25) satisfying def. 26, crit. (iv). We call this:*

$$\mathcal{D}(x_0, \{x_{w-1}\}_{w\in\mathbf{V}}, \mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi)) = \{d(x_w, x_{w-1}) : w \in \mathbf{V}\}$$

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(b) *Using def. 26, crit. (v), normalize $\mathcal{D}$ into a discrete probability distribution. This is defined as:*

$$\mathbb{P}(\mathcal{D}(x_0, \{x_{w-1}\}_{w\in\mathbf{V}}, \mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi))) = \left\{ y \middle/ \left(\sum_{z\in\mathcal{D}(x_0,\{x_{w-1}\}_{w\in\mathbf{V}},\mathcal{S}(\mathscr{U}(\varepsilon,F_r^\star,\omega),\psi))} z \right) : y \in \mathcal{D}(x_0, \{x_{w-1}\}_{w\in\mathbf{V}}, \mathcal{S}(\mathscr{U}(\epsilon, F_r^\star, \omega), \psi)) \right\} \tag{3.1.3}$$

(c) *Take the entropy of def. 26, crit. (b), (for further reading, see [10, p.61-95]). This is defined as:*

$$E(\mathcal{D}(x_0, \{x_{w-1}\}_{w\in\mathbf{V}}, \mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi))) = -\sum_{x\in\mathbb{P}(\mathcal{D}(x_0,\{x_{w-1}\}_{w\in\mathbf{V}},\mathcal{S}(\mathscr{U}(\varepsilon,F_r^\star,\omega),\psi)))} x \log_2 x \tag{3.1.4}$$

(d) *Take $x_0 \in \mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi))$ where $E(\mathcal{D}(x_0, \{x_{w-1}\}_{w\in\mathbf{V}}, \mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi)))$ is maximized. Call this, $E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi)))$ where:*

$$E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi))) = \sup_{x_0\in\mathcal{S}(\mathscr{U}(\varepsilon,F_r^\star,\omega),\psi)} E(\mathcal{D}(x_0, \{x_{w-1}\}_{w\in\mathbf{V}}, \mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi))) \tag{3.1.5}$$

with eq. 3.1.5 the entropy of the sample of uniform ε coverings of $F_r^\star$.

Definition 27 (Starred-Sequence of sets converging Sublinearly, Linearly, or Superlinearly to A compared to that of another $\star$-Sequence). *Suppose we define starred-sequences of sets $(F_r^\star)_{r\in\mathbb{N}}$ and $(F_j^{\star\star})_{j\in\mathbb{N}}$, where for a **constant** $\varepsilon \in range(H^h)$ greater than zero and variable $r \in \mathbb{N}$, we say:*

(a) *Using def. 25 and 26, suppose we have:*

$$\underline{|\mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi)|} = \sup\left\{ |\mathcal{S}(\mathscr{U}(\varepsilon, F_j^{\star\star}, \omega'), \psi')| : j \in \mathbb{N},\ \omega' \in \Omega,\ \psi' \in \Psi_\omega,\ E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon, F_j^{\star\star}, \omega'), \psi'))) \le E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi))) \right\} \tag{3.1.6}$$

then (using $\underline{|\mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi)|}$) we get

$$\overline{\alpha}(\varepsilon, r, \omega, \psi) = |\mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi))| / \sup_{\omega\in\Omega} \sup_{\psi\in\Psi_\omega} \underline{|\mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi)|} \tag{3.1.7}$$

(b) *From def. 25 and 26, suppose we have:*

$$\overline{|\mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi)|} = \inf\left\{ |\mathcal{S}(\mathscr{U}(\varepsilon, F_j^{\star\star}, \omega'), \psi')| : j \in \mathbb{N},\ \omega' \in \Omega,\ \psi' \in \Psi_\omega,\ E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon, F_j^{\star\star}, \omega'), \psi'))) \ge E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi))) \right\} \tag{3.1.8}$$

then (using $\overline{|\mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi)|}$) we have:

$$\underline{\alpha}(\varepsilon, r, \omega, \psi) = |\mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi)| / \sup_{\omega\in\Omega} \sup_{\psi\in\Psi_\omega} \overline{|\mathcal{S}(\mathscr{U}(\varepsilon, F_r^\star, \omega), \psi)|} \tag{3.1.9}$$

(1) *If using $\overline{\alpha}(\epsilon, r, \omega, \psi)$ and $\underline{\alpha}(\epsilon, r, \omega, \psi)$ we have that:*

$$\sup_{\omega\in\Omega} \sup_{\psi\in\Psi_\omega} \limsup_{\varepsilon\to 0} \limsup_{r\to\infty} \overline{\alpha}(\varepsilon, r, \omega, \psi) = \inf_{\omega\in\Omega} \inf_{\psi\in\Psi_\omega} \liminf_{\varepsilon\to 0} \liminf_{r\to\infty} \underline{\alpha}(\varepsilon, r, \omega, \psi) = 0$$

*we say $(F_r^\star)_{r\in\mathbb{N}}$ converges to A at a rate **superlinear** to that of $(F_j^{\star\star})_{j\in\mathbb{N}}$.*

(2) *If using equations $\overline{\alpha}(\varepsilon, j, \omega, \psi)$ and $\underline{\alpha}(\varepsilon, j, \omega, \psi)$ (where we swap $(F_r^\star)_{r\in\mathbb{N}}$ in $\overline{\alpha}(\epsilon, r, \omega, \psi)$ and $\underline{\alpha}(\epsilon, r, \omega, \psi)$ with $(F_j^{\star\star})_{j\in\mathbb{N}}$) we have that:*

$$\sup_{\omega\in\Omega} \sup_{\psi\in\Psi_\omega} \limsup_{\varepsilon\to 0} \limsup_{j\to\infty} \overline{\alpha}(\varepsilon, j, \omega, \psi) = \inf_{\omega\in\Omega} \inf_{\psi\in\Psi_\omega} \liminf_{\varepsilon\to 0} \liminf_{j\to\infty} \underline{\alpha}(\varepsilon, j, \omega, \psi) = 0$$

*we then say $(F_r^\star)_{r\in\mathbb{N}}$ converges to A at a rate **sublinear** to that of $(F_j^{\star\star})_{j\in\mathbb{N}}$.*

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(3) *If using equations* $\overline{\alpha}(\varepsilon, r, \omega, \psi)$, $\underline{\alpha}(\varepsilon, r, \omega, \psi)$, $\overline{\alpha}(\varepsilon, j, \omega, \psi)$, *and* $\underline{\alpha}(\varepsilon, j, \omega, \psi)$ *(such for the two latter, we swap* $(F_r^{\star})_{r\in\mathbb{N}}$ *in* $\overline{\alpha}(\varepsilon, r, \omega, \psi)$ *and* $\underline{\alpha}(\varepsilon, r, \omega, \psi)$ *with* $(F_j^{\star\star})_{j\in\mathbb{N}}$*) we have* ***both****:*

   (a) $\sup\limits_{\omega\in\Omega}\sup\limits_{\psi\in\Psi_\omega}\limsup\limits_{\varepsilon\to 0}\limsup\limits_{r\to\infty}\overline{\alpha}(\varepsilon, r, \omega, \psi)$ *or* $\inf\limits_{\omega\in\Omega}\inf\limits_{\psi\in\Psi_\omega}\liminf\limits_{\varepsilon\to 0}\liminf\limits_{r\to\infty}\underline{\alpha}(\varepsilon, r, \omega, \psi)$ *does not equal zero*

   (b) $\sup\limits_{\omega\in\Omega}\sup\limits_{\psi\in\Psi_\omega}\limsup\limits_{\varepsilon\to 0}\limsup\limits_{j\to\infty}\overline{\alpha}(\varepsilon, j, \omega, \psi)$ *or* $\inf\limits_{\omega\in\Omega}\inf\limits_{\psi\in\Psi_\omega}\liminf\limits_{\varepsilon\to 0}\liminf\limits_{j\to\infty}\underline{\alpha}(\varepsilon, j, \omega, \psi)$ *does not equal zero*

and say $(F_r^{\star})_{r\in\mathbb{N}}$ *converges to A at a rate* ***linear*** *to that of* $(F_j^{\star\star})_{j\in\mathbb{N}}$.

4. Attempt to Answer Main Question Of Section 2.4 (In Current Form)

4.1. Choice Function

Suppose we define the following:

(1) $(F_k^{\star\star\star})_{k\in\mathbb{N}}$ is a starred-sequence of sets (def. 12) which satisfies (1), (2), and (3) of the main question in §2.4

(2) $\mathbb{S}'(G)$, where G is the graph of f; i.e.,

$$G = \{(x_1, \cdots, x_n, f(x_1, \cdots, x_n)) : (x_1, \cdots, x_n) \in A\}$$

is the set of the starred-sequences of sets that have finite *generalized mean* (def. 13).

(3) $(F_j^{\star\star})_{j\in\mathbb{N}}$ is an element $\mathbb{S}'(G)$ but **not** an element in the set of equivalent starred-sequences of sets (def. 18) of $(F_k^{\star\star\star})_{k\in\mathbb{N}}$ where using note 23, we can represent this criteria as:

$$(F_j^{\star\star})_{j\in\mathbb{N}} \in \mathbb{S}'(G) \setminus \sim (F_k^{\star\star\star})_{k\in\mathbb{N}} \tag{4.1.1}$$

Further note, from def. 27, if we take:

$$\overline{|\mathcal{S}(\mathscr{U}(\varepsilon, F_k^{\star\star\star}, \omega), \psi)|} = \tag{4.1.2}$$
$$\inf\left\{|\mathcal{S}(\mathscr{U}(\varepsilon, F_j^{\star\star}, \omega'), \psi')| : j \in \mathbb{N},\ \omega' \in \Omega,\ \psi' \in \Psi_\omega,\ E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon, F_j^{\star\star}, \omega'), \psi'))) \geq E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon, F_k^{\star\star\star}, \omega), \psi)))\right\}$$

and from def. 27, we take:

$$\underline{|\mathcal{S}(\mathscr{U}(\varepsilon, F_k^{\star\star\star}, \omega), \psi)|} = \tag{4.1.3}$$
$$\sup\left\{|\mathcal{S}(\mathscr{U}(\varepsilon, F_j^{\star\star}, \omega'), \psi')| : j \in \mathbb{N},\ \omega' \in \Omega,\ \psi' \in \Psi_\omega,\ E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon, F_j^{\star\star}, \omega'), \psi'))) \leq E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon, F_k^{\star\star\star}, \omega), \psi)))\right\}$$

Then, when we write def. 25, eq. 4.1.2 and eq. 4.1.3 as:

$$\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}|\mathcal{S}(\mathscr{U}(\varepsilon, F_k^{\star\star\star}, \omega), \psi)| = |\mathcal{S}'(\varepsilon, F_k^{\star\star\star})| = |\mathcal{S}'| \tag{4.1.4}$$

$$\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\overline{|\mathcal{S}(\mathscr{U}(\varepsilon, F_k^{\star\star\star}, \omega), \psi)|} = \overline{|\mathcal{S}'(\varepsilon, F_k^{\star\star\star})|} = \overline{|\mathcal{S}'|} \tag{4.1.5}$$

$$\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\underline{|\mathcal{S}(\mathscr{U}(\varepsilon, F_k^{\star\star\star}, \omega), \psi)|} = \underline{|\mathcal{S}'(\varepsilon, F_k^{\star\star\star})|} = \underline{|\mathcal{S}'|} \tag{4.1.6}$$

the choice function (which we'll later define on pg. 16, thm. 28) should immediately choose $F_k^{\star\star\star}$ when:

(1) For all $m \in \{1, \cdots, n\}$ when defining the set of all values of the m-th coordinate of $(c_1, c_2, \cdots, c_{n+1}) \in F_k^{\star\star\star}$ (i.e., $F_{k,m}^{\star\star\star}$—where, unlike cit. [9, §4], we focus on the domain of $F_k^{\star\star\star}$ to get "n" instead of "$n+1$"), then when $z > 0$, we either want:

   (a) $\sup(F_{k+1,m}^{\star\star\star}) - \sup(F_{k,m}^{\star\star\star}) = z$ and $\inf(F_{k+1,m}^{\star\star\star}) - \inf(F_{k,m}^{\star\star\star}) = -z$.

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(b) $\sup(F^{\star\star\star}_{k+1,m}) - \sup(F^{\star\star\star}_{k,m}) = 0$ and $\inf(F^{\star\star\star}_{k+1,m}) - \inf(F^{\star\star\star}_{k,m}) = -z$.
(c) $\sup(F^{\star\star\star}_{k+1,m}) - \sup(F^{\star\star\star}_{k,m}) = z$ and $\inf(F^{\star\star\star}_{k+1,m}) - \inf(F^{\star\star\star}_{k,m}) = 0$.
(d) $\sup(F^{\star\star\star}_{k+1,m}) - \sup(F^{\star\star\star}_{k,m}) = 0$ and $\inf(F^{\star\star\star}_{k+1,m}) - \inf(F^{\star\star\star}_{k,m}) = 0$.

(2) If the center of the universe is a chosen point $Z \in \mathbb{R}^{n+1}$, where:

$$Z = (z_1, z_2, \cdots, z_{n+1}) \tag{4.1.7}$$

then for all $m \in \{1, \cdots, n\}$, there exists $q \in \mathbb{N}$, s.t. for all $k \geq q$, when set $F^{\star\star\star}_{k,m}$ is a collection of all the values of the m-th co-ordinate of $(c_1, c_2, \cdots, c_{n+1}) \in F^{\star\star\star}_k$, such that $x_1 \in F^{\star\star\star}_{k,m}$ (again, unlike cit. [9, §4], we focus on the domain of $F^{\star\star\star}_k$ to get "n" instead of "$n+1$"), we must get:

$$\frac{1}{H^h(F^{\star\star\star}_{k,m})} \int_{F^{\star\star\star}_{k,m}} x_1 \, dH^h = z_m \tag{4.1.8}$$

where, using absolute value function $||\cdot||$ and $m \in \{1, 2, \cdots, n\}$, when set $F^{\star\star\star}_{k,m}$ is a collection of all the values of the m-th co-ordinate of $(c_1, c_2, \cdots, c_{n+1}) \in F^{\star\star\star}_k$, for $z > 0$, when we define:

$$\begin{aligned} S(z,k,m) = \Bigg|\Bigg| z - \left(\sup(F^{\star\star\star}_{k+1,m}) - \sup(F^{\star\star\star}_{k,m})\right)\left(\inf(F^{\star\star\star}_{k,m}) - \inf(F^{\star\star\star}_{k+1,m})\right) \\ \Big|\Big|\left(\inf(F^{\star\star\star}_{k,m}) - \inf(F^{\star\star\star}_{k+1,m})\right)\left(\sup(F^{\star\star\star}_{k+1,m}) - \sup(F^{\star\star\star}_{k,m}) - 1\right)\Big|\Big| \Bigg|\Bigg| \end{aligned} \tag{4.1.9}$$

and

$$\begin{aligned} T(z_m,k,m) = &\Big(\left(\sup(F^{\star\star\star}_{k+1,m}) - z_m\right)\left(\inf(F^{\star\star\star}_{k,m}) - z_m\right) - \left(\sup(F^{\star\star\star}_{k,m}) - z_m\right)\left(\inf(F^{\star\star\star}_{k+1,m}) - z_m\right)\Big) \\ &\Big(\left(\inf(F^{\star\star\star}_{k,m}) - z_m\right) - \left(\inf(F^{\star\star\star}_{k+1,m}) - z_m\right) + \left(\sup(F^{\star\star\star}_{k+1,m}) - z_m\right) - \left(\sup(F^{\star\star\star}_{k,m}) - z_m\right) - 1\Big) \\ &\Big(\left(\inf(F^{\star\star\star}_{k,m}) - z_m\right) - \left(\inf(F^{\star\star\star}_{k+1,m}) - z_m\right)\Big)\Big(\left(\sup(F^{\star\star\star}_{k+1,m}) - z_m\right) - \left(\sup(F^{\star\star\star}_{k,m}) - z_m\right)\Big) \end{aligned} \tag{4.1.10}$$

criteria ((1)) is achieved, using eq. 4.1.9, when:

$$S'(z,k) = \frac{1}{n} \sum_{m=1}^{n} S(z,k,m) \tag{4.1.11}$$

such that, for all $k \in \mathbb{N}$:

$$S'(z,k) = 1 \tag{4.1.12}$$

and criteria ((2)) is achieved, using eq. 4.1.7 and 4.1.10, when:

$$T'(Z,k) = \frac{1}{n} \sum_{m=1}^{n} T(z_m,k,m) \tag{4.1.13}$$

such that, for all $k \in \mathbb{N}$:

$$T'(Z,k) = 0 \tag{4.1.14}$$

where we consider the following:

4.2. Question:

How do we create a choice function which solves the question in sec. 2.4 using $\mathcal{S}'$, $\overline{|\mathcal{S}'|}$, $\underline{|\mathcal{S}'|}$, $S'(z,k)$, and $T'(Z,k)$ or equations 4.1.4, 4.1.5, 4.1.6, 4.1.11 and 4.1.13 resp.?

4.3. "Attempt" to Answer the Question

(Note the attempt might be wrong but could offer hints to how the solution would appear).

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Suppose $z = 1$ and the chosen coordinate for the center of the universe (i.e., eq. 4.1.7) is the origin, where $z_m = 0$ for all $m \in \{1, \cdots, n\}$:

$$Z = (z_1, z_2, \cdots, z_{n+1}) \Rightarrow \\ Z = O = (\underbrace{0, 0, \cdots, 0}_{n+1 \text{ times}}) \tag{4.3.1}$$

Using equations $\mathcal{S}'$, $\overline{|\mathcal{S}'|}$, $\underline{|\mathcal{S}'|}$, $S'(z,k)$, and $T'(Z,k)$ (i.e., eq. 4.1.4, 4.1.5, 4.1.6, 4.1.11 and 4.1.13) with the absolute value function $||\cdot||$ and the nearest integer function $[\cdot]$, we define:

$$K(\varepsilon, F_k^{\star\star\star}) = \\ S'(1,k)\left(\left\|\frac{|\mathcal{S}'|\left(1+\left[\frac{|\mathcal{S}'|\left(\underline{|\mathcal{S}'|}+2|\mathcal{S}'|\right)}{\left(\underline{|\mathcal{S}'|}+|\mathcal{S}'|\right)\left(\underline{|\mathcal{S}'|}+|\mathcal{S}'|+\overline{|\mathcal{S}'|}\right)}\right]\right)\left(1+\left[\underline{|\mathcal{S}'|}/|\mathcal{S}'|\right]\right)}{\left(1+\left[|\mathcal{S}'|/\overline{|\mathcal{S}'|}\right]\right)\left(1+\left[\underline{|\mathcal{S}'|}/\overline{|\mathcal{S}'|}\right]\right)} - |\mathcal{S}'|\right\| + |\mathcal{S}'|\right) - T'(O,k) \tag{4.3.2}$$

where using $K(\varepsilon, F_k^{\star\star\star})$, the choice function should be the following:

Theorem 28. *If we define:*

$$\mathcal{M}(\varepsilon, F_k^{\star\star\star}) = |\mathcal{S}'(\varepsilon, F_k^{\star\star\star})|\left(K(\varepsilon, F_k^{\star\star\star}) - |\mathcal{S}'(\varepsilon, F_k^{\star\star\star})|\right)$$

$$\mathcal{M}(\varepsilon, F_j^{\star\star}) = |\mathcal{S}'(\varepsilon, F_j^{\star\star})|\left(K(\varepsilon, F_j^{\star\star}) - |\mathcal{S}'(\varepsilon, F_j^{\star\star})|\right)$$

where for $\mathcal{M}(\varepsilon, F_k^{\star\star\star})$, we define $\mathcal{M}(\varepsilon, F_k^{\star\star\star})$ to be the same as $\mathcal{M}(\varepsilon, F_j^{\star\star})$ when swapping "$j \in \mathbb{N}$" with "$k \in \mathbb{N}$" (for eq. 4.1.5 & 4.1.6) and sets $F_k^{\star\star\star}$ with $F_j^{\star\star}$ (for eq. 4.1.4–4.3.2), then for constant $v > 0$ and variable $v^* > 0$, if:

$$\overline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star}) = \inf\left(\left\{|\mathcal{S}'(\varepsilon, F_j^{\star\star})| : j \in \mathbb{N}, \mathcal{M}(\varepsilon, F_j^{\star\star}) \geq \mathcal{M}(\varepsilon, F_k^{\star\star\star}) \geq v^*\right\} \cup \{v^*\}\right) + v \tag{4.3.3}$$

and:

$$\underline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star}) = \sup\left(\left\{|\mathcal{S}'(\varepsilon, F_j^{\star\star})| : j \in \mathbb{N}, v^* \leq \mathcal{M}(\varepsilon, F_j^{\star\star}) \leq \mathcal{M}(\varepsilon, F_k^{\star\star\star})\right\} \cup \{-v^*\}\right) + v \tag{4.3.4}$$

then for all $(F_j^{\star\star})_{j\in\mathbb{N}} \in \mathbb{S}'(G) \setminus \sim (F_k^{\star\star\star})_{k\in\mathbb{N}}$ (§4.1, crit. (3)), if:

$$\liminf_{\varepsilon\to 0} \lim_{v^*\to\infty} \lim_{k\to\infty} \frac{|\mathcal{S}'(\varepsilon, F_k^{\star\star\star})| + v}{\overline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star})} = \\ \limsup_{\varepsilon\to 0} \lim_{v^*\to\infty} \lim_{k\to\infty} \frac{|\mathcal{S}'(\varepsilon, F_k^{\star\star\star})| + v}{\underline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star})} = 0 \tag{4.3.5}$$

we choose $(F_k^{\star\star\star})_{k\in\mathbb{N}}$ satisfying eq. 4.3.5. (Note, we want $\sup\emptyset = -\infty$, $\inf\emptyset = +\infty$, and $(F_k^{\star\star\star})_{k\in\mathbb{N}}$ to answer the main question of §2.4) where the answer to the focus[3] *is $\mathbb{E}^{**}[f, F_k^{\star\star\star}]$ in eq. 4.3.6—using def. 13 (when it exists):*

[3] If the set $B^{**} \subseteq B^*$ (def. 9) is the set of all $f \in B^*$, with an unique and "meaningful" extension of the expected value—w.r.t the Hausdorff measure—on bounded functions to bounded/unbounded f taking finite values, then B^{**} should be non-shy subset of B^*

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$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(k\in\mathbb{N})\left(k\geq N\Rightarrow\left|\frac{1}{H^h(dom(F_k^{\star\star\star}))}\int_{dom(F_k^{\star\star\star})}f_k^{\star}\,dH^h-\mathbb{E}^{**}[f,F_k^{\star\star\star}]\right|<\epsilon\right)\tag{4.3.6}$$

Note 29 (Explanation of Theorem 28). *The theorem 28 is similar to the methods used in def. 27 crit. (a) and (b)—$\overline{\alpha}(\varepsilon,r,\omega,\psi)$ and $\underline{\alpha}(\varepsilon,r,\omega,\psi)$—and def. 27 crit. (1) and crit. (3)—linear/superlinear convergence—where:*

$$\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\limsup_{\varepsilon\to 0}\limsup_{r\to\infty}\overline{\alpha}(\varepsilon,r,\omega,\psi)=\inf_{\omega\in\Omega}\inf_{\psi\in\Psi_\omega}\liminf_{\varepsilon\to 0}\liminf_{r\to\infty}\underline{\alpha}(\varepsilon,r,\omega,\psi)=0$$

such that we replace:

$$\begin{aligned}
&E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon,F_r^\star,\omega),\psi)))\mapsto\mathcal{M}(\varepsilon,F_k^{\star\star\star})\\
&E(\mathcal{D}(\mathcal{S}(\mathscr{U}(\varepsilon,F_j^{\star\star},\omega),\psi)))\mapsto\mathcal{M}(\varepsilon,F_j^{\star\star})\\
&|\mathcal{S}(\mathscr{U}(\varepsilon,F_r^\star,\omega),\psi))|\mapsto|\mathcal{S}'(\varepsilon,F_j^{\star\star})|\\
&\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\underline{|\mathcal{S}(\mathscr{U}(\varepsilon,F_r^\star,\omega),\psi)|}\mapsto\underline{\mathcal{S}}(\varepsilon,k,v^*,F_j^{\star\star})\\
&\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\overline{|\mathcal{S}(\mathscr{U}(\varepsilon,F_r^\star,\omega),\psi)|}\mapsto\overline{\mathcal{S}}(\varepsilon,k,v^*,F_j^{\star\star})
\end{aligned}$$

note the changes to def. 27, crit. (1) were made, so $\mathcal{M}(\varepsilon,F_k^{\star\star\star})$ is "large enough" compared to $\mathcal{M}(\varepsilon,F_j^{\star\star})$, with $(F_j^{\star\star})_{j\in\mathbb{N}}$ non-equivalent to $(F_k^{\star\star\star})_{k\in\mathbb{N}}$ (e.g., when $A=\mathbb{Q}$, $(F_k^{\star\star\star})_{k\in\mathbb{N}}$ should be $(\{c/k!:c\in\mathbb{N},1\leq c\leq k!\})_{k\in\mathbb{N}}$ and never give $\mathcal{M}(\varepsilon,F_k^{\star\star\star})$ which increases at a smaller rate than that of "small" $\mathcal{M}(\varepsilon,F_j^{\star\star})$, e.g.:

$$(F_j^{\star\star})_{j\in\mathbb{N}}=(\{u/w:u\in\mathbb{Z},w\in\mathbb{N},w\leq j,-w\cdot j\leq u\leq w\cdot j\})_{j\in\mathbb{N}}$$

or smaller than that of "large" $\mathcal{M}(\varepsilon,F_j^{\star\star})$; e.g.,

$$(F_j^{\star\star})_{j\in\mathbb{N}}=(\{u_1/(6(j!)):u_1\in\mathbb{Z},-6j\cdot j!\leq u_1\leq 6j\cdot j!\})_{j\in\mathbb{N}}$$

Moreover, in $\underline{\mathcal{S}}(\varepsilon,k,v^*,F_j^{\star\star})$ and $\overline{\mathcal{S}}(\varepsilon,k,v^*,F_j^{\star\star})$ of thm. 28, we add constant $v>0$ and variable $v^*>0$ so if either

(1) *$\underline{\mathcal{S}}(\varepsilon,k,v^*,F_j^{\star\star})-v=0$ (i.e., using a related limit to eq. 4.3.5, division by zero is undefined).*
(2) *$\overline{\mathcal{S}}(\varepsilon,k,v^*,F_j^{\star\star})-v=0$ (i.e., using a related limit to eq. 4.3.5, division by zero is undefined).*
(3) *$\inf\left(\left\{|\mathcal{S}'(\varepsilon,F_j^{\star\star})|:j\in\mathbb{N},\mathcal{M}(\varepsilon,F_j^{\star\star})\geq\mathcal{M}(\varepsilon,F_k^{\star\star\star})\right\}\right)=+\infty$ (i.e., similar to $\underline{\mathcal{S}}(\varepsilon,k,v^*,F_j^{\star\star})$ of eq. 4.3.3, with no variable v^* such that $\mathcal{M}(\varepsilon,F_k^{\star\star\star})=0$ and $\exists(J>0)\forall(j_1>0)\exists(j\geq j_1)(\mathcal{M}(\varepsilon,F_j^{\star\star})\leq J)$, where we apply a related limit to eq. 4.3.5 that's undefined due to division by infinity.)*
(4) *$\inf\left(\left\{|\mathcal{S}'(\varepsilon,F_j^{\star\star})|:j\in\mathbb{N},\mathcal{M}(\varepsilon,F_j^{\star\star})\geq\mathcal{M}(\varepsilon,F_k^{\star\star\star})\right\}\right)=\emptyset$ (i.e., similar to $\underline{\mathcal{S}}(\varepsilon,k,v^*,F_j^{\star\star})$ of eq. 4.3.3, with no variable v^* and $\mathcal{M}(\varepsilon,F_j^{\star\star})=0$, where we apply a related limit to eq. 4.3.5 that's undefined since $\inf\left(\left\{|\mathcal{S}'(\varepsilon,F_j^{\star\star})|:j\in\mathbb{N},\mathcal{M}(\varepsilon,F_j^{\star\star})\geq\mathcal{M}(\varepsilon,F_k^{\star\star\star})\right\}\right)$ is an undefined empty set.)*
(5) *$\sup\left(\left\{|\mathcal{S}'(\varepsilon,F_j^{\star\star})|:j\in\mathbb{N},\mathcal{M}(\varepsilon,F_j^{\star\star})\leq\mathcal{M}(\varepsilon,F_k^{\star\star\star})\right\}\right)=+\infty$ (i.e., similar to $\overline{\mathcal{S}}(\varepsilon,k,v^*,F_j^{\star\star})$ of eq. 4.3.4, with no variable v^* and $\mathcal{M}(\varepsilon,F_j^{\star\star})=0$, where we apply a related limit to eq. 4.3.5 that's undefined due to division by infinity.)*
(6) *$\sup\left(\left\{|\mathcal{S}'(\varepsilon,F_j^{\star\star})|:j\in\mathbb{N},\mathcal{M}(\varepsilon,F_j^{\star\star})\leq\mathcal{M}(\varepsilon,F_k^{\star\star\star})\right\}\right)=\emptyset$ (i.e., similar to $\underline{\mathcal{S}}(\varepsilon,k,v^*,F_j^{\star\star})$ of eq. 4.3.3, with no variable v^* and $\mathcal{M}(\varepsilon,F_k^{\star\star})=0$, where we apply a related limit to eq. 4.3.5 that's undefined since $\inf\left(\left\{|\mathcal{S}'(\varepsilon,F_j^{\star\star})|:j\in\mathbb{N},\mathcal{M}(\varepsilon,F_j^{\star\star})\geq\mathcal{M}(\varepsilon,F_k^{\star\star\star})\right\}\right)$ is an undefined empty set.)*
(7) *$|\{z:j,z\in\mathbb{N},\mathcal{M}(\varepsilon,F_{j+z}^{\star\star})\leq\mathcal{M}(\varepsilon,F_j^{\star\star})\}|=+\infty$ (i.e., infinite number succeeding F_j are smaller than original F_j, where such F_j should be eliminated).*

the limit in eq. 4.3.5 still exists.

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4.4. *Questions Regarding* §2.4, §3 *and* §4 *[Revisited]*

(1) How do we use mathematica code to illustrate §3 and 4?
(2) Is there a more efficient solution to §2.4?
(3) If §2.4 should be changed, (see note 23) what else should be §2.4? What is the most efficient solution to the improved version of §2.4?

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