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Posted Date: 12 July 2023

doi: 10.20944/preprints202307.0787.v1

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Conservation Laws and Symmetry Reductions of the Hunter-Saxton Equation via the Double Reduction Method

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Abstract: This study investigates via Lie symmetry analysis the Hunter-Saxton equation, an equation relevant to the theoretical analysis of nematic liquid crystals. We employ the multiplier method to obtain conservation laws of the equation derived from first-order multipliers. Conservation laws of the Hunter-Saxton equation, in combination with the Lie point symmetries of the equation, enable us to perform symmetry reductions through employment of the double reduction method. The method exploits the relationship between Lie point symmetries and conservation laws to effectively reduce both the number of variables and the order of the equation. Five nontrivial conservation laws of the Hunter-Saxton equation are derived, four of which are found to have associated Lie point symmetries. Applying the double reduction method to the Hunter-Saxton equation results in a set of first-order ordinary differential equations, the solutions of which represent invariant solutions for the equation.

Keywords: double reduction; hunter-saxton equation; lie symmetry analysis; conservation law; invariant solution

1. Introduction

In this research article, we focus on the Hunter-Saxton equation [1], which is a significant mathematical model in the theoretical study of nematic liquid crystals. This equation, given by

$$(u_t + uu_x)_x = \frac{1}{2}ux^2, \tag{1}$$

has garnered considerable attention from researchers. Notably, Zhang et al. [2] investigated the well-posedness and global existence of solutions for the equation, shedding light on its mathematical behavior. Xu and Guo [3] explored solitary wave solutions using the modified Hirota's bilinear method, contributing to our understanding of the equation's soliton solutions and their properties.

Several studies on the Hunter-Saxton equation have utilized Lie symmetry analysis to study properties of the equations and to find solutions, and our study follows a similar approach. Zhdanov et al. [4] and Mikodynski and Trynkiewicz [5] determined Lie symmetries of the Hunter-Saxton equation and then used the symmetries to find conservation laws for the equation. Deng et al. [6] employed Lie symmetry analysis to study the stability of travelling wave solutions of the Hunter-Saxton equation. Mikodynski [7] found a family of exact solutions to the Hunter-Saxton equation using Lie symmetry analysis. Barakat and El-Khodary [8] discovered a new class of exact solutions, while Liu et al. [9] obtained a family of exact solutions for a modified version of the equation. Wang et al. [10] found a new class of exact solutions with nonlocal terms using Lie symmetry analysis. Johnpillai and Khalique [11] also utilized Lie symmetry analysis to find a new class of exact solutions for a generalized version of the Hunter-Saxton equation.

Our study focuses on the symmetry reductions of the Hunter-Saxton equation using the double reduction method. We determine Lie point symmetries of the equation, construct conservation laws using the multiplier method, and employ the double reduction method to perform symmetry

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reductions. This research complements existing studies on the Hunter-Saxton equation and contributes to the application of the double reduction method for finding solutions of partial differential equations (PDEs).

The double reduction method, introduced by Sjöberg [12,13], is a technique for solving PDEs based on the use of Lie symmetries and conservation laws. For a $(1+1)$ PDE of order q, the double reduction theory allows for the reduction of the PDE to an ODE of order $q-1$, provided that the PDE possesses a conservation law and an associated symmetry. Generalizations of the double reduction method have been proposed to handle higher-dimensional PDEs and systems of PDEs [14–16]. Anco and Gandarias [17] have introduced a further generalization of the double reduction method to handle partial differential equations (PDEs) with $n \geq 2$ independent variables and a symmetry algebra of dimension at least $n-1$. In their work [17], they present an algorithm for identifying all symmetry-invariant conservation laws that reduce to first integrals for the corresponding ordinary differential equation (ODE) governing symmetry-invariant solutions of the PDE.

Moreover, Anco and Gandarias [17] propose an improved formulation for assessing the symmetry-invariance of conservation laws by utilizing multipliers. This refined formulation enables the direct derivation of symmetry-invariant conservation laws, eliminating the need to first obtain conservation laws and subsequently verify their invariance.

The subsequent sections of this paper are structured as follows: Section 2 provides an overview of the necessary preliminaries and outlines the fundamental principles of the double reduction theorem. In Section 3, we calculate the Lie point symmetries and conservation laws for the Hunter-Saxton equation, determining which conservation laws are associated with symmetries. Section 4 focuses on executing symmetry reductions for the Hunter-Saxton equation. Finally, in Section 5, we present our concluding remarks.

2. Fundamentals of the Double Reduction Theorem

In this section, we present the double reduction routine for a qth-order ($q \geq 1$) partial differential equation with n independent variables $x = (x^1, x^2, \ldots, x^n)$ and one dependent variable $u = u(x)$, namely

$$F(x, u, \ldots, u_{(q)}) = 0, \tag{2}$$

where $u_{(q)}$ denotes the collection $\{u_q\}$ of qth-order partial derivatives. In this connection, we firstly present the following well-known definitions and results (see, e.g., [18–21])

1. The total derivative operator with respect to x^i is

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots, \quad i = 1, 2, \ldots, n, \tag{3}$$

   where u_i denotes the derivative of u with respect to x^i. Similarly u_{ij} denotes the derivative of u with respect to x^i and x^j.
2. An n-tuple $T = \left(T^1, T^2, \ldots, T^n\right), i = 1, 2 \cdots n$, such that

$$D_i T^i = 0 \tag{4}$$

   holds for all solutions of (2) is known as a conservation law of (2).
3. A multiplier Λ for Equation (2) is a non-singular function on the solution space of (2) with the property

$$D_i T^i = \Lambda E \tag{5}$$

   for arbitrary function $u\left(x^1, x^2, \ldots, x^n\right)$.
4. The determining equations for multipliers are obtained by taking the variational derivative:

$$\frac{\delta}{\delta u}(\Lambda E) = 0, \tag{6}$$

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where the Euler operator $\delta/\delta u$ is defined by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_i \frac{\partial}{\partial u_i} + D_{ij}\frac{\partial}{\partial u_{ij}} - D_{ijk}\frac{\partial}{\partial u_{ijk}} + \cdots . \tag{7}$$

5. A Lie symmetry of (2) with infinitesimal generator $X = \xi_i \partial x_i + \eta \partial u$ is said to be associated with a conserved law (4) if the symmetry and the conservation law satisfy the relations [22]

$$\left[T^i, X\right] = X\left(T^i\right) + T^i D_j \xi^j - T^j D_j \xi^i, \quad i = 1, \ldots, n. \tag{8}$$

Suppose that the PDE (2) admits a Lie point symmetry with infinitesimal generator $X = \xi_i \partial x_i + \eta \partial u$ that is associated with a conservation law $D_i T^i = 0$. The following steps constitute the routine of the double reduction method:

I. Find similarity variables $\tilde{x}_i, i = 1, 2, \ldots, n$ and w

$$\begin{aligned} \tilde{x}_i &= \tilde{x}_i\left(x^1, x^2, \ldots, x^n\right), \quad i = 1, 2, \ldots, n \\ w\left(\tilde{x}_1, \ldots, \tilde{x}_{n-1}\right) &= \omega\left(x^1, x^2, \ldots, x^n\right) u \end{aligned}$$

such that in these variables $X = \dfrac{\partial}{\partial \tilde{x}_n}$.

II. Find inverse canonical coordinates

$$\begin{aligned} x^i &= x^i\left(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n\right), \quad i = 1, 2, \ldots, n \\ u\left(x^1, x^2, \ldots, x^n\right) &= \psi\left(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n\right) w \end{aligned}$$

III. Write partial derivatives of u in terms of the similarity variables.

IV. Construct matrices A and A^{-1} as follows:

$$A = \begin{pmatrix} \widetilde{D}_1 x_1 & \widetilde{D}_1 x_2 & \ldots & \widetilde{D}_1 x_n \\ \widetilde{D}_2 x_1 & \widetilde{D}_2 x_2 & \ldots & \widetilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{D}_n x_1 & \widetilde{D}_n x_2 & \ldots & \widetilde{D}_n x_n \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} D_1 \tilde{x}_1 & D_1 \tilde{x}_2 & \ldots & D_1 \tilde{x}_n \\ D_2 \tilde{x}_1 & D_2 \tilde{x}_2 & \ldots & D_2 \tilde{x}_n \\ \vdots & \vdots & \vdots & \vdots \\ D_n \tilde{x}_1 & D_n \tilde{x}_2 & \ldots & D_n \tilde{x}_n \end{pmatrix}$$

V. Write components T^i of the conserved vector in terms of the similarity variables as follows:

$$\begin{pmatrix} \widetilde{T}^1 \\ \widetilde{T}^2 \\ \vdots \\ \widetilde{T}^n \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix}, \tag{9}$$

where $J = \det(A)$. Note that $T^1, \ldots T^n$ in (9) are easily expressed in terms of the similarity variables in light of II and III.

VI. The reduced conservation law becomes

$$D_1 \widetilde{T}^1 + D_2 \widetilde{T}^2 + \ldots + D_{n-1} \widetilde{T}^{n-1} = 0. \tag{10}$$

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3. Symmetries and Conservation Laws of the Hunter-Saxton Equation

The Hunter-Saxton equation (1) is a $(1+1)$ PDE with two independent variables $x = (x^1, x^2) = (t, x)$ and one dependent variable $u = u(t, x)$. It admits the following four symmetries:

$$\begin{aligned} X_1 &= x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} & X_2 &= \frac{\partial}{\partial t} \\ X_3 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} & X_4 &= t^2\frac{\partial}{\partial t} + 2tx\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial u}. \end{aligned} \tag{11}$$

The symmetries are easily computed using MathLie, the symmetry-finding package for Mathematica [23] developed by G. Baumann [24]. We will use the multiplier approach to derive conservation laws for the Hunter-Saxton equation (1). We seek first-order multipliers

$$\Lambda = \Lambda(x, t, u, u_x, u_t) \tag{12}$$

of (1), for which the determining equation according to (6) is

$$\frac{\delta}{\delta u}\left[\Lambda\left((u_t + uu_x)_x - \frac{1}{2}\,ux^2\right)\right] = 0, \tag{13}$$

where the standard Euler operator $\delta/\delta u$, as defined in (7), is

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_t^2\frac{\partial}{\partial u_{tt}} + D_x^2\frac{\partial}{\partial u_{xx}} + D_xD_t\frac{\partial}{\partial u_{tx}} - \cdots, \tag{14}$$

and total derivative operators D_t and D_x using (3) are

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \cdots, \\ D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \cdots. \end{aligned}$$

The determining equation for the multiplier Λ after expansion takes the following form:

$$\Omega_0 + u_{tt}\Omega_1 + u_{tx}\Omega_2 + (u_{tx})^2\,\Omega_3 + u_{xx}\Omega_4 + u_{xx}u_{tt}\Omega_5 = 0, \tag{15}$$

where

$$\begin{aligned} \Omega_0 &= u_x\Lambda_{tu} - \frac{1}{2}u_x^2\Lambda_{tu_t} + \Lambda_{tx} - \frac{1}{2}u_x^3\Lambda_{uu_x} - \frac{1}{2}u_x^2u_t\Lambda_{uu_t} + uu_x^2\Lambda_{uu} + u_xu_t\Lambda_{uu} \\ &\quad + 2uu_x\Lambda_{xu} + u_t\Lambda_{xu} + u\Lambda_{xx} - \frac{1}{2}u_x^2\Lambda_{xu_x} + \frac{3u_x^2\Lambda_u}{2} + u_x\Lambda_x \\ \Omega_1 &= u_x\Lambda_{uu_t} - \frac{1}{2}u_x^2\Lambda_{u_tu_t} + \Lambda_{xu_t} \\ \Omega_2 &= 2uu_x\Lambda_{uu_t} + 2u\Lambda_{xu_t} - u_x^2\Lambda_{u_tu_x} + 2\Lambda_u \\ \Omega_3 &= u\Lambda_{u_tu_t} - \Lambda_{u_tu_x} \\ \Omega_4 &= \Lambda_{tu_x} - u\Lambda_{tu_t} + u_t\Lambda_{uu_x} + u\Lambda_{xu_x} + uu_x\Lambda_{uu_x} - uu_t\Lambda_{uu_t} - \frac{1}{2}u_x^2\Lambda_{u_xu_x} \\ &\quad + 2u\Lambda_u - u_x\Lambda_{u_x} - u_t\Lambda_{u_t} + \Lambda \\ \Omega_5 &= \Lambda_{u_tu_x} - u\Lambda_{u_tu_t} \end{aligned}$$

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The multiplier determining equation (15) splits with respect to different combinations of the derivatives u_{xx}, u_{tx} and u_{tt} yielding an overdetermined linear system of equations for the multiplier. The system of equations was solved using Mathematica [23] to obtain

$$\Lambda = u_t\left(\delta_2 + \delta_3 t - \frac{\delta_1 t^2}{2}\right) + u_x x(\delta_3 - \delta_1 t) + \delta_1 x + \delta_4 u_x + \frac{\delta_5}{u_x^2}, \tag{16}$$

where δ_i are arbitrary constants. From (5) and (16), we have

$$\begin{aligned}&\left[(u_t + uu_x)_x - \frac{1}{2}ux^2\right]\left[u_t\left(\delta_2 + \delta_3 t - \frac{\delta_1 t^2}{2}\right) + u_x x(\delta_3 - \delta_1 t)\right.\\ &\left. + \delta_1 x + \delta_4 u_x + \frac{\delta_5}{u_x^2}\right] = D_t T^t + D_x T^x,\end{aligned} \tag{17}$$

where

$$\begin{aligned}T^t &= u_x^2\left(u\left(\frac{\delta_1 t^2}{4} - \frac{\delta_2}{2} - \frac{\delta_3 t}{2}\right) + x\left(\frac{\delta_3}{2} - \frac{\delta_1 t}{2}\right) + \frac{\delta_4}{2}\right) - \frac{\delta_5}{u_x} + \phi_2(x)\\ &\quad + u_x(\delta_1 x - \delta_1 tu + \phi_1(u))\\ T^x &= u_t^2\left(\frac{\delta_2}{2} + \frac{\delta_3 t}{2} - \frac{\delta_1 t^2}{4}\right) + uu_x^2\left(x\left(\frac{\delta_3}{2} - \frac{\delta_1 t}{2}\right) + \frac{\delta_4}{2}\right) - \frac{\delta_5 u}{u_x} + \frac{3\delta_5 x}{2}\\ &\quad + u_x\left(uu_t\left(\delta_2 + \delta_3 t - \frac{\delta_1 t^2}{2}\right) + \delta_1 ux\right) + u_t(\delta_1 tu - \phi_1(u)) + \phi_3(t)\end{aligned}$$

for arbitrary functions $u(t,x)$. When $u(t,x)$ is a solution of Eq. (1), the left hand side of (17) vanishes and we obtain conservation laws of Hunter-Saxton equation (1) for which the conserved vectors (T_i^1, T_i^2) are given by:

$$\begin{aligned}T_1^1 &= u_x\left(ux - \frac{1}{2}t^2 uu_t\right) - \frac{t^2 u_t^2}{4} - \frac{1}{2}tuu_x^2 x + u_t(tu - \phi_1(u)) + \phi_3(t)\\ T_1^2 &= u_x^2\left(\frac{t^2 u}{4} - \frac{tx}{2}\right) + u_x(x - tu + \phi_1(u)) + \phi_2(x),\end{aligned}$$

$$\begin{aligned}T_2^1 &= \phi_3(t) + uu_x u_t - u_t\phi_1(u) + \frac{u_t^2}{2}\\ T_2^2 &= u_x\phi_1(u) + \phi_2(x) - \frac{uu_x^2}{2}\end{aligned}$$

$$\begin{aligned}T_3^1 &= tuu_x u_t + \frac{tu_t^2}{2} + \phi_3(t) + \frac{1}{2}uu_x^2 x - u_t\phi_1(u)\\ T_3^2 &= u_x^2\left(\frac{x}{2} - \frac{tu}{2}\right) + u_x\phi_1(u) + \phi_2(x),\end{aligned}$$

$$\begin{aligned}T_4^1 &= \phi_3(t) - u_t\phi_1(u) + \frac{uu_x^2}{2}\\ T_4^2 &= \phi_2(x) + u_x\phi_1(u) + \frac{u_x^2}{2},\end{aligned}$$

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$$
\begin{aligned}
T_5^1 &= \phi_3(t) - u_t\phi_1(u) - \frac{u}{u_x} + \frac{3x}{2} \\
T_5^2 &= \phi_2(x) + u_x\phi_1(u) - \frac{1}{u_x}.
\end{aligned}
$$

According to (8), a symmetry X is associated with a conservation law $D_tT^t + D_xT^x = 0$ if the following formula is satisfied:

$$
X\begin{pmatrix} T^t \\ T^x \end{pmatrix} - \begin{pmatrix} D_t\xi^t & D_x\xi^t \\ D_t\xi^x & D_x\xi^x \end{pmatrix}\begin{pmatrix} T^t \\ T^x \end{pmatrix} + (D_t\xi^t + D_x\xi^x)\begin{pmatrix} T^t \\ T^x \end{pmatrix} = 0.
$$

It turns out that the association of symmetries and conservation laws of (1) is obtained in the following cases:

$$
\kappa_1(X_1 + 2X_3) + \kappa_2 X_2 \to \begin{cases} T_2^1 = \frac{u_t^2}{2} - \frac{\delta_1 u_t}{u} + uu_xu_t \\ T_2^2 = \frac{\delta_1 u_x}{u} + \frac{\delta_3}{x} - \frac{uu_x^2}{2} \end{cases}
$$

$$
\kappa_1(X_1 + X_3) + \kappa_2 X_2 \to \begin{cases} T_4^1 = \frac{uu_x^2}{2} - \frac{\delta_1 u_t}{u} \\ T_4^2 = \frac{\delta_1 u_x}{u} + \frac{\delta_3}{x} + \frac{u_x^2}{2} \end{cases}
$$

$$
\kappa_1\left(X_1 - \frac{X_3}{2}\right) + \kappa_2 X_2 \to \begin{cases} T_5^1 = \frac{\delta_2}{2\kappa_2 - \kappa_1 t} - \frac{\delta_1 u_t}{u} - \frac{u}{u_x} + \frac{3x}{2} \\ T_5^2 = \frac{\delta_1 u_x}{u} + \frac{\delta_3}{x} - \frac{1}{u_x} \end{cases}
$$

$$
X_3 \to \begin{cases} T_3^1 = \frac{\delta_1}{t} + tuu_xu_t + \frac{tu_t^2}{2} + \frac{1}{2}uu_x^2x - u_t\phi_1(u) \\ T_3^2 = \frac{\delta_2}{x} + u_x^2\left(\frac{x}{2} - \frac{tu}{2}\right) + u_x\phi_1(u) \end{cases}
$$

4. Double Reduction of the Hunter-Saxton Equation

4.1. Double Reduction of (1) by $\langle \kappa_1(X_1 + 2X_3) + \kappa_2 X_2 \rangle$

We transform the generator $Z = \kappa_1(X_1 + 2X_3) + \kappa_2 X_2$ to its canonical form $Y = 0\frac{\partial}{\partial r} + \frac{\partial}{\partial s} + 0\frac{\partial}{\partial w}$. Therefore canonical coordinates $r = r(t,x)$, $s = s(t,x)$ and $w = w(t,x,u)$ must be found such that $Z(r) = 0, Z(s) = 1$ and $Z(w) = 0$. While the coordinates r and w are obtained from invariants of Z, the coordinate s may be determined by inspection. More systematically, it can be obtained from an invariant $J = v - s(x,y)$ of the extended operator $Z + \partial_v$, where v is an auxiliary variable [21]. We obtain

$$
r = \frac{x}{(2\kappa_1 t + \kappa_2)^{3/2}}, \quad s = \frac{\ln x}{3\kappa_1}, \quad w = \frac{u}{\sqrt{2\kappa_1 t + \kappa_2}}, \quad \kappa_1 \neq 0, \tag{18}
$$

where $w = w(r)$. Inverse canonical coordinates follow from (18) and are given by

$$
t = \frac{e^{2\kappa_1 s} - \kappa_2 r^{2/3}}{2\kappa_1 r^{2/3}}, \quad x = e^{3\kappa_1 s}, \quad u = \frac{we^{\kappa_1 s}}{r^{1/3}}. \tag{19}
$$

Computing A and $\left(A^{-1}\right)^T$, we obtain

$$
A = \begin{pmatrix} D_rt & D_rx \\ D_st & D_sx \end{pmatrix} = \begin{pmatrix} -\frac{e^{2\kappa_1 s}}{3\kappa_1 r^{5/3}} & 0 \\ \frac{e^{2\kappa_1 s}}{r^{2/3}} & 3e^{3\kappa_1 s}\kappa_1 \end{pmatrix}
$$

 doi:10.20944/preprints202307.0787.v1

and

$$\left(A^{-1}\right)^T = \begin{pmatrix} D_t r & D_x r \\ D_t s & D_x s \end{pmatrix} = \begin{pmatrix} -3e^{-2\kappa_1 s}\kappa_1 r^{5/3} & e^{-3\kappa_1 s} r \\ 0 & \frac{e^{-3\kappa_1 s}}{3\kappa_1} \end{pmatrix}.$$

The partial derivatives of u from (19) are given by

$$\begin{aligned} &u_t = \kappa_1 \sqrt[3]{r} e^{-\kappa_1 s}(w - 3rw_r), \quad u_x = r^{2/3} w_r e^{-2\kappa_1 s}, \\ &u_{tx} = -\kappa_1 r^{4/3} e^{-4\kappa_1 s}(3rw_{rr} + 2w_r), \\ &u_{xx} = r^{5/3} w_{rr} e^{-5\kappa_1 s}. \end{aligned} \tag{20}$$

The reduced conserved form is given by

$$\begin{pmatrix} T_2^r \\ T_2^s \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix}, \tag{21}$$

where $J = \det(A) = -\frac{e^{5\kappa_1 s}}{r^{5/3}}$. By substituting (19) and (20) into (21), we obtain

$$\begin{aligned} T_2^r &= \delta_1\kappa_1 + 3\delta_3\kappa_1 + 3\kappa_1^2 rww_r - \frac{9}{2}\kappa_1^2 r^2 w_r^2 - \frac{\kappa_1^2 w^2}{2} + \frac{3}{2}\kappa_1 rww_r^2 - \kappa_1 w^2 w_r, \\ T_2^s &= w_r\left(\kappa_1 w - \frac{\delta_1}{w} - \frac{w^2}{3r}\right) + \frac{\delta_1}{3r} - \frac{\kappa_1 w^2}{6r} + w_r^2\left(w - \frac{3\kappa_1 r}{2}\right), \end{aligned} \tag{22}$$

where the reduced conserved form satisfies

$$D_r T_2^r = 0. \tag{23}$$

From (22) and (23), we have

$$3\kappa_1^2 rww_r - \frac{9}{2}\kappa_1^2 r^2 w_r^2 - \frac{\kappa_1^2 w^2}{2} + \frac{3}{2}\kappa_1 rww_r^2 - \kappa_1 w^2 w_r = k,$$

where k is an arbitrary constant.

4.2. Double Reduction of (1) by $\langle \kappa_1(X_1 + X_3) + \kappa_2 X_2 \rangle$

Canonical coordinated determined from $\langle \kappa_1(X_1 + X_3) + \kappa_2 X_2 \rangle$ are:

$$r = \frac{x}{(\kappa_1 t + \kappa_2)^2}, \quad s = \frac{\ln x}{2\kappa_1}, \quad w = \frac{u}{\sqrt{x}}, \quad \kappa_1 \neq 0, \tag{24}$$

where $w = w(r)$, and the inverse canonical coordinates are given by

$$t = -\frac{\kappa_2\sqrt{r} - e^{\kappa_1 s}}{\kappa_1\sqrt{r}}, \quad x = e^{2\kappa_1 s} \quad u = we^{\kappa_1 s}. \tag{25}$$

Therefore, the partial derivatives of u from (25) are given by

$$\begin{aligned} &u_t = -2\kappa_1 r^{3/2} w_r, \quad u_x = \frac{1}{2}e^{-\kappa_1 s}(2rw_r + w), \\ &u_{tx} = -e^{-2\kappa_1 s}\kappa_1 r^{3/2}(2rw_{rr} + 3w_r, \\ &u_{xx} = -\frac{1}{4}e^{-3\kappa_1 s}(w - 4r(rw_{rr} + w_r)). \end{aligned} \tag{26}$$

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As for A and $\left(A^{-1}\right)^T$, we obtain

$$A = \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} -\frac{e^{\kappa_1 s}}{2\kappa_1 r^{3/2}} & 0 \\ \frac{e^{\kappa_1 s}}{\sqrt{r}} & 2e^{2\kappa_1 s}\kappa_1 \end{pmatrix},$$

and

$$\left(A^{-1}\right)^T = \begin{pmatrix} D_t r & D_x r \\ D_t s & D_x s \end{pmatrix} = \begin{pmatrix} -2e^{-\kappa_1 s}\kappa_1 r^{3/2} & e^{-2\kappa_1 s} r \\ 0 & \frac{e^{-2\kappa_1 s}}{2\kappa_1} \end{pmatrix}.$$

Therefore, from

$$\begin{pmatrix} T_4^r \\ T_4^s \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_4^t \\ T_4^x \end{pmatrix}, \tag{27}$$

where $J = \det(A) = -\frac{e^{3\kappa_1 s}}{r^{3/2}}$, we obtain

$$\begin{aligned} T_4^r &= \delta_1\kappa_1 + 2\delta_3\kappa_1 + \kappa_1 r^2 w_r^2 + \kappa_1 r w w_r + \frac{\kappa_1 w^2}{4} - \frac{1}{2} r^{3/2} w w_r^2 - \frac{w^3}{8\sqrt{r}} - \frac{1}{2}\sqrt{r} w^2 w_r, \\ T_4^s &= -\frac{\delta_1 w_r}{w} - \frac{w^3}{16\kappa_1 r^{3/2}} - \frac{w^2 w_r}{4\kappa_1\sqrt{r}} - \frac{\sqrt{r} w w_r^2}{4\kappa_1}. \end{aligned} \tag{28}$$

From the reduced conservation law $D_r T_4^r = 0$, we obtain

$$\kappa_1 r^2 w_r^2 + \kappa_1 r w w_r + \frac{\kappa_1 w^2}{4} - \frac{1}{2} r^{3/2} w w_r^2 - \frac{w^3}{8\sqrt{r}} - \frac{1}{2}\sqrt{r} w^2 w_r = k,$$

where k is an arbitrary constant.

4.3. Double Reduction of (1) by $\left\langle \kappa_1\left(X_1 - \frac{X_3}{2}\right) + \kappa_2 X_2 \right\rangle$

Canonical coordinated determined from $\left\langle \kappa_1\left(X_1 - \frac{X_3}{2}\right) + \kappa_2 X_2 \right\rangle$ are:

$$r = x(2\kappa_2 - \kappa_1 t), \quad s = \frac{2\ln x}{\kappa_1}, \quad w = \frac{u}{x^2}, \quad \kappa_1 \neq 0 \tag{29}$$

where $w = w(r)$, and the inverse canonical coordinates are given by

$$t = \frac{2\kappa_2 - re^{-\frac{1}{2}\kappa_1 s}}{\kappa_1}, \quad x = e^{\frac{\kappa_1 s}{2}}, \quad u = we^{\kappa_1 s} \tag{30}$$

Therefore, the partial derivatives of u from (30) are given by

$$\begin{aligned} u_t &= -\kappa_1 w_r e^{\frac{3\kappa_1 s}{2}}, \quad u_x = e^{\frac{\kappa_1 s}{2}}(rw_r + 2w), \\ u_{tx} &= -\kappa_1 e^{\kappa_1 s}(rw_{rr} + 3w_r), \\ u_{xx} &= r(rw_{rr} + 4w_r) + 2w \end{aligned} \tag{31}$$

Therefore,

$$A = \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} -\frac{e^{-\frac{1}{2}\kappa_1 s}}{\kappa_1} & 0 \\ \frac{1}{2}e^{-\frac{1}{2}\kappa_1 s} r & \frac{1}{2}e^{\frac{\kappa_1 s}{2}}\kappa_1 \end{pmatrix}$$

and

$$\left(A^{-1}\right)^T = \begin{pmatrix} D_t r & D_x r \\ D_t s & D_x s \end{pmatrix} = \begin{pmatrix} -e^{\frac{\kappa_1 s}{2}}\kappa_1 & e^{-\frac{1}{2}\kappa_1 s} r \\ 0 & \frac{2e^{-\frac{1}{2}\kappa_1 s}}{\kappa_1} \end{pmatrix}.$$

 doi:10.20944/preprints202307.0787.v1

Therefore, from

$$\begin{pmatrix} T_5^r \\ T_5^s \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_5^t \\ T_5^x \end{pmatrix}, \tag{32}$$

where $J = \det(A) = -\frac{1}{2}$, we obtain

$$\begin{aligned} T_5^r &= \frac{2\kappa_1(2\delta_1 r w_r + 4\delta_1 w + \delta_3 r w_r + 2\delta_3 w - 1) - 2\delta_2(r w_r + 2w) - r(3r w_r + 4w)}{4r w_r + 8w}, \\ T_5^s &= -\frac{2\delta_1\kappa_1 r^2 w_r^2 + 4\delta_1\kappa_1 r w w_r + 2\delta_2 r w w_r + 4\delta_2 w^2 + 3r^2 w w_r + 4r w^2}{2\kappa_1 r^2 w w_r + 4\kappa_1 r w^2}. \end{aligned} \tag{33}$$

From the reduced conservation law $D_r T_5^r = 0$, we obtain

$$\frac{2\kappa_1(2\delta_1 r w_r + 4\delta_1 w + \delta_3 r w_r + 2\delta_3 w - 1) - 2\delta_2(r w_r + 2w) - r(3r w_r + 4w)}{4r w_r + 8w} = k,$$

where k is an arbitrary constant.

4.4. Double Reduction of (1) by $\langle X_3 \rangle$

Canonical coordinated determined from X_3 are:

$$r = \frac{x}{t}, \quad s = \ln x \quad w = u, \tag{34}$$

where $w = w(r)$, and the inverse canonical coordinates are given by

$$t = \frac{e^s}{r}, \quad x = e^s \quad u = w. \tag{35}$$

Therefore, the partial derivatives of u from (35) are given by

$$\begin{aligned} u_t &= -r^2 e^{-s} w_r, \quad u_x = r e^{-s} w_r, \\ u_{tx} &= -r^2 e^{-2s}(r w_{rr} + w_r), \\ u_{xx} &= r^2 e^{-2s} w_{rr} \end{aligned} \tag{36}$$

As for A and $\left(A^{-1}\right)^T$, we obtain

$$A = \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} -\frac{e^s}{r^2} & 0 \\ \frac{e^s}{r} & e^s \end{pmatrix},$$

and

$$\left(A^{-1}\right)^T = \begin{pmatrix} D_t r & D_x r \\ D_t s & D_x s \end{pmatrix} = \begin{pmatrix} -e^{-s} r^2 & e^{-s} r \\ 0 & e^{-s} \end{pmatrix}.$$

Therefore, from

$$\begin{pmatrix} T_3^r \\ T_3^s \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_3^t \\ T_3^x \end{pmatrix}, \tag{37}$$

where $J = \det(A) = -\frac{e^{2s}}{r^2}$, we obtain

$$\begin{aligned} T_3^r &= \delta_2 - \delta_1, \\ T_3^s &= \frac{1}{2} w_r^2 (w - r) - \frac{\delta_1}{r} - w_r \phi_1(w). \end{aligned} \tag{38}$$

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It is remarkable that in this case, because T_3^r in (38) is simply a constant, the reduced conservation law $D_r T_3^r = 0$ does not result in an ODE that can be solved for w. Therefore, no invariant solution will arise via the double reduction method from the association of X_3 and the conservation law T_3.

5. Concluding Remarks

In this paper, a study of the Hunter-Saxton equation using Lie symmetry analysis was presented. Symmetry reductions of the equation were carried out by employing the generalised approach to double reduction theory proposed by Bokhari et al [14]. By utilising the multiplier method, nontrivial conservation laws for the Hunter-Saxton equation were derived. These conservation laws, along with the Lie point symmetries of the equation, were employed to perform symmetry reductions via the double reduction method.

Through the analysis, a set of first-order ODEs was obtained, whose solutions represent invariant solutions for the Hunter-Saxton equation. Out of the five nontrivial conservation laws constructed, it was observed that only four had associated Lie point symmetries according to the definition provided by Kara and Mahomed [18]. The conservation law T_1 did not have any linear combination of symmetries associated with it. Additionally, it is noteworthy that despite the conservation law T_3 having an associated Lie point symmetry, X_3, the application of the double reduction method in this case did not yield a symmetry reduction of the Hunter-Saxton equation. This outcome could be attributed to the “collapse" of the first integral, which was expected to represent a reduced ODE for the PDE but instead resulted in a constant value.

Author Contributions: Conceptualization, M.C.K. and W.S.; methodology, M.C.K.; W.S. and P.T.; software, M.C.K. and W.S.; validation, W.S. and P.T; formal analysis, M.C.K.; W.S. and P.T.; writing—original draft preparation, M.C.K. and W.S.; writing—review and editing, M.C.K.; W.S. and P.T.; . All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: This article has no additional data.

Acknowledgments: The authors would like to thank the Directorate of Research Development and Innovation of Walter Sisulu University for continued financial support.

Conflicts of Interest: The authors declare no conflict of interest.

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