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Not peer-reviewed version

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Posted Date: 13 July 2023

doi: 10.20944/preprints202307.0918.v1

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Article

Analogies between Lattice QCD and the Truncated Nambu-Jona-Lasinio Model

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Abstract: A modified Nambu–Jona-Lasinio Model with lattice structure is very instructive. It shows several similar problems and their solutions as the Lattice QCD. We study the limits of the large box size, small cell size and realistic pion mass. In particular, we study the relation of the discrete (bound state) solutions to the physical scattering states, for example the pion-pion scattering.

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1. Introduction

Lattice models as well as few-body models with a finite Hilbert space do not provide a continuum description of the two-body decay channel. Instead, the diagonalization of the Hamiltonian yields a discrete spectrum which hides a lot of information about the relevant continuum which we are trying to extract. As an example, we study approximate methods for $\pi - \pi$ scattering at low energy as well as via the σ meson resonance.

A good lesson can be learned from a truncated Nambu–Jona-Lasinio model (NJL), “the quasispin model”, in which quarks are enclosed in a periodic box \mathcal{V} and have a momentum cutoff Λ where $b = \sqrt[3]{6\pi^2}/\Lambda$ is a parameter analogous to the size of the lattice cell in Lattice QCD.

In Section 2 we recapitulate the formulation and some salient features of our model [1–3]. In Section 3 we discuss the lessons offered by the model. Finally (Section 4), we attempt to estimate the width of sigma meson using an analytic extrapolation.

2. The two-level quasispin model

The model is characterized by a finite number N of quarks occupying a finite number \mathcal{N} of states in the Dirac sea and the same number of states in the valence space. This allows us to use the first quantization and an explicit wavefunction.

We make the following simplifications: (i) a periodic box of volume \mathcal{V} , (ii) a sharp 3-momentum cutoff Λ , (iii) an average value of kinetic energy for all momentum states $|\vec{p}_i| \rightarrow P = \frac{3}{4}\Lambda$, (iv) restriction to one flavour of quarks $n_f = 1$, (v) truncation of interaction (while in the NJL model the interaction conserves the sum of momenta of both quarks we assume that each quark conserves its momentum and only switches from the Dirac level to Fermi level).

The finite number of discrete momentum states is then $\mathcal{N} = n_h n_c n_f n_p$ where n_h, n_c, n_f and $n_p = \mathcal{V}\Lambda^3/6\pi^2$ are the number of quark helicities, colours, flavours and momentum states.

The model Hamiltonian can then be written as

$$H = \sum_{k=1}^N \left(\gamma_5(k) h(k) P + m_0 \beta(k) \right) + \frac{g}{2} \left(\sum_{k=1}^N \beta(k) \sum_{l=1}^N \beta(l) + \sum_{k=1}^N i\beta(k) \gamma_5(k) \sum_{l=1}^N i\beta(l) \gamma_5(l) \right) .$$

where γ_5 and β are Dirac matrices, and $h = \vec{\sigma} \cdot \vec{p}/p$ is helicity.

There are 3 model parameters, $m_0 = 4.58$ MeV is the bare quark mass, $P = \frac{3}{4}\Lambda$ with $\Lambda = 648$ MeV is the average momentum and $g = 4G/\mathcal{V}$ where $G = 40.6$ MeV is the interaction strength in the original (continuum) NJL. These parameters have been fitted to the experimental or phenomenological values of the pion mass $m_\pi = 136$ MeV, constituent quark mass $M = 335$ MeV and quark condensate $Q = 250^3$ MeV³ [3]. The values of our model parameters turn out to be very close to the popular values of full NJL [4,5].

It is usually overlooked that the following operators obey (quasi)spin commutation relations $j_x = \frac{1}{2}\beta$, $j_y = \frac{1}{2}i\beta\gamma_5$, $j_z = \frac{1}{2}\gamma_5$. The (quasi)spin commutation relations are also obeyed by separate sums over quarks with right and left helicity as well as by the total sum ($\alpha = x, y, z$)

$$R_\alpha = \sum_{k=1}^N \frac{1+h(k)}{2} j_\alpha(k), \quad L_\alpha = \sum_{k=1}^N \frac{1-h(k)}{2} j_\alpha(k), \quad J_\alpha = R_\alpha + L_\alpha = \sum_{k=1}^N j_\alpha(k).$$

The model Hamiltonian can then be rewritten as

$$H = 2P(R_z - L_z) + 2m_0 J_x - 2g(J_x^2 + J_y^2) \quad (1)$$

It commutes with R^2 and L^2 but not with R_z and L_z . Nevertheless, it is convenient to work in the basis $|R, L, R_z, L_z\rangle$. The Hamiltonian matrix elements can be easily calculated using the angular momentum algebra. By diagonalisation we then obtain the energy spectrum of the system (Table 1).

Table 1. The spectrum of the quasispin model with $N = 144$ and $N = 192$, and the ground state quantum numbers $R + L = N/4$. All energies are in MeV.

n	Parity	$E - E_0$	ΔE	\bar{V}	$E - E_0$	ΔE	\bar{V}
		N=144	N=144	N=144	N=192	N=192	N=192
8	+	771	4	-11.3	861	59	-8.3
7	-	767	121	-8.8	802	93	-7.3
6	+	646	66	-11.4	709	98	-7.3
6	+	634		(-12.2)	655		(-10.9)
5	-	580	98	-10.0	611	108	-7.2
4	+	482	114	-10.5	503	115	-7.1
3	-	378	117	-10.1	388	122	-7.1
2	+	261	125	-10.3	266	129	-7.1
1	-	136	136		137	137	
0	+	0			0		

The salient features are

1. In the large N limit the exact results of our quasispin model tend in fact to the Hartree-Fock + RPA values which is a popular approximation for full NJL.
2. The spectrum of the "ground state band" (Table 1) is almost equidistant and can be interpreted as multipion states. The energy deficit can be assumed to be due to an attractive average pion-pion interaction: $E - E_0 = nm_\pi + \frac{1}{2}n(n-1)\bar{V}$.
3. This average potential is in fact proportional to the density of each pion, $\bar{V} \propto 1/\mathcal{V} \propto 1/N$.
4. The idea of an average pion-pion potential allows to calculate the pion-pion scattering length a in the first order Born approximation (Lüscher formula [6]) $am_\pi = \frac{m_\pi^2}{4\pi} \int V(\vec{r}) d^3r = \frac{m_\pi}{4\pi} \bar{V}\mathcal{V} = -0.077$ which is qualitatively consistent with the two-flavour experimental analysis of Lesniak, $am_\pi = -0.034$ or -0.044 ([7]).
5. The parity of multipion states alternates. There are, however, intruders which do not follow the alternation. In Table 1 they are written in boldface and the lowest can be interpreted as the σ meson (now called a(500)). Also the states around $n = 7$ may be perturbed by admixtures of $\sigma + \pi$.

3. Some lessons for lattice-like models

From the salient features of the two-level quasispin model we can learn a few lessons.

We get a ground-state band of almost equidistant discrete states with alternating 0^+ and 0^- which suggest multipion states. We expect that also in other lattice-like models the energy deficit with respect to equidistant values is due to an attractive average pion-pion interaction.

Even if we have only discrete states we can mimic scattering states by using the effective pion-pion interaction to calculate scattering amplitudes. Resonances such as σ meson can be recognized from irregularities of the discrete spectrum (the “intruders”).

In our simple model the results depend on the product $\mathcal{N} = n_h n_c n_f n_p$ and not on individual factors (the number of helicities, colours, flavours and momentum states). It is equivalent to have a large number of colours and poor resolution (small n_p) or viceversa. Alternatively, we get the same limit $N \rightarrow \infty$ whether we take the large N_c limit or a large box \mathcal{V} . This fact helps us to appreciate the meaning of the large N_c limit which suggests a good Hartree-Fock approximation and suppression of off-diagonal terms of full NJL Hamiltonian.

For the chosen model parameters and for $N = \mathcal{N} = 192$ the “size of the box” $B = \sqrt[3]{\mathcal{V}} = \sqrt[3]{\pi^2 \mathcal{N}} / \Lambda = 3.7$ fm. The size of the “lattice cell” $b = B / \sqrt[3]{\mathcal{N} / n_h n_c n_f} = B / 32^{1/3} = 1.2$ fm. We see that B is only about three times larger than b , nevertheless the model works well. The explanation is that in one dimension with the same n_p , B would be 32 times larger than b which is nice. Since the Hamiltonian is not very sensitive on the number of dimensions, the momenta $p(i)$ act only as “house numbers” and there are no spacial correlations, the quality of the threedimensional solution is equally good. This is a general feature of Nambu – Jona-lasinio models.

The convergence and the quality of the results in our model seems very good for $N = 194$ but not so good for $N = 144$ indicating the critical number of particles and the corresponding B/b ratio.

4. The width of the sigma meson

In the spectrum in Table 1 one can clearly distinguish the presence of the sigma meson by noticing the doubling of the positive parity states at 634 and 646 MeV for $N = 144$ (655 and 709 MeV for $N = 192$). Moreover, the states at 646 MeV (655 MeV) indicated in boldface have strong one-body transition matrix elements from the ground state.

For its width we are trying to get the complex pole. For that purpose, we explore the method of analytic continuation from the bound state [8]. For this purpose, we vary the bare quark mass m from the region where the σ meson would be bound ($E_\sigma < E_{2\pi}$) down to the physical value of $m \rightarrow m_0$ (where $E_\sigma \gg E_{2\pi}$). The method consists of the following steps:

- Determine the threshold value m_{th} and calculate $\epsilon = E_\sigma - E_{2\pi}$ as a function of m for $m > m_{\text{th}}$.
- Introduce a variable $x = \sqrt{m - m_{\text{th}}}$; calculate $k(x) = i\sqrt{-\epsilon}$ in the bound state region.
- Fit $k(x)$ by a polynomial $k(x) = i(c_0 + c_1 x + c_2 x^2 + \dots + c_M x^{2M})$.
- Construct a Padé approximant: $k(x) = i \frac{a_0 + a_1 x + \dots + a_M x^M}{1 + b_1 x + \dots + b_M x^M}$.
- Analytically continue $k(x)$ to the region $m < m_{\text{th}}$ (i.e. to imaginary x) where $k(x)$ becomes complex.
- Determine the position and the width of the resonance as:
 $E_{\text{res}} = \text{Re} [k^2(m \rightarrow m_0)]$, $\Gamma = -2 \text{Im} [k^2(m \rightarrow m_0)]$.

We notice that the results for E_{res} and Γ in Table 2 deviate strongly for first and second order Padé approximants. This is due to the large stretch for the analytic continuation so that convergence at higher orders cannot be expected. Nevertheless, it is rewarding that the physical values for E_{res} and Γ lie somewhere in the middle between both curves. Intentionally, we have plotted the energy and width of the σ meson as a function of the corresponding pion mass rather than as a function of the model parameter m . This is reminiscent of the extrapolation of pion mass from about 500 MeV towards its physical value in typical lattice calculations.

Table 2. The resonance energy E_{res} and the width Γ of the σ meson as a function of the pion mass - extrapolation using Padé approximants, in [MeV].

pion mass	136	180	254	355	433	499
$E_{res}(\text{order 1})$	779	840	914	959	964	959
$E_{res}(\text{order 2})$	538	613	724	853	925	959
$\Gamma(\text{order 1})$	240	220	178	100	36	0
$\Gamma(\text{order 1})$	940	818	576	242	64	0

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