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Posted Date: 27 July 2023

doi: 10.20944/preprints202307.1844.v1

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Article

Continuous Deutsch Uncertainty Principle and Continuous Kraus Conjecture

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Abstract: Let (Ω,μ), (Δ,ν) be measure spaces and $\{\tau_\alpha\}_{\alpha\in\Omega}$, $\{\omega_\beta\}_{\beta\in\Delta}$ be 1-bounded continuous Parseval frames for a Hilbert space $\mathcal{H}$. Then we show that (1) $\log(\mu(\Omega)\nu(\Delta)) \geq S_\tau(h)+S_\omega(h) \geq -2\log\left(\frac{1+\sup_{\alpha\in\Omega,\beta\in\Delta}|\langle\tau_\alpha,\omega_\beta\rangle|}{2}\right)$, $\forall h \in \mathcal{H}_\tau\cap\mathcal{H}_\omega$, where $\mathcal{H}_\tau := \{h_1 \in \mathcal{H} : \langle h_1,\tau_\alpha\rangle \neq 0, \alpha \in \Omega\}$, $\mathcal{H}_\omega := \{h_2 \in \mathcal{H} : \langle h_2,\omega_\beta\rangle \neq 0, \beta \in \Delta\}$, $S_\tau(h) := -\int_\Omega \left|\left\langle \frac{h}{\|h\|},\tau_\alpha\right\rangle\right|^2 \log\left|\left\langle \frac{h}{\|h\|},\tau_\alpha\right\rangle\right|^2 d\mu(\alpha)$, $\forall h \in \mathcal{H}_\tau$, $S_\omega(h) := -\int_\Delta \left|\left\langle \frac{h}{\|h\|},\omega_\beta\right\rangle\right|^2 \log\left|\left\langle \frac{h}{\|h\|},\omega_\beta\right\rangle\right|^2 d\nu(\beta)$, $\forall h \in \mathcal{H}_\omega$. We call Inequality (1) as **Continuous Deutsch Uncertainty Principle**. Inequality (1) improves the uncertainty principle obtained by Deutsch *[Phys. Rev. Lett., 1983]*. We formulate Kraus conjecture for 1-bounded continuous Parseval frames. We also derive continuous Deutsch uncertainty principles for Banach spaces.

Keywords: uncertainty principle; Hilbert space; frame; Banach space; Deutsch uncertainty; Kraus conjecture

MSC: 42C15

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1. Introduction

Let $\mathcal{H}$ be a finite dimensional Hilbert space. Given an orthonormal basis $\{\omega_j\}_{j=1}^n$ for $\mathcal{H}$, the **(finite) Shannon entropy** at a point $h \in \mathcal{H}_\tau$ is defined as

$$S_\tau(h) := -\sum_{j=1}^{n}\left|\left\langle \frac{h}{\|h\|},\tau_j\right\rangle\right|^2 \log\left|\left\langle \frac{h}{\|h\|},\tau_j\right\rangle\right|^2 \geq 0,$$

where $\mathcal{H}_\tau := \{h \in \mathcal{H} : \langle h,\tau_j\rangle \neq 0, 1 \leq j \leq n\}$. In 1983, Deutsch derived following uncertainty principle for Shannon entropy [3].

Theorem 1.1. *(**Deutsch Uncertainty Principle**) [3] Let $\{\tau_j\}_{j=1}^n$, $\{\omega_j\}_{j=1}^n$ be two orthonormal bases for a finite dimensional Hilbert space $\mathcal{H}$. Then*

$$2\log n \geq S_\tau(h)+S_\omega(h) \geq -2\log\left(\frac{1+\max_{1\leq j,k\leq n}|\langle\tau_j,\omega_k\rangle|}{2}\right) \geq 0, \quad \forall h \in \mathcal{H}_\tau\cap\mathcal{H}_\omega. \qquad (2)$$

In 1988, followed by a conjecture of Kraus [9] made in 1987, Maassen and Uffink improved Inequality (2) [12].

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Theorem 1.2. *(**Kraus Conjecture/Maassen-Uffink Uncertainty Principle**) [9,12] Let $\{\tau_j\}_{j=1}^n$, $\{\omega_j\}_{j=1}^n$ be two orthonormal bases for a finite dimensional Hilbert space $\mathcal{H}$. Then*

$$2\log n \geq S_\tau(h) + S_\omega(h) \geq -2\log\left(\max_{1\leq j,k\leq n} |\langle \tau_j, \omega_k\rangle|\right) \geq 0, \quad \forall h \in \mathcal{H}_\tau \cap \mathcal{H}_\omega.$$

In 2013, Ricaud and Torrésani [13] showed that Theorem 1.2 holds for Parseval frames.

Theorem 1.3. *(**Maassen-Uffink-Ricaud-Torrésani Uncertainty Principle**) [13] Let $\{\tau_j\}_{j=1}^n$, $\{\omega_j\}_{j=1}^n$ be two Parseval frames for a finite dimensional Hilbert space $\mathcal{H}$. Then*

$$2\log n \geq S_\tau(h) + S_\omega(h) \geq -2\log\left(\max_{1\leq j,k\leq n} |\langle \tau_j, \omega_k\rangle|\right) \geq 0, \quad \forall h \in \mathcal{H}_\tau \cap \mathcal{H}_\omega.$$

Recently, Banach space versions of Deutsch uncertainty principle have been derived in [11]. To formulate them, we need some notions. Given a Parseval p-frame $\{f_j\}_{j=1}^n$ for $\mathcal{X}$, we define the **(finite) p-Shannon entropy** at a point $x \in \mathcal{X}_f$ as

$$S_f(x) := -\sum_{j=1}^n \left|f_j\left(\frac{x}{\|x\|}\right)\right|^p \log\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p \geq 0,$$

where $\mathcal{X}_f := \{x \in \mathcal{X} : f_j(x) \neq 0, 1 \leq j \leq n\}$. On the other way, given a Parseval p-frame $\{\tau_j\}_{j=1}^n$ for $\mathcal{X}^*$, we define the **(finite) p-Shannon entropy** at a point $f \in \mathcal{X}_\tau^*$ as

$$S_\tau(f) := -\sum_{j=1}^n \left|\frac{f(\tau_j)}{\|f\|}\right|^p \log\left|\frac{f(\tau_j)}{\|f\|}\right|^p \geq 0,$$

where $\mathcal{X}_\tau^* := \{f \in \mathcal{X}^* : f(\tau_j) \neq 0, 1 \leq j \leq n\}$.

Theorem 1.4. *[11] (**Functional Deutsch Uncertainty Principle**) Let $\{f_j\}_{j=1}^n$ and $\{g_k\}_{k=1}^m$ be Parseval p-frames for a finite dimensional Banach space $\mathcal{X}$. Then*

$$\frac{1}{(nm)^{\frac{1}{p}}} \leq \sup_{y\in\mathcal{X}, \|y\|=1}\left(\max_{1\leq j\leq n, 1\leq k\leq m} |f_j(y)g_k(y)|\right)$$

and

$$\log(nm) \geq S_f(x) + S_g(x) \geq -p\log\left(\sup_{y\in\mathcal{X}_f\cap\mathcal{X}_g, \|y\|=1}\left(\max_{1\leq j\leq n, 1\leq k\leq m} |f_j(y)g_k(y)|\right)\right) > 0, \quad \forall x \in \mathcal{X}_f \cap \mathcal{X}_g.$$

Theorem 1.5. *(**Functional Deutsch Uncertainty Principle**) Let $\{\tau_j\}_{j=1}^n$ and $\{\omega_k\}_{k=1}^m$ be two Parseval p-frames for the dual $\mathcal{X}^*$ of a finite dimensional Banach space $\mathcal{X}$. Then*

$$\frac{1}{(nm)^{\frac{1}{p}}} \leq \sup_{g\in\mathcal{X}^*, \|g\|=1}\left(\max_{1\leq j\leq n, 1\leq k\leq m} |g(\tau_j)g(\omega_k)|\right)$$

and

$$\log(nm) \geq S_\tau(f) + S_\omega(f) \geq -p\log\left(\sup_{g\in\mathcal{X}_\tau^*\cap\mathcal{X}_\omega^*, \|g\|=1}\left(\max_{1\leq j\leq n, 1\leq k\leq m} |g(\tau_j)g(\omega_k)|\right)\right) > 0, \quad \forall f \in \mathcal{X}_\tau^* \cap \mathcal{X}_\omega^*.$$

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In this paper, we derive continuous versions of Theorem 1.1, Theorem 1.4 and Theorem 1.5. We also formulate a conjecture based on Theorem 1.2. We wish to say that functional continuous uncertainty principles are derived in [10].

2. Continuous Deutsch Uncertainty Principle and Continuous Kraus Conjecture

In the paper, $\mathbb{K}$ denotes $\mathbb{C}$ or $\mathbb{R}$ and $\mathcal{H}$ (resp. $\mathcal{X}$) denotes a Hilbert space (resp. Banach space) (need not be finite dimensional) over $\mathbb{K}$. We use (Ω, μ) to denote a measure space. Continuous frames are introduced independently by Ali, Antoine and Gazeau [1] and Kaiser [8]. In the paper, $\mathbb{K}$ denotes $\mathbb{C}$ or $\mathbb{R}$ and $\mathcal{H}$ denotes a finite dimensional Hilbert space.

Definition 2.1. *[1,8] Let (Ω, μ) be a measure space. A collection $\{\tau_\alpha\}_{\alpha\in\Omega}$ in a Hilbert space $\mathcal{H}$ is said to be a **continuous Parseval frame** for $\mathcal{H}$ if the following conditions hold.*

(i) For each $h \in \mathcal{H}$, the map $\Omega \ni \alpha \mapsto \langle h, \tau_\alpha\rangle \in \mathbb{K}$ is measurable.
(ii)

$$\|h\|^2 = \int\limits_{\Omega} |\langle h, \tau_\alpha\rangle|^2 \, d\mu(\alpha), \quad \forall h \in \mathcal{H}.$$

We consider the following subclass of continuous Parseval frames.

Definition 2.2. *A continuous Parseval frame $\{\tau_\alpha\}_{\alpha\in\Omega}$ for $\mathcal{H}$ is said to be **1-bounded** if*

$$\|\tau_\alpha\| \leq 1, \quad \forall \alpha \in \Omega.$$

Note that if $\{\tau_j\}_{j=1}^n$ is a Parseval frame for a Hilbert space $\mathcal{H}$, then $\|\tau_j\| \leq 1$, $\forall 1 \leq j \leq n$ (see Remark 3.12 in [7]). We are unable to derive this for continuous frames. Given a continuous 1-bounded Parseval frame $\{\tau_\alpha\}_{\alpha\in\Omega}$ for $\mathcal{H}$, we define the **continuous Shannon entropy** at a point $h \in \mathcal{H}_\tau$ as

$$S_\tau(h) := -\int\limits_{\Omega} \left|\left\langle \frac{h}{\|h\|}, \tau_\alpha \right\rangle\right|^2 \log \left|\left\langle \frac{h}{\|h\|}, \tau_\alpha \right\rangle\right|^2 d\mu(\alpha) \geq 0,$$

where $\mathcal{H}_\tau := \{h \in \mathcal{H} : \langle h, \tau_\alpha\rangle \neq 0, \alpha \in \Omega\}$. Following is the first fundamental result of this paper.

Theorem 2.3. *(**Continuous Deutsch Uncertainty Principle**) Let (Ω, μ), (Δ, ν) be measure spaces and $\{\tau_\alpha\}_{\alpha\in\Omega}$, $\{\omega_\beta\}_{\beta\in\Delta}$ be 1-bounded continuous Parseval frames for a Hilbert space $\mathcal{H}$. Then*

$$\log(\mu(\Omega)\nu(\Delta)) \geq S_\tau(h) + S_\omega(h) \geq -2\log\left(\frac{1 + \sup\limits_{\alpha\in\Omega, \beta\in\Delta} |\langle \tau_\alpha, \omega_\beta\rangle|}{2}\right) \geq 0, \quad \forall h \in \mathcal{H}_\tau \cap \mathcal{H}_\omega. \tag{3}$$

Proof. Since $1 = \int\limits_{\Omega} \left|\left\langle \frac{h}{\|h\|}, \tau_\alpha \right\rangle\right|^2 d\mu(\alpha)$ for all $h \in \mathcal{H} \setminus \{0\}$, $1 = \int\limits_{\Delta} \left|\left\langle \frac{h}{\|h\|}, \omega_\beta \right\rangle\right|^2 d\nu(\beta)$ for all $h \in \mathcal{H} \setminus \{0\}$ and log is concave, using Jensen's inequality (cf. [6]) we get

$$\begin{aligned}
S_\tau(h) + S_\omega(h) &= \int\limits_{\Omega} \left|\left\langle \frac{h}{\|h\|}, \tau_\alpha \right\rangle\right|^2 \log\left(\frac{1}{\left|\left\langle \frac{h}{\|h\|}, \tau_\alpha \right\rangle\right|^2}\right) d\mu(\alpha) + \int\limits_{\Delta} \left|\left\langle \frac{h}{\|h\|}, \omega_\beta \right\rangle\right|^2 \log\left(\frac{1}{\left|\left\langle \frac{h}{\|h\|}, \omega_\beta \right\rangle\right|^2}\right) d\nu(\beta) \\
&\leq \log\left(\int\limits_{\Omega} \left|\left\langle \frac{h}{\|h\|}, \tau_\alpha \right\rangle\right|^2 \frac{1}{\left|\left\langle \frac{h}{\|h\|}, \tau_\alpha \right\rangle\right|^2} d\mu(\alpha)\right) + \log\left(\int\limits_{\Delta} \left|\left\langle \frac{h}{\|h\|}, \omega_\beta \right\rangle\right|^2 \frac{1}{\left|\left\langle \frac{h}{\|h\|}, \omega_\beta \right\rangle\right|^2} d\nu(\beta)\right) \\
&= \log(\mu(\Omega)) + \log\nu(\Delta)) = \log(\mu(\Omega)\nu(\Delta)), \quad \forall h \in \mathcal{H}_\tau \cap \mathcal{H}_\omega.
\end{aligned}$$

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Let $h \in \mathcal{H}_\tau \cap \mathcal{H}_\omega$. Then using Buzano inequality [2,5] we get

$$\begin{aligned}
S_\tau(h)+S_\omega(h) &= -\int\limits_{\Delta}\int\limits_{\Omega}\left|\left\langle \frac{h}{\|h\|},\tau_\alpha\right\rangle\right|^2\left|\left\langle \frac{h}{\|h\|},\omega_\beta\right\rangle\right|^2\left[\log\left|\left\langle \frac{h}{\|h\|},\tau_\alpha\right\rangle\right|^2+\log\left|\left\langle \frac{h}{\|h\|},\omega_\beta\right\rangle\right|^2\right]d\mu(\alpha)\,d\nu(\beta)\\
&= -\int\limits_{\Delta}\int\limits_{\Omega}\left|\left\langle \frac{h}{\|h\|},\tau_\alpha\right\rangle\right|^2\left|\left\langle \frac{h}{\|h\|},\omega_\beta\right\rangle\right|^2\log\left|\left\langle \frac{h}{\|h\|},\tau_\alpha\right\rangle\left\langle \frac{h}{\|h\|},\omega_\beta\right\rangle\right|^2 d\mu(\alpha)\,d\nu(\beta)\\
&= -2\int\limits_{\Delta}\int\limits_{\Omega}\left|\left\langle \frac{h}{\|h\|},\tau_\alpha\right\rangle\right|^2\left|\left\langle \frac{h}{\|h\|},\omega_\beta\right\rangle\right|^2\log\left|\left\langle \frac{h}{\|h\|},\tau_\alpha\right\rangle\left\langle \frac{h}{\|h\|},\omega_\beta\right\rangle\right| d\mu(\alpha)\,d\nu(\beta)\\
&\geq -2\int\limits_{\Delta}\int\limits_{\Omega}\left|\left\langle \frac{h}{\|h\|},\tau_\alpha\right\rangle\right|^2\left|\left\langle \frac{h}{\|h\|},\omega_\beta\right\rangle\right|^2\log\left(\left\|\frac{h}{\|h\|}\right\|^2\frac{\|\tau_\alpha\|\|\omega_\beta\|+|\langle\tau_\alpha,\omega_\beta\rangle|}{2}\right)d\mu(\alpha)\,d\nu(\beta)\\
&= -2\int\limits_{\Delta}\int\limits_{\Omega}\left|\left\langle \frac{h}{\|h\|},\tau_\alpha\right\rangle\right|^2\left|\left\langle \frac{h}{\|h\|},\omega_\beta\right\rangle\right|^2\log\left(\frac{1+|\langle\tau_\alpha,\omega_\beta\rangle|}{2}\right)d\mu(\alpha)\,d\nu(\beta)\\
&\geq -2\int\limits_{\Delta}\int\limits_{\Omega}\left|\left\langle \frac{h}{\|h\|},\tau_\alpha\right\rangle\right|^2\left|\left\langle \frac{h}{\|h\|},\omega_\beta\right\rangle\right|^2\log\left(\frac{1+\sup\limits_{\alpha\in\Omega,\beta\in\Delta}|\langle\tau_\alpha,\omega_\beta\rangle|}{2}\right)d\mu(\alpha)\,d\nu(\beta)\\
&= -2\log\left(\frac{1+\sup\limits_{\alpha\in\Omega,\beta\in\Delta}|\langle\tau_\alpha,\omega_\beta\rangle|}{2}\right)\int\limits_{\Delta}\int\limits_{\Omega}\left|\left\langle \frac{h}{\|h\|},\tau_\alpha\right\rangle\right|^2\left|\left\langle \frac{h}{\|h\|},\omega_\beta\right\rangle\right|^2 d\mu(\alpha)\,d\nu(\beta)\\
&\geq -2\log\left(\frac{1+\sup\limits_{\alpha\in\Omega,\beta\in\Delta}|\langle\tau_\alpha,\omega_\beta\rangle|}{2}\right).
\end{aligned}$$

□

Theorem 2.3 promotes following question.

Question 2.4. *Let (Ω,μ), (Δ,ν) be measure spaces, $\mathcal{H}$ be a Hilbert space. For which pairs of 1-bounded continuous Parseval frames $\{\tau_\alpha\}_{\alpha\in\Omega}$ and $\{\omega_\beta\}_{\beta\in\Delta}$ for $\mathcal{H}$, we have equality in Inequality (3)?*

Based on Theorems 1.3 and Theorem 2.3 we formulate following conjecture.

Conjecture 2.5. ***(Continuous Kraus Conjecture) Let (Ω,μ), (Δ,ν) be measure spaces and $\{\tau_\alpha\}_{\alpha\in\Omega}$, $\{\omega_\beta\}_{\beta\in\Delta}$ be 1-bounded continuous Parseval frames for a Hilbert space $\mathcal{H}$. Then***

$$S_\tau(h)+S_\omega(h)\geq -2\log\left(\sup_{\alpha\in\Omega,\beta\in\Delta}|\langle\tau_\alpha,\omega_\beta\rangle|\right)\geq 0,\quad \forall h\in\mathcal{H}_\tau\cap\mathcal{H}_\omega.$$

Next we derive continuous Deutsch uncertainty for Banach spaces. We need a definition.

Definition 2.6. *[4] Let (Ω,μ) be a measure space and $\mathcal{X}$ be a Banach space over $\mathbb{K}$. A collection $\{f_\alpha\}_{\alpha\in\Omega}$ in $\mathcal{X}^*$ is said to be a **continuous Parseval p-frame** ($1\leq p<\infty$) for $\mathcal{X}$ if the following conditions hold.*

(i) *For each $x\in\mathcal{X}$, the map $\Omega\ni\alpha\mapsto f_\alpha(x)\in\mathbb{K}$ is measurable.*
(ii)

$$\|x\|^p=\int\limits_{\Omega}|f_\alpha(x)|^p\,d\mu(\alpha),\quad \forall x\in\mathcal{X}.$$

If $\|\tau_\alpha\|\leq 1$, $\forall\alpha\in\Omega$, then we say that the frame $\{f_\alpha\}_{\alpha\in\Omega}$ is 1-bounded.

Similar to Definition 2.2, we set the following.

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Definition 2.7. *A continuous Parseval p-frame $\{f_\alpha\}_{\alpha\in\Omega}$ for $\mathcal{X}$ is said to be* ***1-bounded*** *if*

$$\|f_\alpha\| \leq 1, \quad \forall \alpha \in \Omega.$$

Given a 1-bounded continuous Parseval p-frame $\{f_\alpha\}_{\alpha\in\Omega}$ for $\mathcal{X}$, we define the **continuous p-Shannon entropy** at a point $x \in \mathcal{X}_f$ as

$$S_f(x) := -\int\limits_{\Omega} \left|f_\alpha\left(\frac{x}{\|x\|}\right)\right|^p \log\left|f_\alpha\left(\frac{x}{\|x\|}\right)\right|^p d\mu(\alpha) \geq 0,$$

where $\mathcal{X}_f := \{x \in \mathcal{X} : f_\alpha(x) \neq 0, \alpha \in \Omega\}$.

Theorem 2.8. *(**Functional Continuous Deutsch Uncertainty Principle**) Let (Ω,μ), (Δ,ν) be measure spaces and $\{f_\alpha\}_{\alpha\in\Omega}$, $\{g_\beta\}_{\beta\in\Delta}$ be 1-bounded continuous Parseval p-frames for a Banach space $\mathcal{X}$. Then*

$$\frac{1}{(\mu(\Omega)\nu(\Delta))^{\frac{1}{p}}} \leq \sup_{y\in\mathcal{X},\|y\|=1}\left(\sup_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(y)g_\beta(y)|\right)$$

and

$$S_f(x) + S_g(x) \geq -p\log\left(\sup_{y\in\mathcal{X},\|y\|=1}\left(\sup_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(y)g_\beta(y)|\right)\right) \geq 0, \quad \forall x \in \mathcal{X}_f \cap \mathcal{X}_g. \tag{4}$$

Proof. Let $z \in \mathcal{X}$ be such that $\|z\| = 1$. Then

$$\begin{aligned}
1 &= \left(\int\limits_{\Omega} |f_\alpha(z)|^p\, d\mu(\alpha)\right)\left(\int\limits_{\Delta} |g_\beta(z)|^p\, d\nu(\beta)\right) = \int\limits_{\Omega}\int\limits_{\Delta} |f_\alpha(z)g_\beta(z)|^p\, d\nu(\beta)\, d\mu(\alpha) \\
&\leq \int\limits_{\Omega}\int\limits_{\Delta}\left(\sup_{y\in\mathcal{X},\|y\|=1}\left(\sup_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(y)g_\beta(y)|\right)\right)^p d\nu(\beta)\, d\mu(\alpha) \\
&= \left(\sup_{y\in\mathcal{X},\|y\|=1}\left(\sup_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(y)g_\beta(y)|\right)\right)^p \mu(\Omega)\nu(\Delta)
\end{aligned}$$

which gives

$$\frac{1}{\mu(\Omega)\nu(\Delta)} \leq \left(\sup_{y\in\mathcal{X},\|y\|=1}\left(\sup_{\alpha\in\Omega,\beta\in\Delta} |f_\alpha(y)g_\beta(y)|\right)\right)^p.$$

Let $x \in \mathcal{X}_f \cap \mathcal{X}_g$. Then

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$$
\begin{aligned}
S_f(x)+S_g(x) &= -\int_{\Omega}\int_{\Delta}\left|f_\alpha\left(\frac{x}{\|x\|}\right)\right|^p\left|g_\beta\left(\frac{x}{\|x\|}\right)\right|^p\left[\log\left|f_\alpha\left(\frac{x}{\|x\|}\right)\right|^p+\log\left|g_\beta\left(\frac{x}{\|x\|}\right)\right|^p\right]d\nu(\beta)\,d\mu(\alpha)\\
&= -\int_{\Omega}\int_{\Delta}\left|f_\alpha\left(\frac{x}{\|x\|}\right)\right|^p\left|g_\beta\left(\frac{x}{\|x\|}\right)\right|^p\log\left|f_\alpha\left(\frac{x}{\|x\|}\right)g_\beta\left(\frac{x}{\|x\|}\right)\right|^p d\nu(\beta)\,d\mu(\alpha)\\
&= -p\int_{\Omega}\int_{\Delta}\left|f_\alpha\left(\frac{x}{\|x\|}\right)\right|^p\left|g_\beta\left(\frac{x}{\|x\|}\right)\right|^p\log\left|f_\alpha\left(\frac{x}{\|x\|}\right)g_\beta\left(\frac{x}{\|x\|}\right)\right| d\nu(\beta)\,d\mu(\alpha)\\
&\geq -p\int_{\Omega}\int_{\Delta}\left|f_\alpha\left(\frac{x}{\|x\|}\right)\right|^p\left|g_\beta\left(\frac{x}{\|x\|}\right)\right|^p\log\left(\sup_{y\in\mathcal{X},\|y\|=1}\left(\sup_{\alpha\in\Omega,\beta\in\Delta}|f_\alpha(y)g_\beta(y)|\right)\right)d\nu(\beta)\,d\mu(\alpha)\\
&= -p\log\left(\sup_{y\in\mathcal{X},\|y\|=1}\left(\sup_{\alpha\in\Omega,\beta\in\Delta}|f_\alpha(y)g_\beta(y)|\right)\right)\int_{\Omega}\int_{\Delta}\left|f_\alpha\left(\frac{x}{\|x\|}\right)\right|^p\left|g_\beta\left(\frac{x}{\|x\|}\right)\right|^p d\nu(\beta)\,d\mu(\alpha)\\
&= -p\log\left(\sup_{y\in\mathcal{X},\|y\|=1}\left(\sup_{\alpha\in\Omega,\beta\in\Delta}|f_\alpha(y)g_\beta(y)|\right)\right).
\end{aligned}
$$

□

Corollary 2.9. *Theorem 1.1 follows from Theorem 2.8.*

Proof. Let (Ω,μ), (Δ,ν) be measure spaces and $\{\tau_\alpha\}_{\alpha\in\Omega}$, $\{\omega_\beta\}_{\beta\in\Delta}$ be 1-bounded continuous Parseval frames for a Hilbert space $\mathcal{H}$. Define

$$
f_\alpha:\mathcal{H}\ni h\mapsto\langle h,\tau_\alpha\rangle\in\mathbb{K};\quad\forall\alpha\in\Omega,
$$
$$
g_\beta:\mathcal{H}\ni h\mapsto\langle h,\omega_\beta\rangle\in\mathbb{K},\quad\forall\beta\in\Delta.
$$

Now by using Buzano inequality [2,5] we get

$$
\begin{aligned}
\sup_{h\in\mathcal{H},\|h\|=1}\left(\sup_{\alpha\in\Omega,\beta\in\Delta}|f_\alpha(h)g_\beta(h)|\right) &= \sup_{h\in\mathcal{H},\|h\|=1}\left(\sup_{\alpha\in\Omega,\beta\in\Delta}|\langle h,\tau_\alpha\rangle||\langle h,\omega_\beta\rangle|\right)\\
&\leq \sup_{h\in\mathcal{H},\|h\|=1}\left(\sup_{\alpha\in\Omega,\beta\in\Delta}\left(\|h\|^2\frac{\|\tau_\alpha\|\|\omega_\beta\|+|\langle\tau_\alpha,\omega_\beta\rangle|}{2}\right)\right)\\
&= \frac{1+\sup_{\alpha\in\Omega,\beta\in\Delta}|\langle\tau_j,\omega_k\rangle|}{2}.
\end{aligned}
$$

□

Theorem 2.8 brings the following question.

Question 2.10. *Let (Ω,μ), (Δ,ν) be measure spaces, $\mathcal{X}$ be a Banach space and $p>1$. For which pairs of continuous Parseval p-frames $\{f_\alpha\}_{\alpha\in\Omega}$, and $\{g_\beta\}_{\beta\in\Delta}$ for $\mathcal{X}$, we have equality in Inequality (4)?*

Next we derive a dual inequality of (4). For this we need dual of Definition 2.6.

Definition 2.11. *Let (Ω,μ) be a measure space and $\mathcal{X}$ be a Banach space over $\mathbb{K}$. A collection $\{\tau_\alpha\}_{\alpha\in\Omega}$ in $\mathcal{X}$ is said to be a* ***continuous Parseval p-frame*** *$(1\leq p<\infty)$ for $\mathcal{X}^*$ if the following conditions hold.*

(i) *For each $\phi\in\mathcal{X}^*$, the map $\Omega\ni\alpha\mapsto\phi(\tau_\alpha)\in\mathbb{K}$ is measurable.*
(ii)

$$
\|f\|^p=\int_{\Omega}|f(\tau_\alpha)|^p\,d\mu(\alpha),\quad\forall f\in\mathcal{X}^*.
$$

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f $\|\tau_\alpha\| \leq 1, \forall \alpha \in \Omega$*, then we say that the frame* $\{\tau_\alpha\}_{\alpha\in\Omega}$ *is 1-bounded.*

Given a 1-bounded continuous Parseval p-frame $\{\tau_\alpha\}_{\alpha\in\Omega}$ for $\mathcal{X}^*$, we define the **continuous p-Shannon entropy** at a point $f \in \mathcal{X}^*_\tau$ as

$$S_f(x) := -\int\limits_{\Omega} \left|\frac{f(\tau_\alpha)}{\|f\|}\right|^p \log\left|\frac{f(\tau_\alpha)}{\|f\|}\right|^p d\mu(\alpha) \geq 0,$$

where $\mathcal{X}^*_\tau := \{f \in \mathcal{X} : f(\tau_\alpha) \neq 0, \alpha \in \Omega\}\}$. We now have the following dual to Theorem 2.8.

Theorem 2.12. *(**Continuous Deutsch Uncertainty Principle for Banach spaces**) Let* (Ω,μ), (Δ,ν) *be measure spaces and* $\{\tau_\alpha\}_{\alpha\in\Omega}$, $\{\omega_\beta\}_{\beta\in\Delta}$ *be 1-bounded continuous Parseval p-frames for the dual* $\mathcal{X}^*$ *of a Banach space* $\mathcal{X}$*. Then*

$$\frac{1}{(\mu(\Omega)\nu(\Delta))^{\frac{1}{p}}} \leq \sup_{g\in\mathcal{X}^*,\|g\|=1}\left(\sup_{\alpha\in\Omega,\beta\in\Delta}|g(\tau_\alpha)g(\omega_\beta)|\right)$$

and

$$S_\tau(f) + S_\omega(f) \geq -p\log\left(\sup_{g\in\mathcal{X}^*,\|g\|=1}\left(\sup_{\alpha\in\Omega,\beta\in\Delta}|g(\tau_\alpha)g(\omega_\beta)|\right)\right) \geq 0, \quad \forall f \in \mathcal{X}^*_\tau \cap \mathcal{X}^*_\omega. \tag{5}$$

Proof. Let $h \in \mathcal{X}^*$ be such that $\|h\| = 1$. Then

$$\begin{aligned}1 &= \left(\int\limits_{\Omega}|h(\tau_\alpha)|^p\,d\mu(\alpha)\right)\left(\int\limits_{\Delta}h(\omega_\beta)|^p\,d\nu(\beta)\right) = \int\limits_{\Omega}\int\limits_{\Delta}|h(\tau_\alpha)h(\omega_\beta)|^p\,d\nu(\beta)\,d\mu(\alpha)\\ &\leq \int\limits_{\Omega}\int\limits_{\Delta}\left(\sup_{g\in\mathcal{X}^*,\|g\|=1}\left(\sup_{\alpha\in\Omega,\beta\in\Delta}|g(\tau_\alpha)g(\omega_\beta)|\right)\right)^p d\nu(\beta)\,d\mu(\alpha)\\ &= \left(\sup_{g\in\mathcal{X}^*,\|g\|=1}\left(\sup_{\alpha\in\Omega,\beta\in\Delta}|g(\tau_\alpha)g(\omega_\beta)|\right)\right)^p \mu(\Omega)\nu(\Delta)\end{aligned}$$

which gives

$$\frac{1}{\mu(\Omega)\nu(\Delta)} \leq \left(\sup_{g\in\mathcal{X}^*,\|g\|=1}\left(\sup_{\alpha\in\Omega,\beta\in\Delta}|g(\tau_\alpha)g(\omega_\beta)|\right)\right)^p.$$

Let $f \in \mathcal{X}^*_\tau \cap \mathcal{X}^*_\omega$. Then

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$$
\begin{aligned}
S_\tau(f)+S_\omega(f) &= -\int_\Omega\int_\Delta \left|\frac{f(\tau_\alpha)}{\|f\|}\right|^p \left|\frac{f(\omega_\beta)}{\|f\|}\right|^p \left[\log\left|\frac{f(\tau_\alpha)}{\|f\|}\right|^p + \log\left|\frac{f(\omega_\beta)}{\|f\|}\right|^p\right] d\nu(\beta)\, d\mu(\alpha) \\
&= -\int_\Omega\int_\Delta \left|\frac{f(\tau_\alpha)}{\|f\|}\right|^p \left|\frac{f(\omega_\beta)}{\|f\|}\right|^p \log\left|\frac{f(\tau_\alpha)}{\|f\|}\frac{f(\omega_\beta)}{\|f\|}\right|^p d\nu(\beta)\, d\mu(\alpha) \\
&= -p\int_\Omega\int_\Delta \left|\frac{f(\tau_\alpha)}{\|f\|}\right|^p \left|\frac{f(\omega_\beta)}{\|f\|}\right|^p \log\left|\frac{f(\tau_\alpha)}{\|f\|}\frac{f(\omega_\beta)}{\|f\|}\right| d\nu(\beta)\, d\mu(\alpha) \\
&\geq -p\int_\Omega\int_\Delta \left|\frac{f(\tau_\alpha)}{\|f\|}\right|^p \left|\frac{f(\omega_\beta)}{\|f\|}\right|^p \log\left(\sup_{g\in\mathcal{X}^*,\|g\|=1}\left(\sup_{\alpha\in\Omega,\beta\in\Delta}|g(\tau_\alpha)g(\omega_\beta)|\right)\right) d\nu(\beta)\, d\mu(\alpha) \\
&= -p\log\left(\sup_{g\in\mathcal{X}^*,\|g\|=1}\left(\sup_{\alpha\in\Omega,\beta\in\Delta}|g(\tau_\alpha)g(\omega_\beta)|\right)\right)\int_\Omega\int_\Delta \left|\frac{f(\tau_\alpha)}{\|f\|}\right|^p \left|\frac{f(\omega_\beta)}{\|f\|}\right|^p d\nu(\beta)\, d\mu(\alpha) \\
&= -p\log\left(\sup_{g\in\mathcal{X}^*,\|g\|=1}\left(\sup_{\alpha\in\Omega,\beta\in\Delta}|g(\tau_\alpha)g(\omega_\beta)|\right)\right).
\end{aligned}
$$

□

Theorem 2.12 again gives the following question.

Question 2.13. *Let (Ω,μ), (Δ,ν) be measure spaces, $\mathcal{X}$ be a Banach space and $p>1$. For which pairs of continuous Parseval p-frames $\{\tau_\alpha\}_{\alpha\in\Omega}$, and $\{\omega_\beta\}_{\beta\in\Delta}$ for $\mathcal{X}^*$, we have equality in Inequality (5)*

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