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Stochastic Orderings of the Idle time of Inactive Standby Systems

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Abstract: In this paper we consider a failed cold standby systems and obtain some stochastic bounds on the idle time of such systems. It is assumed that, at a time t the failure of a cold standby system is observed, but the actual time of the failure is a time before t. The idle time of the inactive cold standby system is measured as the time between the exact time of failure, which is a random variable, and time t. We use the idle times of the components of the system to obtain lower and upper bounds on the lifetime of the standby system to which they are assembled. We bring some examples to show that the results and the conditions imposed to get them are achievable.

Key words and phrases: cold-standby systems; reliability; stochastic order, likelihood ratio order; inactivity time

MSC subject classication: 62N05, 90B25, 62E10, 65C20

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1. Introduction and Preliminaries

The lifespan of cold standby systems can be described as longer than other types of backup systems because they are not actively operated and are therefore subject to less wear and tear. This can result in cost savings for organizations because they do not need to replace or upgrade the system as frequently. However, the life of the system can also be affected by factors such as the quality of the hardware and software used, maintenance practices, and frequency of use. Cold standby systems are commonly used in reliability to support critical systems and processes. Some applications of cold standby systems in reliability include:

1) Data backup and recovery: cold standby systems can be used to store backup data and ensure its availability in the event of a system failure or malfunction.

2) Emergency power supply: Cold standby systems can be used as a backup power supply in the event of a power failure or malfunction to ensure that critical systems remain operational.

3) Network redundancy: Cold standby systems can be used as network redundancy to ensure that critical network services remain available in the event of a network outage or failure.

4) Server redundancy: Cold standby systems can be used as server redundancy to ensure that critical applications and services remain available in the event of a server failure or defect.

5) Disaster recovery: Cold standby systems can be used as part of a disaster recovery plan to provide backup support for critical systems and processes in the event of a natural disaster or other catastrophic event.

Overall, cold standby systems are an important tool for ensuring the reliability and availability of critical systems and processes. They can help organizations minimize downtime and maintain continuity in the event of unexpected events. Cold-standby systems have recently attracted the attention of many researchers in the field of reliability enginerring (see, e.g., Wang and Ye [23], Ramezani Dobani et al. [19], Danjuma et al. [5], Malhotra et al. [16] and Lin et al. [15]).

In contrast to the residual life of a system, one of the aspects of engineering systems is their idle time. The idle time is also called the inactivity time or reversed residual lifetime. Stochastic properties of coherent systems, a large and well-known class of systems in reliability, in view of their idle time

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have been studied during the past decades (see, for instance, Bayramoglu and Ozkut [3], Zhang and Balakrishnan [24], Navarro et al. [17], Kayid et al. [10], Navarro and Calì [18], Salehi and Tavangar [20], Toomaj and Di Crescenzo [22], Amini-Seresht et al. [2], Guo et al. [7] and Kayid and Shrahili [11]).

Recently, Kayid and Alshehri [12] developed stochastic comparisons between the lifetime of a used cold standby system (which is still functioning) and sum of the residual lifetimes of the components in the system. Here in this paper we consider idle time of a failed cold standby system to obtain some lower/upper bound for it in terms of the idle time of the failed components in the system. As pointed out and clearly shown by Ahmad et al. [1], the results for the idle variable (the inactivity time) cannot be concluded from the similar results on the residual lifetime variable. This is the reason why we start our study in this paper. Another reason is that the properties that we obtain in this paper are quite different and cannot be acquired from the results in Kayid and Alshehri [12]. An interesting point is that in general the uncertainty in the past lifetime of an inactive coherent system is less than the uncertainty in the future lifetime and especially the remaining lifetime of a functioning coherent system. This is because the idle variable takes values at $[0, t)$ but the residual lifetime variable takes values at $(t, +\infty)$, so the former is obviously more predictable than the latter. Therefore, the problem of finding stochastic bounds on the idle time of a cold standby system have a complement role to refine our knowledge about the lifetime of the system.

In the continuing part of the paper several notations are used. Let $\underset{\sim}{X} = (X_1, \ldots, X_n)$ be a random vector and denote $S_n = \sum_{i=1}^{n} X_i$ and Consider a cold standby system comprising of n components. Firstly, one component begins to work and the residual $n-1$ components are standby and ready to lie in the system. Upon the failure of the first component, the components that are in queue in standby mode are replaced one by one until all components become inactive and, thus, the cold standby system fails. Suppose that $X_1, \ldots, X_n$ represent the lifetimes of the n described components having cumulative distribution functions (CDF's) $F_{X_1}, \ldots, F_{X_n}$. It will be assumed that $X_1, \ldots, X_n$ are independent. The lifetime of the cold standby system is then identified as

$$S_n = X_1 + X_2 + \ldots + X_n.$$

The CDF of the lifetime of the cold standby system is

$$F_{S_n}(t) = P(X_1 + X_2 + \ldots + X_n \leq t) = F_{X_1}(t) * F_{X_2}(t) * \ldots * F_{X_n}(t), \tag{1}$$

in which $*$ stands for the convolution operator. It is acknowledged that when X_i and X_j, for $i \neq j$ are independent, then $F_{X_i}(t) * F_{X_j}(t) = \int_{-\infty}^{+\infty} F_{X_i}(t-x) f_{X_j}(x)\, dx$, where f_{X_j} is the probability density function (PDF) of X_j, which is the CDF of the convolution of X_i and X_j, that is the CDF of $X_i + X_j$. From (1), we can develop that

$$F_{S_n}(t) = F_{S_{n-1}}(t) * F_{X_n}(t) = \int_{-\infty}^{+\infty} F_{S_{n-1}}(t-x) f_{X_n}(x)\, dx. \tag{2}$$

Suppose X denotes the life length of a lifetime organism. We assume that at the time t at which an inspection has been carried out, it is found that the organism is not alive or it is not working. This situation can be frequently encountered in different contexts, as there is no process to accurately determine the time point at which the organism failed, and maybe there will be signs by which it is realized that the organism has failed. There are many situations where the observation of events is postponed. The time t, thus, may be the first time at which a sign has observed. We will utilize the random variable (RV) $X_{(t)} = (t - X | X \leq t)$, which is well-defined for every $t \geq 0$ for which $F_X(t) > 0$.

The RV $X_{(t)}$ is called the inactivity time of the organism (which has random lifetime X) at the time t. The RV $X_{(t)}$ has cdf

$$F_{X_{(t)}}(x) = \begin{cases} 0, & x < 0 \\ 1 - \frac{F_X(t-x)}{F_X(t)}, & 0 \leq x < t \\ 1, & x \geq t \end{cases} \tag{3}$$

The associated sf of $X_{(t)}$ is obviously as

$$\bar{F}_{X_{(t)}}(x) = \begin{cases} 1, & x \geq 0 \\ \frac{F_X(t-x)}{F_X(t)}, & 0 \leq x < t \\ 0, & x \geq t \end{cases} \tag{4}$$

The pdf of $X_{(t)}$ is also derived as

$$f_{X_{(t)}}(x) = \begin{cases} \frac{f_X(t-x)}{F_X(t)} & : 0 \leq x < t \\ 0 & : o.w. \end{cases} \tag{5}$$

There are two well-known reliability measures which are constructed using the RV $X_{(t)}$, namely, the reversed hazard rate (RHR) function, and the mean inactivity (MIT) time function. The RHR of a lifetime RV ($F_X(0^-) = 0$), in view of (5,) is defined as

$$\widetilde{h}_X(t) = f_{X_{(t)}}(0) = \frac{f_X(t)}{F_X(t)}, \quad \text{for all } t : F_X(t) > 0.$$

The MIT function of X, as the mathematical expectation of the RV $X_{(t)}$, is derived as

$$\widetilde{m}_X(t) = E[X_{(t)}] = \frac{\int_0^t F_X(x)dx}{F_X(t)}, \quad \text{for all } t : F_X(t) > 0.$$

For the preliminary properties of the RHR function we refer the readers to Block et al. [4] and Finkelstein [6] and for initial aspects and properties of MIT functions one can see Kayid and Izadkhah [9] and Khan et al. [13].

Stochastic orders which have been utilized to compare probability distributions provide useful procedures to compare reliability systems. Two reputable and frequently used stochastic orders are adopted in this paper for making stochastic orders between idle times of inactive systems. The following definition is adopted from Shaked and Shanthikumar [21].

Definition 1. *Suppose that X and Y are two non-negative RVs with PDFs f_X and f_Y, and RFs S_X and S_Y, respectively. Then, we say X is smaller or equal than Y in the*

(i) likelihood ratio order (denoted as $X \leq_{lr} Y$) if $\frac{f_Y(t)}{f_X(t)}$ is increasing in $t \geq 0$.

(ii) usual stochastic order (denoted as $X \leq_{st} Y$) if $S_X(t) \leq S_Y(t)$, for all $t \geq 0$.

The two stochastic orders given in Definition 1 are in a relation with each other so that $X \leq_{lr} Y$ yields $X \leq_{st} Y$ (see, for instance, Theorem 1.C.1 in Shaked and Shanthikumar [21]). In accordance with comparison between $X_{(t_1)}$ and $X_{(t_2)}$ for all $t_1 \leq t_2 \in \mathbb{R}^+$ according to the likelihood ratio order and the usual stochastic order, new classes of lifetime distributions are generated. In the sequel of the paper, the following classes are used.

Definition 2. *The RV X with PDF f_X is said to have*

(i) Increasing likelihood ratio property (denoted by $X \in ILR$) whenever $f_X(t)$ is log-concave in $t \geq 0$.

(ii) Decreasing reversed hazard rate (denoted by $X \in DRHR$) whenever $F_X(t)$ is log-concave in $t \geq 0$.

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For instance the exponential distribution, as a standard lifetime distribution, fulfils both ILR and $DRHR$ properties. The weibull distribution with shape parameter α and scale parameter λ, with RF $S_X(t) = \exp(-(\lambda t)^\alpha)$, has ILR and $DRHR$ properties if $\alpha > 1$. For further properties of the ILR and the $DRHR$ classes of lifetime distributions the readers are referred to Lai and Xie [14]. The following definition is due to Karlin [8].

Definition 3. *Let $g(x, y)$ be a non-negative function defined for $x \in \mathfrak{X} \subseteq \mathbb{R}$ and $y \in \mathfrak{Y} \subseteq \mathbb{R}$. Then, $g(x, y)$ is said to be totally positive of order 2 (denoted as TP_2) in $(x, y) \in \mathfrak{X} \times \mathfrak{Y}$, provided that $g(x_1, y_1)g(x_2, y_2) \geq g(x_1, y_2)g(x_2, y_1)$ for all $x_1 \leq x_2 \in \mathfrak{X}$ and for all $y_1 \leq y_2 \in \mathfrak{Y}$.*

2. Main Results

The idle time of a cold-standby system with n components (one is active with random lifetime X_1 and the residual $n-1$ ones with random lifetimes $X_2, \ldots, X_n$ are ready to function upon the failure of the active component) is characterized by the random variable

$$(S_n)_{(t)} := (t - S_n \mid S_n \leq t), \quad \text{for all } t : F_{S_n}(t) > 0 \tag{6}$$

where t is the first time at which the failure of the system is observed.

In the following theorem, we derive a lower bound for the inactivity time of a standby system, in the sense of the usual stochastic order, in terms of the inactivity time of its components.

Theorem 1. *Let $X_1, X_2, \ldots, X_n$ be independent and non-negative rvs such that $X_{1(t)}, X_{2(t)}, \ldots, X_{n(t)}$ are also independent. Then, for fixed $t \geq 0$,*

$$\left(\sum_{i=1}^{n} X_i\right)_{(t)} \geq_{st} \sum_{i=1}^{n} X_{i(t)} - (n-1)t. \tag{7}$$

Proof. We proceed and prove the theorem by the method of induction. Let $n = 2$ and denote $W_n = (\sum_{i=1}^n X_i)_{(t)}$ and $V_n = \sum_{i=1}^n X_{i(t)} - (n-1)t$, where $t \geq 0$ is a fixed time. We show, firstly, that $\bar{F}_{W_2}(a) \geq \bar{F}_{V_2}(a)$, for all $a \in \mathbb{R}$. Since for every $n \in \mathbb{N}$, $S_{W_n} = [0, t]$ and $S_{V_n} = [-(n-1)t, t]$, thus for all $a \leq 0$, and also for all $a \geq t$, we have $\bar{F}_{W_2}(a) \geq \bar{F}_{V_2}(a)$. Hence, it is enough to show that $\bar{F}_{W_2}(a) \geq \bar{F}_{V_2}(a)$ holds true for all $a \in [0, t]$. This is also sufficient to claim that

$$W_2 = (X_1 + X_2)_{(t)} \geq_{st} X_{1(t)} + X_{2(t)} - t. \tag{8}$$

For $a \in [0, t]$, we can get

$$\begin{aligned}
\bar{F}_{W_2}(a) &= \frac{F_{X_1+X_2}(t-a)}{F_{X_1+X_2}(t)} \\
&= \frac{\int_0^{t-a} F_{X_2}(t-a-x_1) f_{X_1}(x_1) dx_1}{\int_0^t F_{X_2}(t-x_1) f_{X_1}(x_1) dx_1} \\
&= \frac{\int_a^t F_{X_2}(x-a) f_{X_1}(t-x) dx}{\int_0^t F_{X_2}(x) f_{X_1}(t-x) dx},
\end{aligned}$$

where the first identity followed from equation (3), the second identity obtained by total probability formula and the fact that X_1 and X_2 are independent, and the third identity acquired after the change of variable $x = t - x_1$. On the other hand,

$$\begin{aligned}\bar{F}_{V_2}(a) &= P(X_{1(t)} + X_{2(t)} > t + a) \\ &= \int_0^t \bar{F}_{X_{2(t)}}(a + t - x_1) f_{X_{1(t)}}(x_1)dx_1 \\ &= \frac{\int_a^t F_{X_2}(x_1 - a) f_{X_1}(t - x_1)dx_1}{F_{X_1}(t)F_{X_2}(t)},\end{aligned}$$

in which the second equality is due to the total probability formula and that $X_{1(t)}$ and $X_{2(t)}$ are independent and the second equality obtained by using equations (4) and (5). Since, for $x \leq t$, one has $F_{X_2}(x) \leq F_{X_2}(t)$, thus,

$$\begin{aligned}\int_0^t F_{X_2}(x) f_{X_1}(t - x)dx &\leq \int_0^t F_{X_2}(t) f_{X_1}(t - x)dx \\ &= F_{X_2}(t) \int_0^t f_{X_1}(t - x)dx = F_{X_2}(t)F_{X_1}(t).\end{aligned}$$

Therefore, for all $a \in [0, t]$,

$$\begin{aligned}\bar{F}_{W_2}(a) &= \frac{\int_a^t F_{X_2}(x - a) f_{X_1}(t - x)dx}{\int_0^t F_{X_2}(x) f_{X_1}(t - x)dx} \\ &\geq \frac{\int_a^t F_{X_2}(x - a) f_{X_1}(t - x)dx}{F_{X_2}(t)F_{X_1}(t)} = \bar{F}_{V_2}(a).\end{aligned}$$

Let us assume now that (7) holds for $n = m$, that is

$$\left(\sum_{i=1}^{m} X_i\right)_{(t)} \geq_{st} \sum_{i=1}^{m} X_{i(t)} - (m - 1)t. \tag{9}$$

We then prove that (7) also holds for $n = m + 1$. We observe that

$$\begin{aligned}\left(\sum_{i=1}^{m+1} X_i\right)_{(t)} &=_{st} \left(\sum_{i=1}^{m} X_i + X_{m+1}\right)_{(t)} \\ &\geq_{st} \left(\sum_{i=1}^{m} X_i\right)_{(t)} + X_{m+1(t)} - t \\ &\geq_{st} \sum_{i=1}^{m} X_{i(t)} - (m - 1)t + X_{m+1(t)} - t \\ &=_{st} \sum_{i=1}^{m+1} X_{i(t)} - mt,\end{aligned}$$

where the first stochastic order is due to (8), the second stochastic order follows from (9) and $=_{st}$ means equality in distribution. Thus, the proof is validated. □

The lower bound in (7) in Theorem 1 can take negative values for some $t \geq 0$. If the lower bound is negative for a given $t \geq 0$, the result of Theorem 1 becomes trivial. However, in the following example we present a situation where (7) produces a meaningful lower bound for the MIT of a gamma distributed random lifetime.

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Example 1. *Suppose that $X_i \sim E(\lambda), i = 1, 2$, an exponential distribution with mean $1/\lambda_i$. Then, X_i has MIT*

$$\widetilde{m}_{X_i}(t) = \frac{t}{1 - \exp(-\lambda t)} - \frac{1}{\lambda}.$$

It is known that $X_1 + X_2$ has gamma distribution with shape $\alpha = 2$ and scale λ. From Theorem 1, since the usual stochastic order implies the expectation order, thus we obtain a lower bound for the MIT of $X_1 + X_2$ as follows:

$$\begin{aligned} E(X_1 + X_2)_{(t)} &\geq E(X_{1(t)}) + E(X_{2(t)}) - t \\ &= \widetilde{m}_{X_1}(t) + \widetilde{m}_{X_2}(t) - t \\ &= \frac{2t}{1 - \exp(-\lambda t)} - \frac{2}{\lambda} - t. \end{aligned}$$

We show that the obtained lower bound for the MIT of $X_1 + X_2$ is non-negative. It is readily seen that $\frac{2t}{1-\exp(-\lambda t)} - \frac{2}{\lambda} - t \geq 0$, for all $t \geq 0$, if and only if, $(2 + \lambda t)\exp(-\lambda t) - (2 - \lambda t) \geq 0$, for all $t \geq 0$. Since $\exp(x) \geq (1 + x)$, for all $x \geq 0$, thus $1 - (1 + x)\exp(-x) \geq 0$, for all $x \geq 0$. Therefore,

$$(2 + \lambda t)\exp(-\lambda t) - (2 - \lambda t) = \int_0^{\lambda t} (1 - (1 + x)\exp(-x))dx \geq 0, \ \text{for all } t \geq 0.$$

The following example clarifies another utilization of the result of Theorem 1.

Example 2. *It is said that X has a gamma distribution with shape parameter $\alpha > 0$ and the scale parameter λ and denote it by $X \sim G(\alpha, \lambda)$, whenever X has pdf $f_X(x) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}$. Considering two lifetime organisms or devices working one after another in a system (standby structure) and letting X_i denote the lifetime of the ith organism, for $i = 1, 2$. We will consider two situations: firstly when both organisms failed before time $t = 1$, and secondly, when the system failed before time $t = 1$. Suppose that $X_1 \sim G(2, 3)$ and $X_2 \sim G(3, 3)$. Note that if $X \sim G(k, \lambda)$, where $k \in \mathbb{N}$, the the cdf of X is derived via*

$$F_X(x) = 1 - \sum_{n=0}^{k} \exp(-\lambda x)\frac{(\lambda x)^n}{n!}. \tag{10}$$

Denote $P_4(a) = 27a^4 - 144a^3 + 306a^2 - 312a + 131$. In spirit of the equation (10), we get the cdf of the RV $W_2 := (X_1 + X_2)_{(t)}$, when $t = 1$, as follows:

$$\begin{aligned} \bar{F}_{W_2}(a) &= \frac{\int_0^{1-a} F_{X_2}(1 - a - x) f_{X_1}(x)dx}{\int_0^1 F_{X_2}(1 - x) f_{X_1}(x)dx} \\ &= \frac{P_4(a)\exp(3(a-1)) - 8}{131\exp(-3) - 8}, \end{aligned}$$

and on the other hand, using (10), the RV $V_2 := X_{1(t)} + X_{2(t)} - t$, for $t = 1$, has RF

$$\begin{aligned} \bar{F}_{V_2}(a) &= \frac{\int_a^1 F_{X_2}(x - a) f_{X_1}(1 - x)dx}{F_{X_1}(1)F_{X_2}(1)} \\ &= \frac{P_4(a)\exp(3a) - 8\exp(3)}{8(4 - \exp(3))(68 - \exp(3))}. \end{aligned}$$

From Theorem 1, $W_2 \geq_{st} V_2$ and thus, $\bar{F}_{W_2}(a) \geq \bar{F}_{V_2}(a)$, for all $a \in \mathbb{R}$. In Figure 1, we plot the graphs of $\bar{F}_{W_2}(a)$ and $\bar{F}_{V_2}(a)$ for $a \in [0, 1]$ (for the values out of $[0, 1]$ the ordering relation is obviously fulfilled) to exhibit the stochastic ordering property.

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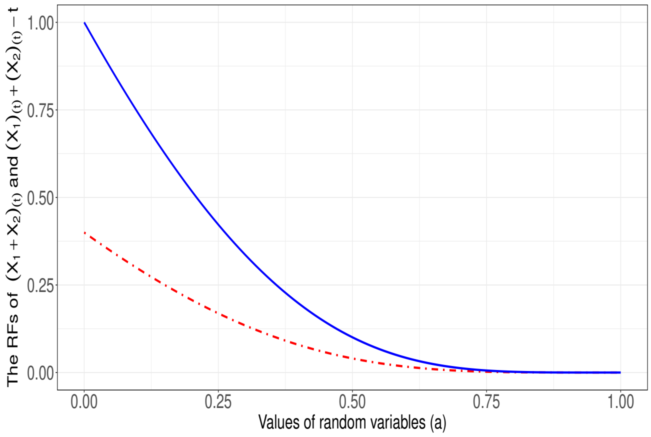

Figure 1. Plot of the reliability function of $(X_1 + X_2)_{(t)}$ (solid line) and the reliability function of $X_{1(t)} + X_{2(t)} - t$ (dot-dashed line) for $t = 1$ and for $x \in (0, 1)$.

Before stating the next result, we introduce some notation. In what follows, we will take

$$\begin{aligned} S_n &= \sum_{i=1}^{n} X_i \\ S_{n(\boldsymbol{t})} &= \sum_{i=1}^{n} X_{i(t_i)} \\ (S_n)_{(t)} &= (\sum_{i=1}^{n} X_i)_{(t)} \\ S_n(t) &= \left(\sum_{i=1}^{n} X_i \mid \sum_{i=1}^{n} X_i \leq t\right) \\ S_{n,\boldsymbol{t}} &= \sum_{i=1}^{n} X_i(t_i). \end{aligned}$$

In the following result, it is demonstrated that given that at a certain time t, both a cold standby system and all the components in it are inactive, the overall idle time of the components is stochastically (with respect to likelihood ratio order) greater than the idle time of the cold standby unit.

Theorem 2. *Let $X_1, X_2, \ldots, X_n$ be independent rvs such that X_n follows exponential distribution with parameter λ_n. Let $X_{1(t)}, X_{2(t)}, \ldots, X_{n(t)}$ are also independent for a fixed $t \geq 0$. Then,*

$$(\sum_{i=1}^{n} X_i)_{(t)} \leq_{lr} \sum_{i=1}^{n} X_{i(t)}.$$

Proof. Notice that $(S_n)_{(t)} := (\sum_{i=1}^{n} X_i)_{(t)}$ has support $[0, t]$, while $S_{n(t)} := \sum_{i=1}^{n} X_{i(t)}$ has support $[0, nt]$. Thus, we need to prove that $\frac{f_{S_{n(t)}}(s)}{f_{(S_n)_{(t)}}(s)}$ is increasing in $s \geq 0$, for every $s \leq t$, since for $s > t$, one has $\frac{f_{S_{n(t)}}(s)}{f_{(S_n)_{(t)}}(s)} = +\infty$. We have

$$f_{(S_n)_{(t)}}(s) = \frac{f_{S_n}(t-s)}{F_{S_n}(t)} I[0 \leq s \leq t], \text{ and } f_{S_{n(t)}}(s) = \int_0^{+\infty} f_{X_{n(t)}}(s-y) f_{S_{n-1(t)}}(y) dy$$

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, Therefore, for every $s \in [0, t]$, we have

$$\frac{f_{S_{n(t)}}(s)}{f_{(S_n)_{(t)}}(s)} = F_{S_n}(t).\frac{\int_0^{+\infty} f_{X_{n(t)}}(s-y) f_{S_{n-1(t)}}(y)dy}{\int_0^{+\infty} f_{X_n}(t-s-y) f_{S_{n-1}}(y)dy}$$
$$= \frac{F_{S_n}(t)}{F_{X_n}(t)}.\frac{\int_0^{s} f_{X_n}(t-s+y) f_{S_{n-1(t)}}(y)dy}{\int_0^{t-s} f_{X_n}(t-s-y) f_{S_{n-1}}(y)dy}$$
$$= \frac{F_{S_n}(t)}{F_{X_n}(t)}.\frac{\int_0^{s} f_{X_n}(t-s+y) f_{S_{n-1(t)}}(y)dy}{\int_s^{t} f_{X_n}(u-s) f_{S_{n-1}}(t-u)du},$$

where the change of variable $u = t - y$ has been made to derive the last equality. Since $X_n \sim E(\lambda_n)$, thus

$$\frac{f_{S_{n(t)}}(s)}{f_{(S_n)_{(t)}}(s)} = \frac{F_{S_n}(t)}{F_{X_n}(t)}.\frac{\int_0^{s} \lambda_n \exp(-\lambda_n(t-s+y)) f_{S_{n-1(t)}}(y)dy}{\int_s^{t} \lambda_n \exp(-\lambda_n(u-s)) f_{S_{n-1}}(t-u)du}$$
$$= \frac{\exp(\lambda_n t).F_{S_n}(t)}{F_{X_n}(t)}.\frac{\int_0^{s} \exp(-\lambda_n y) f_{S_{n-1(t)}}(y)dy}{\int_s^{t} \exp(-\lambda_n u) f_{S_{n-1}}(t-u)du}.$$

Hence, for all $0 \le s \le t$, we get

$$\frac{\partial}{\partial s}\log\left(\frac{f_{S_{n(t)}}(s)}{f_{(S_n)_{(t)}}(s)}\right) = \frac{\partial}{\partial s}\log\left(\frac{\int_0^{s} \exp(-\lambda_n y) f_{S_{n-1(t)}}(y)dy}{\int_s^{t} \exp(-\lambda_n u) f_{S_{n-1}}(t-u)du}\right)$$
$$= \frac{\exp(-\lambda_n s) f_{S_{n-1(t)}}(s)}{\int_0^{s} \exp(-\lambda_n y) f_{S_{n-1(t)}}(y)dy} + \frac{\exp(-\lambda_n s) f_{S_{n-1}}(t-s)}{\int_s^{t} \exp(-\lambda_n y) f_{S_{n-1}}(t-y)dy},$$

which is non-negative.Thus, the proof of theorem is completed. □

By considering a cold standby system composed of a single component with a general continues lifetime distribution equipped with one additional component with exponential distribution lifetime, it is realized by Theorem 2 that the idle time of the cold standby system is smaller, with respect to the likelihood ratio order, than the sum of the inactivity times of both components measured at a specified time t at which the system and its components are inactive and not functioning. The following technical lemma is useful and will be applied in the sequel.

Lemma 1. *Suppose that $\tau_1, \tau_2, \ldots, \tau_m$ are m points of time with mean $\bar{\tau}$, and that $X_1, X_2, \ldots, X_m$ are m non-negative rvs then $m\bar{\tau} - (S_m)(m\bar{\tau}) =_{st} S_{m(\tau)}$, where $=_{st}$ means equality in distribution.*

Theorem 3. *Let $X_1, X_2, \ldots X_m$ be independent non-negative rvs, such that $X_1(t_1), X_2(t_2), \ldots, X_m(t_m)$ and also $X_1(\tau_1), X_2(\tau_2), \ldots, X_n(\tau_n)$, where $m > n$, are also independent where $t_1, t_2, \ldots, t_m$ are m fixed time points and so are $\tau_1, \tau_2, \ldots, \tau_n$. Let $X_i \in ILR, i = 1, 2, \ldots, n$. If*

(i) $m\bar{t} \ge n\bar{\tau}$ where $t_m \ge n\bar{\tau}$ in which $\bar{\tau} = \frac{\sum_{i=1}^{n}\tau_i}{n}$ and $\bar{t} = \frac{\sum_{i=1}^{m}t_i}{m}$;
(ii) X_m and X_n have identical distributions;
(iii) For and $t_i \ge 0, i = 1, 2, \ldots, m$, $\sum_{i=1}^{m-1} X_i(t_i) \ge_{lr} \sum_{i=1}^{n} X_i$,

then:

$$(\sum_{i=1}^{n} X_i)_{(n\bar{\tau})} \ge_{lr} \sum_{i=1}^{m} X_{i(t_i)} - m\bar{t} + n\bar{\tau}. \tag{11}$$

Proof. We first prove that $S_n(n\bar{\tau}) := (\sum_{i=1}^{n} X_i)(n\bar{\tau}) \leq_{lr} \sum_{i=1}^{m} X_i(t_i) := S_{m,\boldsymbol{t}}$. The result is then proved following what we discuss here. From Theorem 1.C.8 in Shaked and Shanthikumar (2007), if ψ is a decreasing function then $X \leq_{lr} Y$ implies $\psi(X) \geq_{lr} \psi(Y)$. Thus, if $\psi(x) = m\bar{t} - x$, since $S_n(n\bar{\tau}) \leq_{lr} S_{m,\boldsymbol{t}}$, thus it follows that $m\bar{t} - S_n(n\bar{\tau}) \geq_{lr} m\bar{t} - S_{m,\boldsymbol{t}}$. Using Lemma 1, it is realized that $m\bar{t} - S_n(n\bar{\tau}) =_{st} m\bar{t} - n\bar{\tau} + (S_n)_{(n\bar{\tau}))}$ and, in addition, $m\bar{t} - S_{m,\boldsymbol{t}} =_{st} S_{m(\boldsymbol{t})}$. Hence, it follows that $(S_n)_{(n\bar{\tau})} \geq_{lr} S_{m(\boldsymbol{t})} - m\bar{t} + n\bar{\tau}$. Thus it is sufficient to prove that $\frac{f_{S_{m,\boldsymbol{t}}}(s)}{f_{S_n(n\bar{\tau})}(s)}$ is increasing in $0 < s \leq n\bar{\tau}$. We assume that $I[x \in A]$ is the indicator function of the set $[x \in A]$, which is equal with one if $x \in A$ and it is equal with 0 if $x \in A^c$, where A^c is the complement of the set A. Let us write for every $s \in (0, n\bar{\tau}]$,

$$\begin{aligned}
\frac{f_{S_{m,\boldsymbol{t}}}(s)}{f_{S_n(n\bar{\tau})}(s)} &= \frac{F_{S_n}(n\bar{\tau}) \int_0^{+\infty} f_{X_m(t_m)}(s-y) f_{S_{m-1,\boldsymbol{t}}}(y)dy}{\int_0^{+\infty} f_{X_n}(s-y) f_{S_{n-1}}(y)dy} \\
&= \frac{F_{S_n}(n\bar{\tau})}{F_{X_m}(t_m)} \cdot \frac{\int_0^{+\infty} f_{X_m}(s-y) I[s-y \leq t_m] f_{S_{m-1,\boldsymbol{t}}}(y) I[y > 0]dy}{\int_0^{+\infty} f_{X_n}(s-y) f_{S_{n-1}}(y)dy} \\
&= \frac{F_{S_n}(n\bar{\tau})}{F_{X_m}(t_m)} \cdot \frac{\int_{(s-t_m)\vee 0}^{+\infty} f_{X_m}(s-y) f_{S_{m-1,\boldsymbol{t}}}(y)dy}{\int_0^{+\infty} f_{X_n}(s-y) f_{S_{n-1}}(y)dy} \\
&= \frac{F_{S_n}(n\bar{\tau})}{F_{X_m}(t_m)} \cdot \frac{\int_0^{+\infty} f_{X_m}(s-y) f_{S_{m-1,\boldsymbol{t}}}(y)dy}{\int_0^{+\infty} f_{X_n}(s-y) f_{S_{n-1}}(y)dy},
\end{aligned}$$

where $a \vee b$ stands for the maximum of a and b, and the last identity follows from the fact that for all $s \in (0, n\bar{t}]$, from assumption (i), one has $s - t_m \leq s - n\bar{t} \leq 0$, thus $(s - t_m) \vee 0 = 0$. Therefore, for every $s \in (0, n\bar{t}]$ since X_m and X_n are identical in distribution (assumption (ii)), thus

$$\begin{aligned}
\frac{f_{S_{m,\boldsymbol{t}}}(s)}{f_{S_n(n\bar{\tau})}(s)} &= \frac{F_{S_n}(n\bar{\tau})}{F_{X_m}(t_m)} \cdot \int_0^{+\infty} \frac{f_{X_m}(s-y)}{f_{X_n}(s-y)} \cdot \frac{f_{S_{m-1,\boldsymbol{t}}}(y)}{f_{S_{n-1}}(y)} \frac{f_{X_n}(s-y) f_{S_{n-1}}(y)}{\int_0^{+\infty} f_{X_n}(s-y) f_{S_{n-1}}(y)dy} dy \\
&= \frac{F_{S_n}(n\bar{\tau})}{F_{X_m}(t_m)} \int_0^{+\infty} \frac{f_{S_{m-1,\boldsymbol{t}}}(y)}{f_{S_{n-1}}(y)} f_{X_n|S_n=s}(s-y|s)dy \\
&= \frac{F_{S_n}(n\bar{\tau})}{F_{X_m}(t_m)} \int_0^{+\infty} \Psi(y) f_{S_{n-1}|S_n=s}(y|s)dy \\
&= \frac{F_{S_n}(n\bar{\tau})}{F_{X_m}(t_m)} E\left(\Psi(S_{n-1}) \mid S_n = s\right),
\end{aligned} \tag{12}$$

where $\Psi(y) = \frac{f_{S_{m-1,\boldsymbol{t}}}(y)}{f_{S_{n-1}}(y)}$. From assumption, $X_i \in ILR$ for all $i = 1, 2, \ldots, n$, thus since the convolution of log-concave densities is log-concave, $S_{n-1} \in ILR$. From Theorem 1.C.53 of Shaked and Shanthikumar (2007), we conclude that

$$[S_{n-1}|S_n = s_1] \leq_{lr} [S_{n-1}|S_n = s_2], \text{ for all } s_1 \leq s_2 \in \mathbb{R}^+,$$

and since $\leq_{lr}$ implies $\leq_{st}$, thus one has

$$[S_{n-1}|S_n = s_1] \leq_{st} [S_{n-1}|S_n = s_2], \text{ for all } s_1 \leq s_2 \in \mathbb{R}^+. \tag{13}$$

Furthermore, from assumption (iii) the function $\Psi(y)$ is increasing in $y > 0$. From (12) and (13), it follows that $\frac{f_{S_{m,\boldsymbol{t}}}(s)}{f_{S_n(n\bar{\tau})}(s)}$ is increasing in $s > 0$, which validates the proof. □

The following lemma is useful in proving next result.

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Lemma 2. *Let $X_1, X_2, \ldots, X_m$ be m dependent rvs which are non0negative and, further, let $X_i(\tau_i), i = 1, 2, \ldots, m$ be independent. Then,*

$$\sum_{i=1}^{m} X_i(\tau_i) =_{st} (\sum_{i=1}^{m} X_i \mid X_1 \leq \tau_1, \cdots, X_m \leq \tau_m),$$

in which $\tau_1, \ldots, \tau_m$ are m points of time such that $\prod_{i=1}^{m} F_{X_i}(\tau_i) > 0$.

Proof. It suffices to prove that $S_m(\boldsymbol{\tau}) := \sum_{i=1}^{m} X_i(\tau_i)$ and $(S_m \mid X_i \leq \tau_i, i = 1, \ldots, m)$ have a common moment generating function. Denote $\boldsymbol{X} = (X_1, \ldots, X_m)$ and let $(\boldsymbol{X}|X_i \leq \tau_i, i = 1, \ldots, m)$ have conditional pdf $f_{(\boldsymbol{X}|X_i \leq \tau_i, i=1,\ldots,m)}$. Since X_i's are independent, thus, for all $\boldsymbol{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ we have

$$\begin{aligned} f_{(\boldsymbol{X}|X_i \leq \tau_i, i=1,\ldots,m)}(x_1, \ldots, x_m) &= \frac{f_{\boldsymbol{X}}(\boldsymbol{x}).I[x_1 \leq \tau_1, \ldots, x_m \leq \tau_m]}{P(X_1 \leq \tau_1, \cdots, X_m \leq \tau_m)} \\ &= \prod_{i=1}^{m} \left(\frac{f_{X_i}(x_i)}{F_{X_i}(\tau_i)} \right) I[x_1 \leq \tau_1, \ldots, x_m \leq \tau_m]. \end{aligned}$$

Now, let us write

$$\begin{aligned} M_{(S_m|X_i \leq \tau_i, i=1,\ldots,m)}(t) &= E(\exp(tS_m)|X_i \leq \tau_i, i = 1, \ldots, m) \\ &= E(\prod_{i=1}^{m} \exp(tX_i) \mid X_i \leq \tau_i, i = 1, \ldots, m) \\ &= \int_0^{+\infty} \cdots \int_0^{+\infty} \left(\prod_{i=1}^{m} \exp(tx_i) \right) f_{(\boldsymbol{X}|X_i \leq \tau_i, i=1,\ldots,m)}(x_1, \ldots, x_m)\, dx_m \ldots dx_1 \\ &= \int_0^{\tau_1} \cdots \int_0^{\tau_m} \prod_{i=1}^{m} \left(\exp(tx_i) \frac{f_{X_i}(x_i)}{F_{X_i}(\tau_i)} \right) \\ &= \prod_{i=1}^{m} \left(\int_0^{\tau_i} \frac{\exp(tx_i) f_{X_i}(x_i)}{F_{X_i}(\tau_i)} dx_i \right) \\ &= \prod_{i=1}^{m} E(\exp(tX_i(\tau_i))) \\ &= M_{S_m(\boldsymbol{\tau})}(t), \text{ for all } t \in \mathbb{R}, \end{aligned}$$

where the last identity follows from the assumption that $X_1(\tau_1), \cdots, X_m(\tau_m)$ are independent. The proof is completed. □

The following result presents sufficient condition for the usual stochastic ordering between idle time of an inactive standby system of size two and the sum of the idle times of inactive components. It is worth mentioning that as the standby systems of size two are very important as an effective redundancy method in engineering reliability systems the previous results are also of particular interest when $n = 2$.

Theorem 4. *Let X_1 and X_2 be two independent rvs such that $X_1(\tau_1)$ and $X_2(\tau_n)$ are also independent for $\tau_1 \leq \tau_2$ as two points of time. Suppose that $\widetilde{h}_{X_2}(s - \tau_1) \leq \widetilde{h}_{X_1+X_2}(s)$, for all $s \in [\tau_1, \tau_1 + \tau_2]$. Then,*

$$(X_1 + X_2)_{(\tau_1+\tau_2)} \leq_{st} X_{1(\tau_1)} + X_{2(\tau_2)}.$$

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Proof. In a similar manner as in the proof of Theorem 3, it is sufficient to show that $(S_2)(\tau_1+\tau_2) \geq_{st} S_{2,\boldsymbol{\tau}}$. From Lemma 2, when $m=2$, since X_1 and X_2 and also $X_1(\tau_1)$ and $X_2(\tau_2)$ are independent, thus we have $S_{2,\boldsymbol{\tau}} =_{st} (S_2 \mid X_1 \leq \tau_1, X_2 \leq \tau_2)$. Thus we prove that

$$(S_2)(\tau_1+\tau_2) \geq_{st} (S_2 \mid X_1 \leq \tau_1, X_2 \leq \tau_2). \tag{14}$$

By routine calculation, it is evidently observed that (14) is satisfied if, and only if,

$$P(S_2 \leq s \mid S_2 \leq \tau_1+\tau_2) \leq P(S_2 \leq s \mid X_1 \leq \tau_1, X_2 \leq \tau_2), \text{ for all } s \in (0, \tau_1+\tau_2]$$

which holds if, and only if,

$$P(X_1 \leq \tau_1, X_2 \leq \tau_2 \mid S_2 \leq \tau_1+\tau_2) \leq P(X_1 \leq \tau_1, X_2 \leq \tau_2 \mid S_2 \leq s), \text{ for all } s \in (0, \tau_1+\tau_2]. \tag{15}$$

The inequality given in (15) holds true for $s \in (0, \tau_1]$ as it can be obviously seen that

$$\begin{aligned} P(X_1 \leq \tau_1, X_2 \leq \tau_2 \mid S_2 \leq \tau_1+\tau_2) &= \frac{P(X_1 \leq \tau_1, X_2 \leq \tau_2)}{P(S_2 \leq \tau_1+\tau_2)} \\ &\leq \frac{P(X_1 \leq \tau_1, X_2 \leq \tau_2)}{P(S_2 \leq s)} \\ &= P(X_1 \leq \tau_1, X_2 \leq \tau_2 \mid S_2 \leq s). \end{aligned}$$

Thus, we only need to prove that the inequality given in (15) is satisfied for $s \in (\tau_1, \tau_1+\tau_2]$. Note that

$$\begin{aligned} \int_0^{\tau_2} \frac{f_{X_2}(x_2) I[x_2 \leq s-x_1]}{F_{S_2}(s)} dx_2 &= P(X_2(s-x_1) \leq \tau_2) F_{X_2}(s-x_1) \\ &= F_{X_2}((s-x_1) \wedge \tau_2), \end{aligned}$$

where $a \vee b = \min\{a, b\}$. Thus, we can get

$$\begin{aligned} P(X_1 \leq \tau_1, X_2 \leq \tau_2 \mid S_2 \leq s) &= \int_0^{\tau_1} \int_0^{\tau_2} \frac{f_{X_1}(x_1) f_{X_2}(x_2) I[x_1+x_2 \leq s]}{F_{X_1+X_2}(s)} dx_2 dx_1 \\ &= \int_0^{\tau_1} f_{X_1}(x_1) \left(\int_0^{\tau_2} \frac{f_{X_2}(x_2) I[x_2 \leq s-x_1]}{F_{X_1+X_2}(s)} dx_2 \right) dx_1 \\ &= \int_0^{\tau_1} f_{X_1}(x_1) \frac{F_{X_2}((s-x_1) \wedge \tau_2)}{F_{X_1+X_2}(s)} dx_1. \end{aligned}$$

Therefore, to prove (14), it is sufficient to show that for all $s \in (\tau_1, \tau_1+\tau_2]$,

$$\int_0^{\tau_1} f_{X_1}(x_1) \frac{F_{X_2}((s-x_1) \wedge \tau_2)}{F_{S_2}(s)} dx_1 \geq \int_0^{\tau_1} f_{X_1}(x_1) \frac{F_{X_2}((\tau_1+\tau_2-x_1) \wedge \tau_2)}{F_{S_2}(\tau_1+\tau_2)} dx_1.$$

Since, for all $x_1 \leq \tau_1$ we have $(\tau_1+\tau_2-x_1) \wedge \tau_2 = \tau_2$, thus it suffices to prove that

$$\int_0^{\tau_1} f_{X_1}(x_1) \frac{F_{X_2}((s-x_1) \wedge \tau_2)}{F_{S_2}(s)} dx_1 \geq \int_0^{\tau_1} f_{X_1}(x_1) \frac{F_{X_2}(\tau_2)}{F_{S_2}(\tau_1+\tau_2)} dx_1,$$

holds for all $s \in (\tau_1, \tau_1+\tau_2]$. That is, it is enough to defenestrate that for all $s \in (\tau_1, \tau_1+\tau_2]$,

$$\int_0^{\tau_1} f_{X_1}(x_1) \Phi(x_1, s, \tau_1, \tau_2) dx_1 \geq 0,$$

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where

$$\Phi(x_1, s, \tau_1, \tau_2) = \left[\frac{F_{X_2}((s-x_1)\wedge\tau_2)}{F_{S_2}(s)} - \frac{F_{X_2}(\tau_2)}{F_{S_2}(\tau_1+\tau_2)}\right].$$

Let us assume that $\tau_1 \leq s < \tau_2$. Then, we can show that

$$\int_0^{\tau_1} f_{X_1}(x)\Psi(x_1, s, \tau_1, \tau_2)dx_1 \geq 0, \text{ for all } s \in [\tau_1, \tau_2).$$

Since $s < \tau_2$, thus, $s - x_1 \leq \tau_2 - x_1 \leq \tau_2$, for all $x_1 \in (0, \tau_1)$. Hence, $(s - x_1) \wedge \tau_2 = s - x_1$. Therefore,

$$\begin{aligned}\Psi(x_1, s, \tau_1, \tau_2) &= \left[\frac{F_{X_2}(s-x_1)}{F_{S_2}(s)} - \frac{F_{X_2}(\tau_2)}{F_{S_2}(\tau_1+\tau_2)}\right] \\ &\geq \left[\frac{F_{X_2}(s-\tau_1)}{F_{S_2}(s)} - \frac{F_{X_2}(\tau_2)}{F_{S_2}(\tau_1+\tau_2)}\right],\end{aligned}$$

which is non-negative if $\frac{F_{X_2}(s-\tau_1)}{F_{S_2}(s)}$ is decreasing in $s \in [\tau_1, \tau_2)$. This is also satisfied if the assumption "$\widetilde{h}_{X_2}(s-\tau_1) \leq \widetilde{h}_{X_1+X_2}(s)$, for all $s \in [\tau_1, \tau_1+\tau_2]$" holds. Now, assume that $\tau_2 \leq s < \tau_1 + \tau_2$. Then,

$$\int_0^{s-\tau_2} \Psi(x_1, s, \tau_1, \tau_2)dx_1 \geq \int_0^{s-\tau_2} F_{X_2}(\tau_2) f_{X_1}(x_1)\left(\frac{1}{F_{S_2}(s)} - \frac{1}{F_{S_2}(\tau_1+\tau_2)}\right)dx_1 \geq 0.$$

In a similar manner to the case when $\tau_1 \leq s_2 < \tau_2$, now we can also established that

$$\int_{s-\tau_2}^{\tau_1} \Psi(x_1, s, \tau_1, \tau_2)dx_1 \geq 0.$$

As a result, we have $\int_0^{\tau_1} \Psi(x_1, s, \tau_1, \tau_2)dx_1 \geq 0$, for all $s \in [\tau_2, \tau_1 + \tau_2]$. The proof of the theorem is now completed. □

In the context of the conditions of Theorem 4, one may question whether the assumption "$\widetilde{h}_{X_2}(s - \tau_1) \leq \widetilde{h}_{X_1+X_2}(s)$, for all $s \in [\tau_1, \tau_1 + \tau_2]$" is attainable. The following remark clarifies the issue.

Remark 1. *Suppose that X_1 and X_2 represent, respectively, the lifetime of a component and the lifetime of the standby unit which are assumed to be independent. Note that for all $x < 0$, $f_{X_i}(x) = F_{X_i}(x) = 0, i = 1, 2$. We can write*

$$\begin{aligned}\widetilde{h}_{X_1+X_2}(s) &:= \frac{f_{X_1+X_2}(s)}{F_{X_1+X_2}(s)} \\ &= \frac{\int_0^{+\infty} f_{X_2}(s-x_1) f_{X_1}(x_1)dx_1}{\int_0^{+\infty} F_{X_2}(s-x_1) f_{X_1}(x_1)dx_1} \\ &= \int_0^s \frac{f_{X_2}(s-x_1)}{F_{X_2}(s-x_1)} \frac{F_{X_2}(s-x_1) f_{X_1}(x_1)}{\int_0^{+\infty} F_{X_2}(s-x_1) f_{X_1}(x_1)dx_1} dx_1 \\ &= \int_0^s \widetilde{h}_{X_2}(s-x_1) f^\star(x_1|s)dx_1 \\ &= E[\widetilde{h}_{X_2}(s - X_s^\star)],\end{aligned}$$

where $X_s^\star$ is a non-negative RV with pdf $f^\star(\cdot|s)$, which is given by

$$f^\star(x_1|s) = \frac{F_{X_2}(s-x_1) f_{X_1}(x_1)}{\int_0^{+\infty} F_{X_2}(s-x_1) f_{X_1}(x_1)dx_1}. \quad (16)$$

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Note that $E(X_s^*) \leq s$. Now, consider an $s_0 > 0$ such that $s_0 < s < \tau_1 + \tau_2$, for which $\tau_1 = E(X_{s_0}^*) \leq E(X_s^*)$. We suppose that X_2 has a decreasing convex reversed hazard rate function. On using Jensen's inequality, for all $s < \tau_1 + \tau_2$

$$\begin{aligned}\widetilde{h}_{X_1+X_2}(s) &= E[\widetilde{h}_{X_2}(s - X_s^\star)] \\ &\geq \widetilde{h}_{X_2}(s - E(X_s^\star)) \\ &\geq \widetilde{h}_{X_2}(s - \tau_1).\end{aligned}$$

Now, Theorem 4 is applicable as the sufficient condition of this theorem is satisfied.

We give the next example to fulfill the conditions in Theorem 4 in the case of heterogenous exponential components.

Example 3. *Suppose that X_1 and X_2 have exponential distribution with parameters λ_1 and λ_2, respectively. It can be checked that the reversed hazard rate function of exponential distribution is decreasing and convex. Thus, $\widetilde{h}_{X_2}(x) = \frac{\lambda_2}{\exp(\lambda_2 x)-1}$ is a decreasing function in x which is further convex in x. The RV $X_s^\star$, as introduced in Remark 1 with pdf (16), has pdf*

$$f^\star(x_1|s) = \frac{\exp(-\lambda_1 x) - \exp(-\lambda_2 s + (\lambda_2 - \lambda_1)x)}{\frac{1}{\lambda_1}(1 - \exp(-\lambda_1 s)) + \frac{1}{\lambda_2 - \lambda_1}(\exp(-\lambda_2 s) - \exp(-\lambda_1 s))}, \quad 0 \leq x \leq s.$$

It is straightforward that, if $X_2 \in DRHR$, then $F_{X_2}(s - x_1)$ is TP_2 in $(x_1, s) \in \mathbb{R}^+ \times \mathbb{R}^+$. As a result, from (16), $f^*(x_1|s)$ is also TP_2 in $(x_1, s) \in \mathbb{R}^+ \times \mathbb{R}^+$, that is $X_{s_1}^* \leq_{lr} X_{s_2}^*$ for every $0 < s_1 \leq s_2$. Since $\leq_{lr}$ implies $\leq_{st}$, thus one can conclude that $X_{s_1}^* \leq_{st} X_{s_2}^*$, for all $0 < s_1 \leq s_2$. Therefore, $E(X_{s_1}^*) \leq E(X_{s_2}^*)$, for all $s_1 \leq s_2 \in \mathbb{R}^+$. For example if $\lambda_1 = 2$ and $\lambda_2 = 3$, then after some calculation

$$E(X_s^\star) = \frac{\exp(3s) + (3 - 6s)\exp(s) - 4}{2\exp(3s) - 6\exp(s) + 4},$$

which is an increasing function. Hence, is one chooses $\tau_1 = E(X_0^*) = 0$, following the discussions in Remark 1 and since for all $s > \tau_1$, $E(X_s^*) \geq E(X_{\tau_1}^*) \geq 0$, the assumption "$\widetilde{h}_{X_2}(s - \tau_1) \leq \widetilde{h}_{X_1+X_2}(s)$, for all $s \in [\tau_1, \tau_1 + \tau_2]$"in Theorem 4 is fulfilled on that account and, consequently, $(X_1 + X_2)_{(\tau_1+\tau_2)} \leq_{st} X_{1(\tau_1)} + X_{2(\tau_2)}$, or more accurately, $(X_1 + X_2)_{(\tau_2)} \leq_{st} X_{2(\tau_2)}$.

3. Concluding Remarks

In this study, we have presented some results to obtain upper and lower stochastic bounds for the idle time of standby systems after their failure in the context of random lifetimes. These bounds are in fact some functions of the idle times of the components of the standby system. Thus, we can evaluate whether it is reasonable to equip a component with redundant standby units to minimize the idle time of a standby system. We used two informative and commonly applied stochastic orderings, namely the likelihood ratio ordering and the usual stochastic ordering. The problem of maintaining systems and replacing them with new systems and the same problem of tuning their components plays an important role in reliability engineering. This is because there are some systems (e.g., systems that gradually fail under certain degradation processes) that need to be in operation uniformly and it is very beneficial to have less idle time. The results obtained in this work may be useful in identifying situations where the idle time of inactive standby systems and the idle time (or a function thereof) of components in the system have a stochastic ordering property. Since the number of standby units can play a key role in reducing or even increasing costs and preventing further losses from early component failures, it becomes increasingly important to identify whether a standby system is highly survivable or equivalently has lower inactivity compared to components that failed earlier.

In the future of this study, we will consider two standby systems with different component lifetimes and possibly different numbers of components, and investigate some stochastic comparisons between the idle time of one system and the sum of the idle times of the components in the other system. We will also look for conditions for stochastic comparisons between the idle times of two standby systems.

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