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Shape Preserving Properties of Parametric Szász Type Operators on Unbounded Intervals

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Abstract: In this paper, we investigate some elementary properties of two kinds modified Szász type basis functions, depending on non-negative parameters. We study some shape preserving properties of these operators concerning monotonicity, convexity, starlikeness, semi-additivity and smoothness.

Keywords: Szász type operators; shape preserving property; monotonicity; convexity

1. Introduction

The famous Bernstein operator has become powerful tool for curve and surface design and representation because of its good shape-preserving properties. It has been effectively applied to computer aided geometric design. In recent years, the shape-preserving properties of various operators have been deeply studied [1–10].

In order to be more effectively applied to CAGD on infinite intervals, two kinds of Szász operators based on non-negative parameters were introduced [11,12]. The aim of this paper is to show some shape preserving properties of these parametric Szász type operators. The complete structure of the manuscript constitutes five sections. The rest of this paper is constructed as follows. In Section 2, the fundamental facts are summarized for use in the sequel. In Section 3, we shall prove that the operators preserve monotonicity, convexity, starlikeness and semi-additivity. In Section 4, we investigate the preservation of smoothness. Finally, in Section 5, some conclusions are provided.

2. Fundamental Properties of the Basis Functions and the Operators

In this section, some basic facts that will be used in the following sections are given.

Let $C[0,\infty)$ denote the space of continuous functions on $[0,\infty)$, $C_B[0,\infty)$ be the space of bounded functions in the space of $C[0,\infty)$ endowed with the norm $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$. Let $A_i(i = 0,1,2,3)$ denote the subclasses of $C[0,\infty)$:
$A_0 := \{f \in C[0,\infty) : f(0) = 0\}$;
$A_1 := \{f \in A_0 : f(x) \text{ is convex on } [0,\infty)\}$;
$A_2 := \{f \in A_0 : x^{-1}f(x) \text{ is increasing on } (0,\infty)\}$;
$A_3 := \{f \in A_0 : f(x) \text{ is super-additive on } [0,\infty)\}$,
where $f(x)$ is said to be super-additive on $[0,\infty)$, if for any $x_1, x_2 \in [0,\infty)$, $f(x_1 + x_2) \geq f(x_1) + f(x_2)$. In addition, by [9 Theorem 5], we find that $A_1 \subset A_2 \subset A_3$. On the other hand, $A_i^*(i = 1,2,3)$ stand for the other three subsets:
$A_1^* := \{f \in A_0 : f(x) \text{ is concave on } [0,\infty)\}$;
$A_2^* := \{f \in A_0 : x^{-1}f(x) \text{ is decreasing on } (0,\infty)\}$;
$A_3^* := \{f \in A_0 : f(x) \text{ is semi-additive on } [0,\infty)\}$,
where $f(x)$ is said to be semi-additive on $[0,\infty)$, if for any $x_1, x_2 \in [0,\infty)$, $f(x_1 + x_2) \leq f(x_1) + f(x_2)$.

The parametric Szász type operators are defined by [11,12],

$$S_n^{\mu}(f;x) = \sum_{k=0}^{\infty} f(\frac{k}{n}) s_{n,k}(x),$$

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where

$$s_{n,k}(x) = e^{-n\alpha(x)} \frac{(n\alpha(x))^k}{k!},$$

(1) when $\alpha(x) = \alpha_1(x) = \frac{\mu x}{n(1-e^{\frac{\mu}{n}})}$, the operators $S_n^{\mu}(f;x)$ preserve the functions 1 and $e^{-\mu x}(\mu > 0)$;
(2) when $\alpha(x) = \alpha_2(x) = \frac{2\mu x}{n(e^{\frac{2\mu}{n}}-1)}$, the operators $S_n^{\mu}(f;x)$ preserve the functions 1 and $e^{2\mu x}(\mu > 0)$.

Remark 1. [11 *Lemma* 3; 12 *Remark* 1] $\lim\limits_{n\to\infty} \alpha(x) = x$.

Remark 2.[11 *Lemma* 2; 12 *Lemma* 2.1, *Lemma* 2.2] For $x \in [0,\infty), \mu > 0, S_n^{\mu}(1;x) = 1, S_n^{\mu}(t;x) = \alpha(x)$.

Remark 3. [11 *Theorem* 3; 12 *Theorem* 3.1] For $f(x) \in C^*[0,\infty) := \{f \in C[0,\infty) : \lim\limits_{n\to\infty} f(x)$ exists and is finite$\}$, one has the sequence $\{S_n^{\mu}(f;x)\}$ converges to $f(x)$ uniformly.

For $\mu > 0$, the new basis functions $s_{n,k}(x)$ are definied by

$$s_{n,k}(x) = e^{-n\alpha(x)} \frac{(n\alpha(x))^k}{k!}, k = 0, 1, \cdots.$$

Figures 1–3 show the Szász type basis with $\alpha(x) = \alpha_1(x)$ of degree 4, Figures 4–6 show the Szász type basis with $\alpha(x) = \alpha_2(x)$ of degree 4.

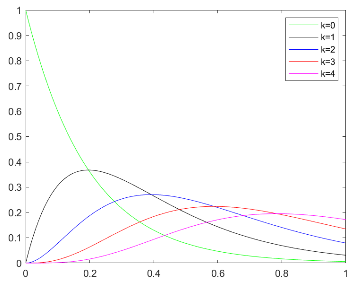

Fig.1 The quartic Szász basis with $\alpha(x) = \alpha_1(x)$, $\mu = 0.2, k = 0, 1, 2, 3, 4$

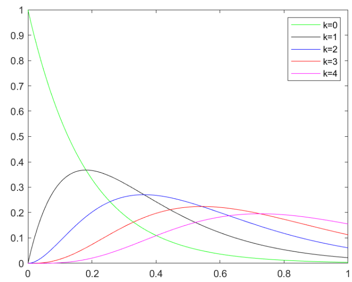

Fig.2 The quartic Szász basis with $\alpha(x) = \alpha_1(x)$,$\mu = 1, k = 0, 1, 2, 3, 4$

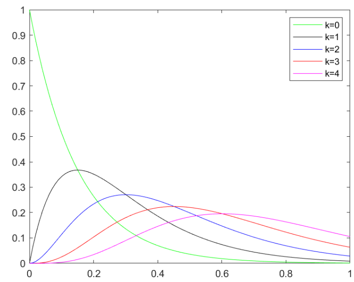

Fig.3 The quartic Szász basis with $\alpha(x) = \alpha_1(x)$,$\mu = 3, k = 0, 1, 2, 3, 4$

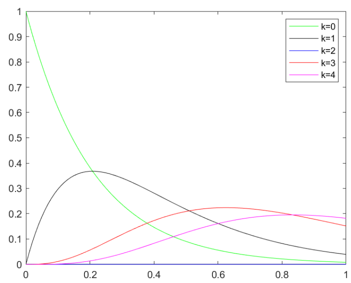

Fig.4 The quartic Szász basis with $\alpha(x) = \alpha_2(x)$, $\mu = 0.2, k = 0, 1, 2, 3, 4$

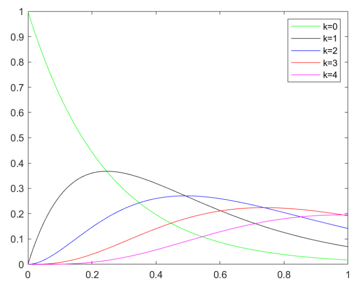

Fig.5 The quartic Szász basis with $\alpha(x) = \alpha_2(x)$, $\mu = 1, k = 0, 1, 2, 3, 4$

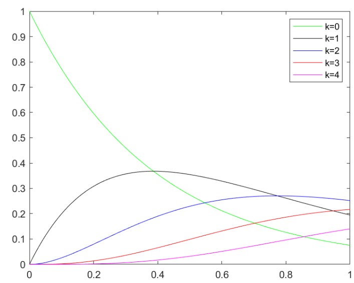

Fig.6 The quartic Szász basis with $\alpha(x) = \alpha_2(x)$, $\mu = 3, k = 0, 1, 2, 3, 4$

The new basis functions $s_{n,k}(x)$ hold the following properties.
(1). Non-negativity: For $x \in [0,\infty), \mu > 0, s_{n,k}(x) \geq 0$.
(2). Partition of unity: $\sum_{k=0}^{\infty} s_{n,k}(x) = 1$.
(3). Properties at the endpoint:

$$s_{n,k}(0) = \begin{cases} 1, & k = 0; \\ 0, & k \neq 0. \end{cases}$$

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(4). Integrals:

$$\int_0^{\infty} s_{n,k}(x)dx = \begin{cases} \frac{1-e^{-\frac{\mu}{n}}}{\mu}, & \alpha(x) = \alpha_1(x); \\ \frac{e^{\frac{2\mu}{n}}-1}{2\mu}, & \alpha(x) = \alpha_2(x). \end{cases}$$

(5). Derivative: $s'_{n,k}(x) = \frac{n\alpha(x)}{x}[s_{n,k-1}(x) - s_{n,k}(x)], x \in (0,\infty)$.

(6). Maximum value: $s_{n,k}(x)$ has only one local maximum at $x = \frac{k(1-e^{-\frac{\mu}{n}})}{\mu}$ and $\frac{k(e^{\frac{2\mu}{n}}-1)}{2\mu}$ for $\alpha_1(x)$ and $\alpha_2(x)$ respectively.

For the operators $S_n^{\mu}(f;x)$, we can deduce the following geometric properties from those of the Szász basis functions.

(1). Normativity and boundedness: $S_n^{\mu}(1;x) = 1$.
(2). Endpoint interpolation: $S_n^{\mu}(f;0) = f(0)$.
(3). Linearity: For all real numbers λ_1 and λ_2, and functions $f(x)$, $g(x)$, one has

$$S_n^{\mu}(\lambda_1 f + \lambda_2 g; x) = \lambda_1 S_n^{\mu}(f;x) + \lambda_2 S_n^{\mu}(g;x).$$

3. Shape Preservation

First, let us consider the monotonicity and convexity for the operators $S_n^{\mu}(f;x)$.

Theorem 3.1.(Monotonicity) Let $f \in C[0,\infty)$, if $f(x)$ is monotonically increasing (or decreasing) on $[0,\infty)$, for $n \in N$, so are all the operators $S_n^{\mu}(f;x)$.

Proof. We write

$$\frac{d}{dx}S_n^{\mu}(f;x) = \frac{n\alpha(x)}{x}e^{-n\alpha(x)}\sum_{k=0}^{\infty}\left[f(\frac{k+1}{n}) - f(\frac{k}{n})\right]\frac{(n\alpha(x))^k}{k!}. \tag{1}$$

Noting that if $f(x)$ is monotonically increasing, then the derivative of the operators $S_n^{\mu}(f;x)$ is nonnegative on $[0,\infty)$, and so $S_n^{\mu}(f;x)$ is monotonically increasing.

Similarly, we see that if $f(x)$ is monotonically decreasing on $[0,\infty)$, so are the operators $S_n^{\mu}(f;x)$.

Theorem 3.2.(Convexity) Let $f(x) \in C[0,\infty)$, if $f(x)$ is convex (or concave), so are all the operators $S_n^{\mu}(f;x)$.

Proof. Now let us take the second order derivative of $S_n^{\mu}(f;x)$. It follows from (1) that

$$\frac{d^2}{dx^2}S_n^{\mu}(f;x) = \left(\frac{n\alpha(x)}{x}\right)^2 e^{-n\alpha(x)}\sum_{k=0}^{\infty}\left[f(\frac{k+2}{n}) - 2f(\frac{k+1}{n}) + f(\frac{k}{n})\right]\frac{(n\alpha(x))^k}{k!}. \tag{2}$$

If $f(x)$ is convex on $C[0,\infty)$, all second order derivative of $S_n^{\mu}(f;x)$ in (2) is nonnegative, which implies the convexity of $S_n^{\mu}(f;x)$.

Similarly, we see that if $f(x)$ is concave on $C[0,\infty)$, so are the operators $S_n^{\mu}(f;x)$.

Next, we turn to the starlikeness and semi-additivity.

Theorem 3.3.(Starlikeness) If $f(x) \in A_2^*$ (or A_2), then for $n \in N$, $S_n^{\mu}(f;x) \in A_2^*$ (or A_2).

Proof.

$$\begin{aligned} \frac{d}{dx}(x^{-1}S_n^{\mu}(f;x)) &= -x^{-2}S_n^{\mu}(f;x) + x^{-1}\frac{d}{dx}S_n^{\mu}(f;x) \\ &= -x^{-2}S_n^{\mu}(f;x) + x^{-2}\alpha(x)e^{-n\alpha(x)}\sum_{k=0}^{\infty} n\left[f(\frac{k+1}{n}) - f(\frac{k}{n})\right]\frac{(n\alpha(x))^k}{k!}, \end{aligned}$$

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where

$$\begin{aligned}\sum_{k=0}^{\infty} n\left[f(\frac{k+1}{n})-f(\frac{k}{n})\right]\frac{(n\alpha(x))^k}{k!} &= \left[nf(\frac{1}{n})-nf(\frac{0}{n})\right]+\left[nf(\frac{2}{n})-nf(\frac{1}{n})\right]n\alpha(x)\\ &\quad+\left[nf(\frac{3}{n})-nf(\frac{2}{n})\right]\frac{(n\alpha(x))^2}{2!}+\cdots\\ &= nf(\frac{1}{n})+\left[\frac{n}{2}f(\frac{2}{n})-nf(\frac{1}{n})\right]n\alpha(x)+\frac{n}{2}f(\frac{2}{n})n\alpha(x)\\ &\quad+2\left[\frac{n}{3}f(\frac{3}{n})-\frac{n}{2}f(\frac{2}{n})\right]\frac{(n\alpha(x))^2}{2!}+\frac{n}{3}f(\frac{3}{n})\frac{(n\alpha(x))^2}{2!}+\cdots.\end{aligned}$$

Let

$$I=x^{-2}\alpha(x)e^{-n\alpha(x)}\sum_{k=1}^{\infty}\left[\frac{n}{k+1}f(\frac{k+1}{n})-\frac{n}{k}f(\frac{k}{n})\right]\frac{(n\alpha(x))^k}{k!},$$

which yields,

$$\begin{aligned}\frac{d}{dx}(x^{-1}S_n^{\mu}(f;x)) &= -x^{-2}S_n^{\mu}(f;x)+x^{-2}\alpha(x)e^{-n\alpha(x)}\sum_{k=1}^{\infty}\frac{n}{k}f(\frac{k}{n})\frac{(n\alpha(x))^{k-1}}{(k-1)!}+I\\ &= -x^{-2}S_n^{\mu}(f;x)+x^{-2}e^{-n\alpha(x)}\sum_{k=1}^{\infty}f(\frac{k}{n})\frac{(n\alpha_n(x))^k}{k!}+I\\ &= -x^{-2}S_n^{\mu}(f;x)+x^{-2}S_n^{\mu}(f;x)-x^{-2}e^{-n\alpha(x)}f(\frac{0}{n})+I\\ &= x^{-2}\alpha(x)e^{-n\alpha(x)}\sum_{k=1}^{\infty}\left[\frac{n}{k+1}f(\frac{k+1}{n})-\frac{n}{k}f(\frac{k}{n})\right]\frac{(n\alpha(x))^k}{k!}.\end{aligned}$$

Noting that if $x^{-1}f(x)$ is decreasing on $(0,\infty)$, then $I\le 0$, i.e. the derivative of $x^{-1}S_n^{\mu}(f;x)$ is negative on $(0,\infty)$, and so $x^{-1}S_n^{\mu}(f;x)$ is decreasing.
Similarly, we see that if $x^{-1}f(x)$ is increasing on $(0,\infty)$, then $I\ge 0$, so is $x^{-1}S_n^{\mu}(f;x)$.
Theorem 3.4.(Semi-additivity) If $f(x)\in A_3^*$ (or A_3), then for $n\in N$, $S_n^{\mu}(f;x)\in A_3^*$ (or A_3).
Proof. We only take $\alpha(x)=\alpha_1(x)$ as an example to prove the semi-additivity, the case $\alpha(x)=\alpha_2(x)$ is similar. $\forall x,y\in[0,\infty)$, $\forall n\in N$,

$$\begin{aligned}S_n^{\mu}(f;x+y) &= e^{-\frac{\mu(x+y)}{1-e^{-\frac{\mu}{n}}}}\sum_{k=0}^{\infty}f(\frac{k}{n})\frac{\left(\frac{\mu(x+y)}{1-e^{-\frac{\mu}{n}}}\right)^k}{k!}\\ &= e^{-\frac{\mu(x+y)}{1-e^{-\frac{\mu}{n}}}}\sum_{k=0}^{\infty}f(\frac{k}{n})\frac{\left(\frac{\mu}{1-e^{-\frac{\mu}{n}}}\right)^k}{k!}\sum_{j=0}^{k}\binom{k}{j}x^jy^{k-j}\\ &= e^{-\frac{\mu(x+y)}{1-e^{-\frac{\mu}{n}}}}\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}f(\frac{k}{n})\left(\frac{\mu}{1-e^{-\frac{\mu}{n}}}\right)^k\frac{x^jy^{k-j}}{j!(k-j)!},\end{aligned}$$

let $K=k-j$, then $k=K+j$,

$$S_n^{\mu}(f;x+y)=e^{-\frac{\mu x}{1-e^{-\frac{\mu}{n}}}}\cdot e^{-\frac{\mu y}{1-e^{-\frac{\mu}{n}}}}\sum_{j=0}^{\infty}\sum_{K=0}^{\infty}f(\frac{K+j}{n})\frac{\left(\frac{\mu x}{1-e^{-\frac{\mu}{n}}}\right)^j}{j!}\cdot\frac{\left(\frac{\mu y}{1-e^{-\frac{\mu}{n}}}\right)^K}{K!}. \tag{3}$$

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If $f(x)$ is semi-additive on $[0,\infty)$, one has,

$$\begin{aligned} S_n^{\mu}(f;x+y) &\leq e^{-\frac{\mu x}{1-e^{-\frac{\mu}{n}}}} \cdot e^{-\frac{\mu y}{1-e^{-\frac{\mu}{n}}}} \sum_{j=0}^{\infty}\sum_{K=0}^{\infty} f(\frac{K}{n}) \frac{\left(\frac{\mu x}{1-e^{-\frac{\mu}{n}}}\right)^j}{j!} \cdot \frac{\left(\frac{\mu y}{1-e^{-\frac{\mu}{n}}}\right)^K}{K!} \\ &\quad + e^{-\frac{\mu x}{1-e^{-\frac{\mu}{n}}}} \cdot e^{-\frac{\mu y}{1-e^{-\frac{\mu}{n}}}} \sum_{j=0}^{\infty}\sum_{K=0}^{\infty} f(\frac{j}{n}) \frac{\left(\frac{\mu x}{1-e^{-\frac{\mu}{n}}}\right)^j}{j!} \cdot \frac{\left(\frac{\mu y}{1-e^{-\frac{\mu}{n}}}\right)^K}{K!} \\ &= S_n^{\mu}(f;x) + S_n^{\mu}(f;y) \end{aligned}$$

which implies $S_n^{\mu}(f;x)$ is semi-additive on $[0,\infty)$. Similarly, if $f(x)$ is super-additive on $[0,\infty)$, so is $S_n^{\mu}(f;x)$, and thus, the proof is completed.

4. Preservation of Smoothness

For $t>0$, the continuous modulus is defined as [13]:

$$\omega(f;t) = sup\{|f(x)-f(y)| : |x-y| \leq t, x, y \in [0,\infty)\}.$$

A function $\omega(t)$ on $[0,1]$ s called a modulus of continuity if $\omega(t)$ is continuous, nondecreasing, semi-additive, and $\lim\limits_{t\to 0^+} \omega(t) = \omega(0) = 0.$[4]

Lemma 4.1. [14] For any continuous $\omega(t)$(not identical to 0), there exists a concave continuous modulus $\omega^*(t)$ such that for $t>0$, one has $\omega(t) \leq \omega^*(t) \leq 2\omega(t)$, where the constant 2 can not be any smaller.

Lemma 4.2. Let $\{L_n(f;x); n = 0,1,\cdots\}$ be a sequence of linear positive operators from $C(I)$ to $C(I)$, where I is a finite or infinite interval, and $L_n(1;x) = 1$, $L_n(t;x) = C_n(x)x, 0 \leq C_n(x) \leq 1$. If $g(x)$ is a convave, monotonically increasing and continuous function, then $L_n(g;x) \leq g(x)$.

Remark 4. We can get Lemma 4.2 by imitating the proof of the Lemma in Ref [10], here we omit the details.

Theorem 4.1. For $f(x) \in C[0,\infty)$, $\forall n \in N$, one has $\omega(S_n^{\mu};\delta) \leq S_n^{\mu}(\omega;\delta)$.

Proof. As before, we only prove the case $\alpha(x) = \alpha_1(x)$, the case $\alpha(x) = \alpha_2(x)$ is similar. Since

$$e^{-\frac{\mu x}{1-e^{-\frac{\mu}{n}}}} \sum_{j=0}^{\infty} \frac{\left(\frac{\mu x}{1-e^{-\frac{\mu}{n}}}\right)^j}{j!} = 1,$$

for $h>0$, from (3), we write

$$\begin{aligned} \left|S_n^{\mu}(f;x+h) - S_n^{\mu}(f;x)\right| &\leq e^{-\frac{\mu x}{1-e^{-\frac{\mu}{n}}}} \cdot e^{-\frac{\mu h}{1-e^{-\frac{\mu}{n}}}} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\left(\frac{\mu x}{1-e^{-\frac{\mu}{n}}}\right)^j}{j!} \cdot \frac{\left(\frac{\mu h}{1-e^{-\frac{\mu}{n}}}\right)^k}{k!} \left|f(\frac{k+j}{n}) - f(\frac{j}{n})\right| \\ &\leq e^{-\frac{\mu x}{1-e^{-\frac{\mu}{n}}}} \cdot e^{-\frac{\mu h}{1-e^{-\frac{\mu}{n}}}} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\left(\frac{\mu x}{1-e^{-\frac{\mu}{n}}}\right)^j}{j!} \cdot \frac{\left(\frac{\mu h}{1-e^{-\frac{\mu}{n}}}\right)^k}{k!} \omega(f;\frac{k}{n}), \end{aligned} \quad (4)$$

then the desired result is obtained.

Theorem 4.2.

(i) For $f \in H_{\omega}(A) = \{f : \omega(f;x) \leq A\omega(x)\}$, then $S_n^{\mu}(f;x) \in H_{\omega}(2A)$, $\forall n \in N$;

(ii) If $\lim\limits_{n\to\infty} S_n^{\mu}(f;x) = f(x)$, then $f \in H_{\omega}(A) \Leftrightarrow S_n^{\mu}(f;x) \in H_{\omega}(A)$, here n is big enough.

Proof. (i) As before, we only prove the case $\alpha(x) = \alpha_1(x)$, the case $\alpha(x) = \alpha_2(x)$ is similar. $\forall x_1, x_2 \in I$, $x_1 > x_2$, from (4), and using Lemma 4.1, we have

$$\begin{aligned}\left|S_n^\mu(f;x_1) - S_n^\mu(f;x_2)\right| &\leq e^{-\frac{\mu(x_1-x_2)}{1-e^{-\frac{\mu}{n}}}} \sum_{j=0}^{\infty} \omega(f;\frac{j}{n}) \frac{\left(\frac{\mu(x_1-x_2)}{1-e^{-\frac{\mu}{n}}}\right)^j}{j!} \leq AL_n(\omega; x_1 - x_2) \\ &\leq AL_n(\omega^*; x_1 - x_2) \leq A\omega^*(x_1 - x_2) \leq 2A\omega(x_1 - x_2),\end{aligned}$$

here we use Lemma 4.2 for $\omega^*(t)$.
(ii) If $lim_{n\to\infty} S_n^\mu(f;x) = f(x)$, then

$$\lim_{n\to\infty} \left|S_n^\mu(f;x_1) - S_n^\mu(f;x_2)\right| = |f(x_1) - f(x_2)|,$$

we have $f \in H_\omega(A) \Leftrightarrow S_n^\mu(f;x) \in H_\omega(A)$, here n is big enough.
Remark 5. If $f \in Lip_K^\alpha = \{f : \omega(f;\delta) \leq K\delta^\alpha, K \geq 0, \alpha \in (0,1]\}$, then $S_n^\mu(f;x) \in Lip_K^\alpha$ here n is big enough.
Theorem 4.3. For any modulus of continuity $\omega(t)$, $S_n^\mu(\omega;t)$ is also a modulus of continuity.
Proof. For any modulus of continuity $\omega(t)$, $S_n^\mu(\omega;t)$ is continuous, nondecreasing, and

$$\lim_{t\to 0^+} S_n^\mu(\omega;t) = S_n^\mu(\omega;0) = \omega(0) = 0.$$

Combining the semi-additivity of $\omega(t)$ and $S_n^\mu(\omega;t)$, we deduce that $S_n^\mu(\omega;t)$ is a modulus of continuity.

5. Conclusions

In this paper, some fundamental facts of two kinds of parametric Szász type operators are presented. The shape preserving properties, such as linearity, monotonicity, convexity, starlikeness, semi-additivity are examined. Finally, with the help of the continuous modulus, the preservation of smoothness are discussed. We think that these newly operators are appliable to CAGD. For future work, we propose to obtain some inverse theorems of these kinds of operators.

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