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Representations by Beurling Systems

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Abstract: We say that a system $\{z^m F(z)\}_{m=0}^{\infty}$ is a Beurling system if F is an outer function. Beurling's approximation theorem asserts that if F is an outer function from $H^2(\mathbb{D})$ then the system $\{z^m F(z)\}_{m=0}^{\infty}$ is complete in the space $H^2(\mathbb{D})$. We prove that a Beurling system with $F \in H^p(\mathbb{D}), 1 \le p < \infty$ is an $M-$bases in $H^p(\mathbb{D})$ with an explicit dual system. Any function $f \in H^p(\mathbb{D}), 1 \le p < \infty$ can be expanded as a series by the system $\{z^m F(z)\}_{m=0}^{\infty}$. For different methods of summation we characterize outer functions F for which the expansion converges to f. Related results for weighted Hardy spaces in the unit disc are studied. Particularly we prove Rosenblum's hypothesis.

Keywords: summation basis; Hardy spaces; outer function; Beurling system; kernels; representation of functions

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0. Introduction

We say that a system $\{z^m F(z)\}_{m=0}^{\infty}$ is a Beurling system if F is an outer function. In his fundamental work [3] Beurling particularly proved that if F is an outer function from $H^2(\mathbb{D})$ then the system $\{z^m F(z)\}_{m=0}^{\infty}$ is complete in the space $H^2(\mathbb{D})$. This result can be easily extended for the spaces $H^p(\mathbb{D}), 1 \le p < \infty$ (see [4]). In the present paper we study questions of representations of functions from the spaces $H^p(\mathbb{D}), 1 \le p < \infty$ by series with respect to Beurling systems. The key result is that Beurling systems are $M-$bases in $H^p(\mathbb{D}), 1 \le p < \infty$ spaces with a dual system which is explicitely written. Afterwards, it is natural to characterize outer functions F for which the system $\{z^m F(z)\}_{m=0}^{\infty}$ is a basis in $H^p(\mathbb{D})$ in one sense or another. In the theory of $H^p(\mathbb{D})$ spaces the most interesting case is to characterize the functions F for which the corresponding Beurling system $\{z^m F(z)\}_{m=0}^{\infty}$ is an $A-$summation basis in $H^p(\mathbb{D}), 1 \le p < \infty$.

The obtained results can be interpreted in terms of weighted H^p spaces with weights which we call admissible weight functions. A non-negative function w defined on the boundary such that $\ln w$ is integrable is called an admissible weight function. That the system $\{e^{int}\}_{n=0}^{\infty}$ is minimal in $H^p(\mathbb{T}, w)$, $1 \le p < \infty$ was mentioned in [10] without proof. The author did it by purpose with the hope to find afterwards the dual system which will permit to indicate the corresponding kernel for representation of any holomorphic function from the weighted $H^p(\mathbb{D})$ space by its boundary values. Thus one can extend for weighted norm spaces results known for the H^p spaces. Moreover, this approach can be helpful for extensions of those results for more general domains. The obtained results permit us to study the systems $e^{ikt}\Psi_{L,M}$ in the spaces $L^p(\mathbb{T})$, where

$$\Psi_{L,M} = \{\overline{L(t)}e^{int}\}_{n=-\infty}^{-1} \cup \{M(t)e^{int}\}_{n=0}^{\infty}$$

and L, M are boundary values of some outer functions defined in $\mathbb{D}$. Times are changing and it is not surprising that the last study was published [11] before the present research.

The paper is divided into two parts. In the first part will be given results for the Beurling systems and the second part will be dedicated to the study of weighted H^p spaces.

0.1. Preliminarely Results, Definitions and Notations

We say that $w \geq 0$ is a weight function on a measurable set $E \subseteq \mathbb{R}$ if w is integrable on E. A function $\varphi \in L^p(E, w), 1 \leq p < \infty$ if $\varphi : E \to \mathbb{C}$ is measurable on E and the norm is defined by

$$\|\varphi\|_{L^p(E,w)} := \left(\int_E |\varphi(t)|^p w(t) dt \right)^{\frac{1}{p}} < +\infty.$$

when $w \equiv 1$ we write $L^p(E)$. Denote $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and identify $\mathbb{T}$ with any 2π length semi-open interval on the real line. For $1 < p < \infty$ the conjugate number p' is defined from the equation $\frac{1}{p} + \frac{1}{p'} = 1$ and $p' = \infty$ if $p = 1$. The set of integers is denoted by $\mathbb{Z}$ and $\mathbb{N} = \{1, 2, \dots\}$.

By $S[f](t)$ we denote the Fourier series of a function $f \in L^1(\mathbb{T})$. For any $n \in \mathbb{N}$

$$S_n[\phi](t) = \sum_{j=-n}^{n} c_j(\phi) e^{ijt}, \qquad c_j(\phi) = \frac{1}{2\pi} \int_{\mathbb{T}} \phi(\theta) e^{-ij\theta} d\theta.$$

The space of continuous functions on $\mathbb{T}$ with the maximum norm is denoted by $C(\mathbb{T})$. For $1 \leq p \leq \infty$ we put

$$H^p(\mathbb{T}) = \{\phi \in L^p(\mathbb{T}) : \int_{\mathbb{T}} \phi(t) e^{int} dt = 0 \, \text{for all} \, n \in \mathbb{N}\}.$$

The spaces $H^p(\mathbb{T}), 1 \leq p \leq \infty$ are Banach spaces of functions defined on $\mathbb{T}$. The Cauchy kernel is defined as follows:

$$C_r(\theta) = \sum_{n=0}^{+\infty} r^n e^{in\theta} \quad 0 < r < 1, \theta \in \mathbb{T},$$

and also the Poisson and conjugate Poisson kernels:

$$P_r(\theta) = \sum_{n=-\infty}^{+\infty} r^{|n|} e^{in\theta} = \frac{1 - r^2}{1 - 2r\cos\theta + r^2},$$

$$Q_r(\theta) = \operatorname{Im} H_r(\theta), \quad P_r(\theta) = \operatorname{Re} H_r(\theta),$$

where

$$H_r(\theta) = 2C_r(\theta) - 1 = \frac{1 + re^{i\theta}}{1 - re^{i\theta}} \quad (0 < r < 1, \theta \in \mathbb{T}).$$

We denote $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and its closure by $\overline{\mathbb{D}}$. The convolution of functions $g, \varphi \in L(\mathbb{T})$ is denoted by

$$g * \varphi(t) = \frac{1}{2\pi} \int_{\mathbb{T}} g(\theta) \varphi(t - \theta) d\theta.$$

Let

$$\ln^+ x = \begin{cases} \ln x, & \text{if} \quad x \geq 1 \\ 0, & \text{if} \quad x < 1. \end{cases}$$

A holomorphic function $f(z), z \in \mathbb{D}$ is said to be of class **N** if

$$\sup_{0 \leq r < 1} \int_{\mathbb{T}} \ln^+ |f(re^{it})| dt < +\infty,$$

and $f \in H^p(\mathbb{D}), 1 \leq p < \infty$ if

$$\sup_{0 \leq r < 1} \int_{\mathbb{T}} |f(re^{it})|^p dt < +\infty.$$

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Moreover, $f \in H^\infty(\mathbb{D})$ if $\sup_{0\le r<1} \|f(re^{it})\|_{L^\infty(\mathbb{T})} < +\infty$. We also have that $H^p(\mathbb{D}) \subset \mathbf{N}$ for all $1 \le p \le \infty$.

If $f \in \mathbf{N}$ by a well known theorem [15] (see also [4]) f is a quotient of two bounded holomorphic functions. Hence, by Fatou's theorem, the non-tangential limit $f(e^{it})$ exists almost everywhere (a.e.) on the unit circle, and $\ln|f(e^{it})|$ is integrable unless f vanishes everywhere. Moreover, the map $T : f(z) \longrightarrow f(e^{it})$ establishes an isomorphism of $H^p(\mathbb{D}), 1 \le p < \infty$ onto $H^p(\mathbb{T})$. Furtheron facts related with metric properties in the space $H^p(\mathbb{D})$ we use in $H^p(\mathbb{T})$ and vice versa without any special quotation.

Spaces $H^p(\mathbb{D})$ have been studied in several books (e.g. [4,7,12,20] and others). A holomorphic function F in $\mathbb{D}$ is an outer function if

$$F(re^{it}) = e^{i\alpha}e^{\varphi * H_r(t)}, \quad \alpha \in \mathbb{T},$$

where φ is a real-valued integrable function defined on $\mathbb{T}$ [3](see also [6,19]). Evidently F is a non-zero holomorphic function and $F \in H^1(\mathbb{D})$ if and only if $e^{\varphi(t)}$ is integrable. The function F has non-tangential limits a.e. on the unit circle: $F^*(t) := \lim_{z\to e^{it}} F(z)$, and

$$|F^*(t)| = e^{\varphi(t)}.$$

Moreover, $\ln|F(z)|$ is an harmonic function in $\mathbb{D}$ and

$$\frac{1}{2\pi}\int_{\mathbb{T}} \ln|F(re^{it})|dt = \ln|F(0)| \quad 0 \le r < 1.$$

For a complex-valued integrable function g defined on $\mathbb{T}$ such that $\ln|g(t)|$ is integrable we set

$$G_g(re^{it}) = e^{\ln|g| * H_r(t)}. \tag{1}$$

The following statement [7] holds.

Proposition 1. *Let $F \in H^1(\mathbb{D})$ be an outer function. Then*

$$G_{F^*}(z) = e^{i\alpha}F(z), z \in \mathbb{D} \quad \textit{for some} \quad \alpha \in \mathbb{T}.$$

If $F \in H^p(\mathbb{D}), 1 \le p < \infty$ then

$$G_{F^*}(re^{it}) = e^{\ln|F^*| * P_r(t)}e^{i\ln|F^*| * Q_r(t)}$$

and by Fatou's and Luzin-Privalov's theorems [2,20] we have that

$$\ln|F^*(t)| = \lim_{r\to 1-} \ln|F^*| * P_r(t) \qquad \text{a.e. on} \quad \mathbb{T},$$

$$\lim_{r\to 1-} \ln|F^*| * Q_r(t) := \widetilde{\ln|F^*|}(t) \qquad \text{a.e. on} \quad \mathbb{T},$$

where by $\tilde{g}$ is denoted the conjugate function of an integrable function g. Thus we have that a.e. on $\mathbb{T}$

$$G^*_{F^*}(t) = \lim_{r\to 1-} G_{F^*}(re^{it}) = |F^*(t)|e^{i\widetilde{\ln|F^*|}(t)},$$

By Jensen's inequality it follows that for $0 < r < 1$

$$\frac{1}{2\pi}\int_{\mathbb{T}} |G_{F^*}(re^{i\theta})|^p d\theta \le \frac{1}{2\pi}\int_{\mathbb{T}} |F^*(\theta)|^p d\theta,$$

which yields $G_{F^*} \in H^p(\mathbb{D})$ and $G^*_{F^*} \in H^p(\mathbb{T})$. We also have that

$$G_{F^*}(0) = e^{\frac{1}{2\pi}\int_{\mathbb{T}} \ln|F^*(t)|dt} \neq 0.$$

The function $\frac{1}{G_{F^*}(re^{it})}$ is holomorphic in $\mathbb{D}$, has no zeros and belongs to $\mathbf{N}$. Clearly

$$\lim_{r\to 1-}[G_{F^*}(re^{it})]^{-1} = |F^*(t)|^{-1}e^{-i\widetilde{\ln|F^*|}(t)}.$$

Let $\mathbf{B}$ be a separable Banach space with the dual space $\mathbf{B}^*$. The closed linear span in $\mathbf{B}$ of a system of elements $X = \{x_k\}_{x=0}^{\infty} \subset \mathbf{B}$ is denoted by $\overline{\text{span}}_{\mathbf{B}}(X)$. A system $X = \{x_k\}_{x=0}^{\infty} \subset \mathbf{B}$ is complete in $\mathbf{B}$ if $\overline{\text{span}}_{\mathbf{B}}(X) = \mathbf{B}$. A system $X = \{x_k\}_{k=0}^{\infty} \subset \mathbf{B}$ is called minimal, if there exists a system $X^* = \{\phi_n\}_{n=0}^{\infty} \subset \mathbf{B}^*$, such that

$$\phi_n(x_k) = \delta_{nk} \quad (n,k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

where δ_{nk} is the Kronecker symbol ($\delta_{nk} = 0 \quad \text{if} \quad n \neq k \quad \text{and} \quad \delta_{kk} = 1$). The system X^* is called dual to X. It is easy to observe that if X is a complete and minimal system in $\mathbf{B}$ then the dual system X^* is unique [13]. A set $\Psi \subset \mathbf{B}^*$ is called total if

$$\phi(x) = 0 \quad \text{for all} \quad \phi \in \Psi$$

if and only if $x = \mathbf{0}$. A system $X = \{x_k\}_{k=0}^{\infty} \subset \mathbf{B}$ is an $M-$basis in $\mathbf{B}$ if X is complete and minimal in $\mathbf{B}$ and its dual system X^* is total. A complete and minimal system $X = \{x_k\}_{k=0}^{\infty} \subset \mathbf{B}$ with the dual system $X^* = \{\phi_k\}_{k=0}^{\infty} \subset \mathbf{B}^*$ is uniformly minimal if there exists $C > 0$ such that

$$\|x_k\|_{\mathbf{B}}\|\phi_k\|_{\mathbf{B}^*} \leq C \qquad \text{for all} \quad k \in \mathbb{N}_0.$$

We will say that a system of elements $X = \{x_k\}_{k=0}^{\infty} \subset \mathbf{B}$ is an $A-$basis of the Banach space $\mathbf{B}$ if X is closed and minimal in $\mathbf{B}$ and for any $x \in \mathbf{B}$

$$\lim_{r\longrightarrow 1-} \|x - \sum_{k=1}^{\infty} r^k\phi_k^*(x)x_k\|_{\mathbf{B}} = 0,$$

where $X^* = \{\phi_k^*\}_{n=1}^{\infty} \subset B^*$ is the dual system.

0.2. Classes of Weight Functions

Furtheron we will consider only weight functions on $\mathbb{T}$. For any $1 \leq p < \infty$ we denote by $\mathcal{W}_p$ the class of all weight functions $w \geq 0$ integrable on $\mathbb{T}$ and such that

$$\int_{\mathbb{T}} [w(t)]^{-\frac{1}{p-1}}dt < +\infty, \qquad \text{if} \quad p = 1 \qquad \text{then} \quad \frac{1}{w} \in L^{\infty}(\mathbb{T}).$$

Denote

$$e^{\mathcal{W}} := \{w \geq 0 : \ln w \in L^1(\mathbb{T})\}.$$

We say that $w(x) \geq 0$ is an admissible weight function if $w \in e^{\mathcal{W}}$. The class $\mathcal{A}_p, p \geq 1$ contains only weights w which satisfy the following condition: there exists $C_p > 0$ such that

$$\frac{1}{|I|}\int_I w(t)dt \left[\frac{1}{|I|}\int_I w(t)^{-\frac{1}{p-1}}dt\right]^{p-1} \leq C_p$$

holds for any interval $I \subset \mathbb{T}$. Sometimes it is called Muckenhoupt's condition [14]. We note that the class $\mathcal{A}_p \subset \mathcal{W}_p$ in an equivalent form had appeared earlier in M. Rosenblum's article [16], where

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weighted H^p spaces were considered, maybe for the first time. In the same article another class of weight functions $\mathcal{R}$ was studied. We will say that $w \in \mathcal{R}$ if $w \in e^{\mathcal{W}}$ and there exists $C > 0$ such that

$$\frac{1}{2\pi}\int_{\mathbb{T}} \frac{w(t)}{|G_w(re^{it})|} P_r(\theta - t)dt \leq C \qquad \forall \theta \in \mathbb{T} \quad \text{and} \quad 0 \leq r < 1. \tag{2}$$

By (1) it is easy to observe that if $w \in e^{\mathcal{W}}$ then for $q > 0$

$$|G_{w^{1/q}}(re^{it})|^q = |G_w(re^{it})| \qquad \forall t \in \mathbb{T} \quad \text{and} \quad 0 \leq r < 1. \tag{3}$$

Note that (see [16])

$$\mathcal{A}_\infty := \bigcup_{p \geq 1} \mathcal{A}_p \subseteq \mathcal{R} \subseteq e^{\mathcal{W}}. \tag{4}$$

1. On Beurling Systems

Let $p \in [1, +\infty)$ be a fixed number and suppose that $F \in H^p(\mathbb{D})$ is an outer function. By Proposition 1 we have that $G^*_{F^*}(t) = e^{i\alpha}F^*(t)$ a.e. on $\mathbb{T}$. Furtheron we will suppose that

$$G_{F^*}(z) = F(z), \quad z \in \mathbb{D} \quad \text{and} \quad F(0) \in \mathbb{R},$$

for convenience. Observe that if $\{e^{ijt}F^*(t)\}_{j=0}^\infty$ is a basis in one or another sense then the system $\{e^{ijt}cF^*(t)\}_{j=0}^\infty$ for any constant c, $|c| = 1$ will be a basis in the same sense. We write the Fourier series of the function $F^* \in H^p(\mathbb{T})$:

$$F^*(\theta) \sim a_0 + \sum_{n=1}^\infty a_n e^{in\theta}, \qquad a_0 \neq 0, a_0 \in \mathbb{R}.$$

Moreover,

$$F(z) = a_0 + \sum_{n=1}^\infty a_n z^n \quad z \in \mathbb{D}.$$

Let

$$[F(z)]^{-1} = c_0 + \sum_{n=1}^\infty c_n z^n \quad z \in \mathbb{D}. \tag{5}$$

then for $z \in \mathbb{D}$

$$1 = F(z)[F(z)]^{-1} = 1 + \sum_{n=1}^\infty b_n z^n,$$

where

$$b_n = \sum_{j=0}^n a_j c_{n-j} = 0 \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad c_0 = \frac{1}{a_0}. \tag{6}$$

1.1. A Remarkable System of Polynomials

Set $T_0(t) \equiv c_0$ and for $n \in \mathbb{N}$

$$T_n(t) = c_0 e^{in\theta} + \sum_{\nu=0}^{n-1} \bar{c}_{n-\nu} e^{i\nu t}, \tag{7}$$

where $\{c_j\}_{j=0}^{\infty}$ are the corresponding coefficients of the representation of $F^{-1}(z)$. By (6) we obtain that if $j \in \mathbb{N}_0$ and $j \leq n$

$$\frac{1}{2\pi}\int_{\mathbb{T}} e^{ijt}F^*(t)\overline{T_n(t)}dt = \frac{1}{2\pi}\int_{\mathbb{T}}\sum_{k=0}^{n-j} a_k e^{i(j+k)t}\overline{T_n(t)}dt$$
$$= \sum_{k=0}^{n-j} a_k c_{n-j-k} = \delta_{jn}.$$

It is clear that the above integral is equal to zero if $j > n$. The following theorem holds.

Theorem 1. *The system $\{e^{ijt}F^*(t)\}_{j=0}^{\infty}$ is an $M-$basis in $H^p(\mathbb{T})$.*

Proof. We have checked that $\{T_n(t)\}_{n=0}^{\infty}$ is the system dual to $\{e^{ijt}F^*(t)\}_{j=0}^{\infty}$. Suppose that there exists $f \in H^p(\mathbb{T})$ such that

$$\int_{\mathbb{T}} f(t)\overline{T_n(t)}dt = 0 \quad \text{for all} \quad n \in \mathbb{N}_0.$$

From (7) it follows that

$$\int_{\mathbb{T}} e^{-int}f(t)dt = 0 \quad \text{for all} \quad n \in \mathbb{N}_0.$$

Hence, $f(t) = 0$ a.e. on $\mathbb{T}$. □

Theorem 2. *The system $\{e^{ijt}F^*(t)\}_{j=0}^{\infty}$ is uniformly minimal in $H^p(\mathbb{T})$, $1 < p < \infty$ if and only if $[F^*]^{-1} \in H^{p'}(\mathbb{T})$.*

Proof. If $[F^*]^{-1} \in H^{p'}(\mathbb{T}), 1 < p < \infty$ then by Proposition 1 and Jensen's inequality we have that that for $0 < r < 1$

$$\frac{1}{2\pi}\int_{\mathbb{T}} |F(re^{it})|^{-p'}dt = \frac{1}{2\pi}\int_{\mathbb{T}} e^{\ln|F^*|^{-p'} * P_r(t)}dt$$
$$\leq \frac{1}{2\pi}\int_{\mathbb{T}} |F^*(\theta)|^{-p'}d\theta < +\infty,$$

which yields $\frac{1}{F} \in H^{p'}(\mathbb{D})$. Set

$$S_n(t) = \sum_{k=0}^{n} c_k e^{ikt}, \qquad n \in \mathbb{N}. \tag{8}$$

Then we have that $e^{int}\overline{S_n(t)} = T_n(t)$ which means that

$$\|T_n\|_{L^q(\mathbb{T})} = \|S_n\|_{L^q(\mathbb{T})} \qquad \text{for} \quad 1 \leq q \leq \infty,\ n \in \mathbb{N}. \tag{9}$$

Note that $S_n[\frac{1}{F^*}](t) = S_n(t)$ if $\frac{1}{F^*} \in H^{p'}(\mathbb{T})$. Hence, there exists $C_{p'} > 0$ (see [20],vol.2, chapter 7) such that

$$\sup_n \|S_n\|_{L^{p'}(\mathbb{T})} \leq C_{p'}\|[F^*]^{-1}\|_{L^{p'}(\mathbb{T})}.$$

For the proof of the necessity suppose that the system $\{e^{ijt}F^*(t)\}_{j=0}^{\infty}$ is uniformly minimal in $H^p(\mathbb{T})$. The norms in $H^p(\mathbb{T})$ of all elements of the system $\{e^{ijt}F^*(t)\}_{j=0}^{\infty}$ are equal to $\|F^*\|_{L^p(\mathbb{T})}$. On the other hand by (9) it folows that

$$\sup_n \|S_n\|_{L^{p'}(\mathbb{T})} < +\infty,$$

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where $S_n(t)$ are defined by (8). By Banach's theorem (see [1]) on weak* compactness of the closed unit ball in the dual space it follows that there exists a subsequence of natural numbers $\{n_\nu\}_{\nu=1}^\infty$ and $\psi \in L^{p'}(\mathbb{T})$ such that for any $\varphi \in L^p(\mathbb{T})$

$$\lim_{\nu\to\infty} \frac{1}{2\pi}\int_{\mathbb{T}} \varphi(t)\overline{S_{n_\nu}(t)}dt = \frac{1}{2\pi}\int_{\mathbb{T}} \varphi(t)\overline{\psi(t)}dt.$$

Hence, $\psi \in H^{p'}(\mathbb{T})$ and

$$c_j(\psi) = c_j \qquad j \in \mathbb{N}_0.$$

Which means that

$$\psi * P_r(t) = \frac{1}{F(re^{it})} \qquad 0 \le r < 1, t \in \mathbb{T}.$$

Thus we obtain that ψ coincides with $[F^*]^{-1}$ a.e. on $\mathbb{T}$ and $[F^*]^{-1} \in H^{p'}(\mathbb{T})$. $\square$

Theorem 3. *If the system $\{e^{ijt}F^*(t)\}_{j=0}^\infty$ is uniformly minimal in $H^1(\mathbb{T})$ then $[F^*]^{-1} \in H^\infty(\mathbb{T})$. If $[F^*]^{-1} \in H^\infty(\mathbb{T})$ and the partial sums of its Fourier series are uniformly bounded in the $C(\mathbb{T})$ norm then the system $\{e^{ijt}F^*(t)\}_{j=0}^\infty$ is uniformly minimal in $H^1(\mathbb{T})$.*

We skip the proof because it is similar to the proof of the previous theorem.

The following lemma is a useful tool for the further exposition. Close statements can be find in [2,20].

Lemma 1. *Let $f \in H^p(\mathbb{T})$ and $g \in H^{p'}(\mathbb{T})$, where $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $p' = \infty$ if $p = 1$. Then*
1) $S[fg] = S[f]S[g]$, and $c_n(fg) = \sum_{j=0}^n c_j(f)c_{n-j}(g)$;
2) $S_n[fg](t) = \sum_{j=0}^n c_j(f)e^{ijt}S_{n-j}[g](t)$ for any $n = 0, 1, \ldots$.

Proof. We skip the proof of statement 1) because it should be well known. We have

$$\begin{aligned}\sum_{j=0}^n c_j(f)e^{ijt}S_{n-j}[g](t) &= \sum_{j=0}^n\sum_{\nu=0}^{n-j} c_j(f)c_\nu(g)e^{i(j+\nu)t} \\ &= \sum_{k=0}^n e^{ikt}\sum_{j=0}^n c_j(f)c_{k-j}(g).\end{aligned}$$

$\square$

Theorem 4. *Let $|F^*|^p \in \mathcal{A}_p$, $1 < p < \infty$. Then $\{e^{ijt}F^*(t)\}_{j=0}^\infty$ is a Schauder basis in $H^p(\mathbb{T})$.*

Proof. We should check that the conditions of Banach's theorem [1] hold in our case. By Theorem 1 we know that $\{e^{ijt}F^*(t)\}_{j=0}^\infty$ is complete and minimal in $H^p(\mathbb{T})$. Set

$$\begin{aligned}E_n(t,\theta) &= F^*(t)\sum_{k=0}^n e^{ikt}\overline{T_k}(\theta) = F^*(t)\sum_{k=0}^n e^{ikt}\sum_{j=0}^k c_{k-j}e^{-ij\theta} \\ &= F^*(t)\sum_{j=0}^n e^{-ij\theta}\sum_{k=j}^n c_{k-j}e^{ikt} = F^*(t)\sum_{j=0}^n e^{ij(t-\theta)}S_{n-j}(t).\end{aligned}$$

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For $g \in H^p(\mathbb{T})$ let

$$\sigma_n[g](t) = \frac{1}{2\pi}\int_{\mathbb{T}} g(\theta)E_n(t,\theta)d\theta = F^*(t)\sum_{j=0}^{n} c_j(g)e^{ijt}S_{n-j}[[F^*]^{-1}](t)$$
$$= F^*(t)S_n[g[F^*]^{-1}](t),$$

where the last equality we obtain by Lemma 1. Let $w(t) = |F^*(t)|^p$ then by a well known weighted norm inequality [5](see also [9]) we finish the proof.

$$\int_{\mathbb{T}} |\sigma_n[g](t)|^p dt = \int_{\mathbb{T}} |S_n[g[F^*]^{-1}](t)|^p w(t)dt$$
$$\leq B_p \int_{\mathbb{T}} |g(t)[F^*(t)]^{-1}|^p w(t)dt = B_p \int_{\mathbb{T}} |g(t)|^p dt.$$

□

2. Weighted H^p Spaces

Further in this section we will consider that $p, 1 \leq p < \infty$ is fixed. Let w be an admissible weight function. We would like to use the notations of the previous section. By (1) and Proposition 1 let $F = G_{w^{1/p}} \in H^p(\mathbb{D})$ be an outer function. From (0.1)

$$F^*(t) = \lim_{r\to 1-} F(re^{it}) = [w(t)]^{\frac{1}{p}} e^{\frac{i}{p}\varrho(t)}, \quad \text{where} \quad \varrho(t) = \widetilde{\ln w}(t). \tag{10}$$

Set $\mathrm{Y} = \{e^{int}\}_{n=0}^{\infty}$ and $\mathrm{Y}_0 = \{e^{int}\}_{n=1}^{\infty}$. We put

$$\overline{\text{span}}_{L^p(\mathbb{T},w)}(\mathrm{Y}) := H^p(\mathbb{T},w) \quad \text{and} \quad \overline{\text{span}}_{L^p(\mathbb{T},w)}(\mathrm{Y}_0) := H_0^p(\mathbb{T},w).$$

We consider that

$$H^\infty(\mathbb{T},w) := H^\infty(\mathbb{T}) = \{f \in L^\infty(\mathbb{T}) : \int_{\mathbb{T}} f(t)e^{int}dt = 0 \,\text{for all}\, n \in \mathbb{N}\}.$$

Set

$$\overline{H^p}(\mathbb{T},w) = \{\overline{f(t)} : f \in H^p(\mathbb{T},w)\} \quad \text{and} \quad \overline{H_0^p}(\mathbb{T},w) = e^{-it}\overline{H^p}(\mathbb{T},w).$$

In [16] weighted spaces $H^q(\mathbb{D},w)$ were defined for $w \in \mathcal{R}$. A function f holomorphic in $\mathbb{D}$ belongs to $H^q(\mathbb{D},w), 1 \leq q < \infty$ if

$$\|f\|_{H^q(\mathbb{D},w)} = \sup_{0\leq r<1}\left(\int_{\mathbb{T}} |f(re^{it})|^q w(t)dt\right)^{\frac{1}{q}} < \infty.$$

Results on weighted Hardy spaces can be found in [18]. The following statement was formulated by M.Rosenblum in the introduction of the article [16]. In the text the reader can find indications for the proof but the author do not formulated the statement as a theorem. That's why we prefer to formulate the statement as an hypothesis.

Hypothesis 1 (M. Rosenblum)**.** *Let $w \in \mathcal{R}$. Then the operator $\Lambda : f(z) \to f(e^{it})$ is a vector space isomorphism mapping $H^q(\mathbb{D},w)$ onto $H^q(\mathbb{T},w)$ such that Λ and Λ^{-1} are bounded. If Λ is an isometry then $w \equiv c, c > 0$.*

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In the formulated statement one considers that given $f \in H^q(\mathbb{D}, w)$, where

$$f(z) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k z^k,$$

then $f(e^{it}) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k e^{ikt}$ exists and $f(e^{it}) \in H^q(\mathbb{T}, w)$. Further in the paper we show that it is also true when $w \in e^{\mathcal{W}}$(see Proposition 2). In the next section we notonly prove that 1 holds but also give the integral representation of the operator Λ^{-1}.

2.1. On the Dual Space of $H^p(\mathbb{T}, w)$

In this subsection we give the characteriztion of the dual space of $H^p(\mathbb{T}, w), 1 \le p < \infty$ when w is an admissible weight function.

Lemma 2. *Let w be an admissible weight function, $1 \le p < \infty$ and $F \in H^p(\mathbb{D})$ be the outer function defined as above. Then $\phi F^* \in H^p(\mathbb{T})$ for $\phi \in H^p(\mathbb{T}, w)$ and if $\psi \in H^p(\mathbb{T})$ then $\psi[F^*]^{-1} \in H^p(\mathbb{T}, w)$.*

Proof. For the proof we use the relation (10) and the fact that the system $\{e^{int} F^*(t)\}_{n=0}^{\infty}$ is complete in $H^p(\mathbb{T})$. If $\phi \in H^p(\mathbb{T}, w)$ there exists a sequence of trigonometric polynomials $P_n(t) = \sum_{j=0}^{N_n} b_j e^{ijt}$ such that

$$0 = \lim_{n\to\infty} \int_{\mathbb{T}} |\phi(t) - P_n(t)|^p w(t) dt = \lim_{n\to\infty} \int_{\mathbb{T}} |\phi(t)F^*(t) - P_n(t)F^*(t)|^p dt.$$

Thus $\phi F^* \in H^p(\mathbb{T})$. On the other hand if $\psi \in H^p(\mathbb{T})$ we find trigonometric polynomials $\tilde{P}_n(t) = \sum_{j=0}^{\tilde{N}_n} \tilde{b}_j e^{ijt}$ such that

$$0 = \lim_{n\to\infty} \int_{\mathbb{T}} |\psi(t) - \tilde{P}_n(t)F^*(t)|^p dt = \lim_{n\to\infty} \int_{\mathbb{T}} \Big|\frac{\psi(t)}{F^*(t)} - \tilde{P}_n(t)\Big|^p w(t) dt.$$

□

We should describe the annihilator $H^p(\mathbb{T}, w)^\perp$ of $H^p(\mathbb{T}, w)$ in $L^{p'}(\mathbb{T}, w)$:

$$H^p(\mathbb{T}, w)^\perp = \left\{ \psi \in L^{p'}(\mathbb{T}, w) : \int_{\mathbb{T}} \phi(\theta)\overline{\psi(\theta)} w(\theta) d\theta = 0 \quad \forall \phi \in H^p(\mathbb{T}, w) \right\}.$$

Suppose $1 < p < \infty$ and let $\psi \in H^p(\mathbb{T}, w)^\perp$. For any $\phi \in H^p(\mathbb{T}, w)$ we write

$$0 = \int_{\mathbb{T}} \phi(\theta)\overline{\psi(\theta)} w(\theta) d\theta = \int_{\mathbb{T}} \phi(\theta) F^*(\theta) \overline{\psi(\theta)} [w(\theta)]^{\frac{1}{p'}} e^{-\frac{i}{p}\varrho(\theta)} d\theta.$$

It is well known that the annihilator of $H^p(\mathbb{T})$ is $\overline{H_0^{p'}(\mathbb{T})}$ (see e.g. [4]). Hence, by Lemma 2 it follows that

$$\overline{\psi(\theta)} [w(\theta)]^{\frac{1}{p'}} e^{-\frac{i}{p}\varrho(\theta)} \in H_0^{p'}(\mathbb{T}) \quad \text{and} \quad \overline{\psi(\theta)} e^{-i\varrho(\theta)} \in H_0^{p'}(\mathbb{T}, w).$$

Which yields $\psi(\theta) \in e^{-i\varrho(\theta)} \overline{H_0^{p'}}(\mathbb{T}, w)$. Conversely, if

$$\psi(\theta) \in e^{-i\varrho(\theta)} \overline{H_0^{p'}}(\mathbb{T}, w) = e^{-i\varrho(\theta)} e^{-it} \overline{H^{p'}}(\mathbb{T}, w)$$

then by (10) and Lemma 2 $\overline{\psi(\theta)} e^{-\frac{i}{p}\varrho(\theta)} [w(\theta)]^{\frac{1}{p'}} \in H^{p'}(\mathbb{T})$. Hence, for all $\phi \in H^p(\mathbb{T}, w)$

$$\int_{\mathbb{T}} \phi(\theta)\overline{\psi(\theta)} w(\theta) d\theta = \int_{\mathbb{T}} \phi(\theta) F^*(\theta) \overline{\psi(\theta)} [w(\theta)]^{\frac{1}{p'}} e^{-\frac{i}{p}\varrho(\theta)} d\theta = 0.$$

In the case $p = 1$ the proof is similar and we skip it. Thus

$$H^p(\mathbb{T}, w)^\perp = e^{-i\varrho(\theta)}\overline{H_0^{p'}}(\mathbb{T}, w)$$

and from Theorem 7.1 of [4] it follows that $[H^p(\mathbb{T}, w)]^*$ is isometrically isomorphic to $L^{p'}(\mathbb{T}, w)/\overline{H_0^{p'}}(\mathbb{T}, w)$ and for every $\psi \in [H^p(\mathbb{T}, w)]^*$

$$\sup_{\|\phi\|_{H^p(\mathbb{T},w)} \leq 1} \left| \frac{1}{2\pi} \int_{\mathbb{T}} \phi(t)\overline{\psi(t)}w(t)dt \right| = \min_{h \in H^p(\mathbb{T},w)^\perp} \|\psi + h\|_{L^{p'}(\mathbb{T},w)}.$$

As above we check that for $1 < p < \infty$

$$H_0^p(\mathbb{T}, w)^\perp = e^{-i\varrho(\theta)}\overline{H^{p'}}(\mathbb{T}, w).$$

Thus the following statement is proved.

Theorem 5. *For $1 < p < \infty$ the dual space $[H^p(\mathbb{T}, w)]^*$ is a reflexive Banach space isometrically isomorphic to $L^{p'}(\mathbb{T}, w)/\overline{H_0^{p'}}(\mathbb{T}, w)$. Moreover, $[H^1(\mathbb{T}, w)]^*$ is isometrically isomorphic to $L^\infty(\mathbb{T})/\overline{H_0^\infty(\mathbb{T})}$.*

2.2. Summation Basis

The following lemma is the analogue of Banach's theorem for a given system to be an $A-$basis. The proof is similar to the proof of Banach's original theorem, and we will not give it here. References about summation bases can be found in [17].

Lemma 3. *Let $X = \{x_k\}_{x=1}^\infty \subset \mathbf{B}$ is complete and minimal in $\mathbf{B}$ with the dual system $X^* = \{\varphi_n\}_{n=1}^\infty$. Then X is an $A-$basis of $\mathbf{B}$ if and only if there exists a constant $C > 0$ such that for any $x \in \mathbf{B}$*

$$\sup_{0<r<1} \| \sum_{k=1}^\infty r^k \phi_k(x) x_k \|_{\mathbf{B}} \leq C\|x\|_{\mathbf{B}}.$$

In this subsection we suppose that w is an admissible weight function. Recall that in this case $F^{-1} \in \mathbf{N}$ and we have the representation (5). Hence, for any $0 < R < 1$ there exists $N_R \in \mathbb{N}$ such that $|c_n| < \frac{1}{R^n}$ for $n > N_R$. For $0 \leq r < 1$ set

$$K_r(t, \theta) = F^*(t)[c_0 + \sum_{n=1}^\infty r^n e^{int}\overline{T_n(\theta)}]. \tag{11}$$

Note that the following series

$$\sum_{n=1}^\infty r^n \sum_{j=0}^n |c_j|$$

converges uniformly on $[0, \rho]$ for any $0 < \rho < 1$. Indeed, for $\rho < R < 1$ and $n > N_R$ we have that

$$\sum_{j=0}^n |c_j| \leq \sum_{j=0}^{N_R} |c_j| + R^{-N_R-1} \sum_{\nu=0}^{n-N_R-1} R^{-\nu} \leq C_R + R^{-n}\frac{1}{1-R}.$$

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Hence, the series (11) converges absolutely on $[0,1)\times\mathbb{T}\times\mathbb{T}$. The absolute convergence of the series permits to write

$$K_r(t,\theta) = F^*(t)\sum_{n=0}^{\infty} r^n e^{int}\sum_{j=0}^{n} c_{n-j}e^{-ij\theta} = F^*(t)\sum_{j=0}^{\infty}\sum_{n=j}^{\infty} r^n c_{n-j}e^{int}e^{-ij\theta}$$
$$= F^*(t)C_r(t-\theta)\sum_{\nu=0}^{\infty} r^\nu c_\nu e^{i\nu t} = F^*(t)C_r(t-\theta)F^{-1}(re^{it}).$$

Thus by Fatou's theorem we obtain

Theorem 6. *Any function $\phi\in H^p(\mathbb{T})$ is the non-tangential limit of*

$$\Phi(re^{it}) := \frac{1}{2\pi}\int_{\mathbb{T}}\phi(\theta)K_r(t,\theta)d\theta.$$

Let $\phi(\theta) = u(\theta)+iv(\theta)\in H^p(\mathbb{T}), 1\le p<\infty$ and $c_0(f)\in\mathbb{R}$. Then it is well known (see e.g. [7]) that

$$\frac{1}{2\pi}\int_{\mathbb{T}}\phi(\theta)C_r(t-\theta)d\theta = \frac{1}{2\pi}\int_{\mathbb{T}}u(\theta)H_r(t-\theta)d\theta$$
$$= \frac{1}{2\pi}\int_{\mathbb{T}}\phi(\theta)P_r(t-\theta)d\theta. \tag{12}$$

Consider the following family of maps

$$\Lambda_r[\phi](t) = F^*(t)F^{-1}(re^{it})\frac{1}{2\pi}\int_{\mathbb{T}}\phi(\theta)P_r(t-\theta)d\theta \quad 0<r<1,\ \phi\in H^p(\mathbb{T}).$$

Theorem 7. *The inequality*

$$\|\Lambda_r[\phi]\|_{L^p(\mathbb{T})} \le C_p\|\phi\|_{H^p(\mathbb{T})} \qquad 0<r<1, \tag{13}$$

holds for all $\phi\in H^p(\mathbb{T}), 1\le p<\infty$ and $C_p>0$ independent of ϕ, if and only if $w\in\mathcal{R}$.

Proof. Let $w\in\mathcal{R}$. If $p=1$ then

$$\|\Lambda_r[\phi]\|_{L(\mathbb{T})} = \frac{1}{4\pi^2}\int_{\mathbb{T}}\left|F^*(t)F^{-1}(re^{it})\int_{\mathbb{T}}\phi(\theta)P_r(t-\theta)d\theta\right|dt$$
$$\le \frac{1}{4\pi^2}\int_{\mathbb{T}}|\phi(\theta|\int_{\mathbb{T}}w(t)|F^{-1}(re^{it})|P_r(t-\theta)dtd\theta \le C\|\phi\|_{H^1(\mathbb{T})},$$

where $C>0$ is the constant in the condition $\mathcal{R}$. If $1<p<\infty$ then for any $\psi\in L^{p'}(\mathbb{T})$ we have

$$\left|\frac{1}{2\pi}\int_{\mathbb{T}}\psi(t)\Lambda_r[\phi](t)dt\right| = \frac{1}{4\pi^2}\left|\int_{\mathbb{T}}\psi(t)F^*(t)F^{-1}(re^{it})\int_{\mathbb{T}}\phi(\theta)P_r(t-\theta)d\theta dt\right|$$
$$\le \frac{1}{4\pi^2}\int_{\mathbb{T}}|\phi(\theta)|\int_{\mathbb{T}}|\psi(t)||F^*(t)F^{-1}(re^{it})|P_r(t-\theta)dtd\theta.$$

By the Hölder inequality we deduce

$$\begin{aligned}
&\int_{\mathbb{T}} |\psi(t)||F^*(t)F^{-1}(re^{it})|P_r(t-\theta)dt \\
&\leq \left(\int_{\mathbb{T}} |\psi(t)|^{p'}P_r(t-\theta)dt\right)^{\frac{1}{p'}} \left(\int_{\mathbb{T}} |F^*(t)F^{-1}(re^{it})|^p P_r(t-\theta)dt\right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{T}} |\psi(t)|^{p'}P_r(t-\theta)dt\right)^{\frac{1}{p'}} \left(\int_{\mathbb{T}} \frac{w(t)}{|F(re^{it})|^p} P_r(t-\theta)dt\right)^{\frac{1}{p}} \\
&\leq (2\pi C)^{\frac{1}{p}} \left(\int_{\mathbb{T}} |\psi(t)|^{p'}P_r(t-\theta)dt\right)^{\frac{1}{p'}}.
\end{aligned}$$

The last inequality folows by (3) and (2). Hence,

$$\begin{aligned}
&\left|\frac{1}{2\pi}\int_{\mathbb{T}} \psi(t)\Lambda_r[\varphi](t)dt\right| \\
&\leq \frac{1}{4\pi^2}(2\pi C)^{\frac{1}{p}} \left(\int_{\mathbb{T}} |\varphi(\theta)|^p d\theta\right)^{\frac{1}{p}} \left(\int_{\mathbb{T}}\int_{\mathbb{T}} |\psi(t)|^{p'}P_r(t-\theta)d\theta dt\right)^{\frac{1}{p'}} \\
&= C^{\frac{1}{p}} \left(\frac{1}{2\pi}\int_{\mathbb{T}} |\varphi(\theta)|^p d\theta\right)^{\frac{1}{p}} \left(\frac{1}{2\pi}\int_{\mathbb{T}} |\psi(t)|^{p'} dt\right)^{\frac{1}{p'}},
\end{aligned}$$

which yields (13).

For the proof of necessity fix some $\rho, 0 < \rho < 1$ and set $\varphi_\theta(t), \theta \in \mathbb{T}$ equal to any branch of $\frac{1}{(1-\rho e^{i(t-\theta)})^{\frac{2}{p}}}$. Then we have that

$$\begin{aligned}
\|\Lambda_r[\varphi_\theta]\|^p_{L^p(\mathbb{T})} &= \frac{1}{2\pi}\int_{\mathbb{T}} \left|\frac{F^*(t)}{F(re^{it})}\right|^p \frac{1}{|1-\rho re^{i(t-\theta)}|^2}dt \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{w(t)}{|F(re^{it})|^p} \frac{1}{|1-\rho re^{i(t-\theta)}|^2}dt
\end{aligned}$$

If we suppose that the maps $\Lambda_r : H^p(\mathbb{T}) \to L^p(\mathbb{T})$ are uniformly bounded then for $r = \rho$ we obtain

$$\frac{1}{2\pi}\int_{\mathbb{T}} \frac{w(t)}{|F(re^{it})|^p} \frac{1}{|1-r^2e^{i(t-\theta)}|^2}dt \leq \frac{C_p^p}{2\pi}\int_{\mathbb{T}} \frac{1}{|1-re^{i(t-\theta)}|^2}dt = \frac{C_p^p}{1-r^2},$$

where the last relation follows by Parseval's equality. By the inequality $|1-r^2e^{ix}|^2 \leq 4|1-re^{ix}|^2$ for all $x \in \mathbb{T}$ and $0 < r < 1$, we finish the proof. □

We would like to formulate the main result of this subsection from another point of view. Let $F \in H^p(\mathbb{D}), 1 \leq p < \infty$ be an outer function and let $F(e^{it})$ be the non-tangential limit of F on the unit circle. Note that Beurling's approximation theorem says that the system $\{F(e^{it})e^{ikt}\}_{k=0}^{\infty}$ is complete in $H^p(\mathbb{T})$. Set $w(t) = |F(e^{it})|^p$ then by Proposition 1 we claim that Theorem 7 yields

Theorem 8. *Let $F \in H^p(\mathbb{D}), 1 \leq p < \infty$ be an outer function and let $F(e^{it})$ be the non-tangential limit of F on the unit circle.*

Then $\{F(e^{it})e^{ikt}\}_{k=0}^{\infty}$ is an $A-$basis in $H^p(\mathbb{T})$ if and only if $|F(e^{it})|^p \in \mathcal{R}$.

2.3. The System $\{e^{ijt}\}_{j=0}^{\infty}$ in the Space $H^p(\mathbb{T}, w)$

The following assertion holds.

Theorem 9. *For any admissible weight function w the system $\{e^{ijt}\}_{j=0}^{\infty}$ is an $M-$basis in $H^p(\mathbb{T}, w)$, $1 \leq p < \infty$.*

Proof. The completeness of the system $\{e^{ijt}\}_{j=0}^{\infty}$ in $H^p(\mathbb{T}, w)$ follows by definition. Set

$$\varphi_n(t) = \frac{\overline{F^*(t)}}{w(t)} T_n(t), \qquad n \in \mathbb{N}_0,$$

where polynomials $T_n(t)$ are defined by (7). As in the proof of Theorem 1 it is easy to check that $\{\varphi_n(t)\}_{n=0}^{\infty}$ is the dual system of $\{e^{ijt}\}_{j=0}^{\infty}$ in $H^p(\mathbb{T}, w)$. Suppose that there exists $f \in H^p(\mathbb{T}, w)$ such that for all $n \in \mathbb{N}_0$

$$0 = \int_{\mathbb{T}} f(t)\overline{\varphi_n(t)} w(t) dt = \int_{\mathbb{T}} f(t) F^*(t) \overline{T_n(t)} dt.$$

By Lemma 2 we have that $fF^* \in H^p(\mathbb{T})$. Hence, by Theorem 1 it follows that $f(t) = 0$ a.e. on $\mathbb{T}$. □

Theorem 10. *The system $\{e^{ijt}\}_{j=0}^{\infty}$ is uniformly minimal in $H^p(\mathbb{T}, w)$, $1 < p < \infty$ if and only if $w \in \mathcal{W}_p$.*

Proof. The statement is an immediate consequence of Theorem 2. By (10) we deduce

$$\frac{1}{2\pi} \int_{\mathbb{T}} |\varphi_n(t)|^{p'} w(t) dt = \frac{1}{2\pi} \int_{\mathbb{T}} |T_n(t)|^{p'} dt.$$

□

The following theorem is a direct consequence of Theorem 4.

Theorem 11. *Let $w \in \mathcal{A}_p$, $1 < p < \infty$. Then the system $\{e^{ijt}\}_{j=0}^{\infty}$ is a Schauder basis in $H^p(\mathbb{T}, w)$.*

Let w be an admissible weight function. We expand any $f \in H^p(\mathbb{T}, w)$ with respect to the system $\{e^{ijt}\}_{j=0}^{\infty}$ and consider the Abel means of the obtained expansion. Let

$$L_r(t, \theta) = \sum_{n=0}^{\infty} r^n e^{int} \overline{\varphi_n(\theta)} = \frac{F^*(\theta)}{w(\theta)} \sum_{n=0}^{\infty} r^n e^{int} \overline{T_n(\theta)}, \quad 0 < r < 1.$$

As in the case of the kernel $K_r(t, \theta)$ we deduce that

$$L_r(t, \theta) = \frac{F^*(\theta)}{w(\theta)} C_r(t - \theta) F^{-1}(re^{it}).$$

Set

$$\sigma_r[f](t) = \frac{1}{2\pi} \int_{\mathbb{T}} f(\theta) L_r(t, \theta) w(\theta) d\theta = \frac{F^{-1}(re^{it})}{2\pi} \int_{\mathbb{T}} f(\theta) F^*(\theta) C_r(t - \theta) d\theta.$$

By Lemma 2 we have that $fF^* \in H^p(\mathbb{T})$. Hence the following theorem holds.

Theorem 12. *Any function $f \in H^p(\mathbb{T}, w)$, $1 \leq p < \infty$ is the non-tangential limit of*

$$\Psi(re^{it}) := \frac{1}{2\pi} \int_{\mathbb{T}} f(\theta) L_r(t, \theta) d\theta.$$

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Afterwards, we write $f(\theta)F^*(\theta) = u_1(\theta) + iv_1(\theta) \in H^p(\mathbb{T}), 1 \leq p < \infty$ and assume that $c_0(fF^*) \in \mathbb{R}$. Then as in (12)

$$\frac{1}{2\pi}\int_{\mathbb{T}} f(\theta)F^*(\theta)C_r(t-\theta)d\theta = \frac{1}{2\pi}\int_{\mathbb{T}} u_1(\theta)H_r(t-\theta)d\theta$$
$$= \frac{1}{2\pi}\int_{\mathbb{T}} f(\theta)F^*(\theta)P_r(t-\theta)d\theta.$$

Theorem 13. *Let $1 \leq p < \infty$. The inequality*

$$\left\| \frac{F^{-1}(re^{it})}{2\pi}\int_{\mathbb{T}} f(\theta)F^*(\theta)P_r(t-\theta)d\theta \right\|_{L^p(\mathbb{T},w)} \leq C_p^*\|f\|_{H^p(\mathbb{T},w)} \qquad 0 < r < 1,$$

holds for all $f \in H^p(\mathbb{T}, w), 1 \leq p < \infty$ and $C_p^* > 0$ independent of f, if and only if $w \in \mathcal{R}$.

Proof. Let $w \in \mathcal{R}$. Then for any $g \in L^{p'}(\mathbb{T}, w)$ we have

$$\frac{1}{4\pi^2}\left|\int_{\mathbb{T}} g(t)F^{-1}(re^{it})\int_{\mathbb{T}} f(\theta)F^*(\theta)P_r(t-\theta)d\theta w(t)dt\right|$$
$$\leq \frac{1}{4\pi^2}\int_{\mathbb{T}} |f(\theta)F^*(\theta)|\int_{\mathbb{T}} |g(t)||F^{-1}(re^{it})|P_r(t-\theta)w(t)dtd\theta.$$

Afterwards, by (10) we obtain

$$\int_{\mathbb{T}} |g(t)||F^{-1}(re^{it})|P_r(t-\theta)w(t)dt$$
$$\leq \left(\int_{\mathbb{T}} |g(t)|^{p'}P_r(t-\theta)w(t)dt\right)^{\frac{1}{p'}}\left(\int_{\mathbb{T}} |F^{-1}(re^{it})|^p w(t)P_r(t-\theta)dt\right)^{\frac{1}{p}}$$
$$\leq (2\pi C)^{\frac{1}{p}}\left(\int_{\mathbb{T}} |g(t)|^{p'}P_r(t-\theta)w(t)dt\right)^{\frac{1}{p'}},$$

where $C > 0$ is the constant in the condition $\mathcal{R}$. Hence,

$$\frac{1}{4\pi^2}\left|\int_{\mathbb{T}} g(t)h_p^{-1}(re^{it})\int_{\mathbb{T}} f(\theta)F^*(\theta)P_r(t-\theta)d\theta w(t)dt\right|$$
$$\leq \frac{1}{4\pi^2}(2\pi C)^{\frac{1}{p}}\left(\int_{\mathbb{T}} |f(\theta)|^p w(\theta)d\theta\right)^{\frac{1}{p}}\left(\int_{\mathbb{T}}\int_{\mathbb{T}} |g(t)|^{p'}w(t)P_r(t-\theta)d\theta dt\right)^{\frac{1}{p'}}$$
$$= C^{\frac{1}{p}}\left(\frac{1}{2\pi}\int_{\mathbb{T}} |f(\theta)|^p w(\theta)d\theta\right)^{\frac{1}{p}}\left(\frac{1}{2\pi}\int_{\mathbb{T}} |g(t)|^{p'}w(t)dt\right)^{\frac{1}{p'}}.$$

For the proof of necessity we fix some $r, 0 < r < 1$ and take $\frac{1}{F^*(t)}\varphi_\theta(t), \theta \in \mathbb{T}$, where $\varphi_\theta(t)$ equals to any branch of $\frac{1}{(1-re^{i(t-\theta)})^{\frac{2}{p}}}$. We skip further details because they are similar to those given in the proof of Theorem 7. □

The following statement gives a representation of the inverse operator Λ^{-1} from Hypothesis 1.

Corollary 1. *Let $w \in \mathcal{R}$ and $f \in H^p(\mathbb{T}, w), 1 \leq p < \infty$. Then the holomorphic function*

$$f(re^{it}) = \frac{F^{-1}(re^{it})}{2\pi}\int_{\mathbb{T}} f(\theta)F^*(\theta)P_r(t-\theta)d\theta$$

belongs to $H^p(\mathbb{D}, w)$.

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Thus by Lemma 3 we obtain

Theorem 14. *The system* $\{e^{ikt}\}_{k=0}^{\infty}$ *is an* $A-$*basis in* $H^p(w), 1 \le p < \infty$ *if and only if* $w \in \mathcal{R}$.

The following proposition is related with our remark after the formulation of Hypothesis 1.

Proposition 2. *Let* $w \in e^{\mathcal{W}}$ *and* $1 \le p < \infty$. *Let*

$$f(z) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k z^k$$

be a holomorphic function in $\mathbb{D}$ *such that*

$$\sup_{0 \le r < 1} \frac{1}{2\pi} \int_{\mathbb{T}} |f(re^{i\theta})|^p w(\theta) d\theta < +\infty.$$

Then there exists $\varphi \in H^p(\mathbb{T}, w)$ *such that for all* $n \in \mathbb{N}_0$

$$\alpha_n = \frac{1}{2\pi} \int_{\mathbb{T}} \varphi(t) \overline{\varphi_n(t)} w(t) dt. \tag{14}$$

Proof. By Lemma 2 we have that $f(re^{it})F^*(t), r \in (0,1)$ is a uniformly bounded family of functions in $H^p(\mathbb{T})$. Thus by Banach's theorem [1] we can find a sequence $0 < r_1 < r_2 < \cdots < 1$ such that $\lim_{j \to \infty} r_j = 1$ and $f(r_j e^{it})F^*(t)$ converges weakly in $H^p(\mathbb{T}), 1 < p < \infty$. In other words there exists $\psi \in H^p(\mathbb{T})$ such that for any $h \in H^{p'}(\mathbb{T})$

$$\lim_{j \to \infty} \frac{1}{2\pi} \int_{\mathbb{T}} f(r_j e^{it}) F^*(t) \overline{h(t)} dt = \frac{1}{2\pi} \int_{\mathbb{T}} \psi(t) \overline{h(t)} dt.$$

If we fix $j \in \mathbb{N}$ then for any $n \in \mathbb{N}$

$$\alpha_n r_j^n = \frac{1}{2\pi} \int_{\mathbb{T}} f(r_j e^{it}) \overline{\varphi_n(t)} w(t) dt = \frac{1}{2\pi} \int_{\mathbb{T}} f(r_j e^{it}) F^*(t) \overline{T_n(t)} dt.$$

Letting $j \to +\infty$, we obtain

$$\alpha_n = \frac{1}{2\pi} \int_{\mathbb{T}} \psi(t) \overline{T_n(t)} dt = \frac{1}{2\pi} \int_{\mathbb{T}} \varphi(t) \overline{\varphi_n(t)} w(t) dt,$$

where $\varphi(t) = \frac{\psi(t)}{F^*(t)} \in H^p(\mathbb{T}, w)$. The proof for the case $p = 1$ is longer but its first part is well known (see e.g. [7], [12]). The set $f(re^{it})F^*(t), r \in (0,1)$ is uniformly bounded in the $L^1(\mathbb{T})$ norm. Afterwards we consider $L^1(\mathbb{T})$ as a subspace of the space of Borel measures, the dual of $C(\mathbb{T})$. Thus as above one can pick an increasing sequence $\{r_j\}_{j=1}^{\infty}, \lim_{j \to \infty} r_j = 1$ such that for some analytic Borel measure μ

$$\lim_{j \to \infty} \frac{1}{2\pi} \int_{\mathbb{T}} f(r_j e^{it}) F^*(t) \overline{g(t)} dt = \frac{1}{2\pi} \int_{\mathbb{T}} \overline{g(t)} d\mu \quad \text{for any} \quad g \in C(\mathbb{T}).$$

By Riesz brothers theorem we obtain that μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}$, $d\mu(t) = \psi(t)dt$, where $\psi \in H^1(\mathbb{T})$. Afterwards we finish the proof as above. □

3. Discussion

The author feels obliged to explain some trivial things. Professional mathematicians may skip following few lines. It is clear that any solved problem is no longer a problem. The key instrument for the present study are polynomials defined in subsection 1.1. It is easy to check that they constitute a dual system for the corresponding Beurling system. The main difficulty is to find out the mentioned system.

Those polynomials are remarkable because expanding elements of the Hardy spaces by a Beurling system we get integral representations with classical kernels. This fact permits to use tools developed for weighted norm inequalities in our research. The obtained results belong to different topics which can be classified as parallel.

On one hand we extend Beurling's approximation theorem showing that Beurling's systems are $M-$bases in the corresponding Hardy spaces. Moreover, we characterise outer functions for which they are uniformly bounded $M-$bases, bases, summation bases. On the other hand we can study weighted norm Hardy spaces. Here we should mention M.Rosenblum's important article [16]. In the introduction of [16] a statement was formulated related with weighted norm H^p spaces. In my talks related with present study we formulated that statement as Rosenblum's theorem. We should remark that in [16] the author do not formulated that statement as a theorem. Hence, after some reflección it seems to us more adequate to formulated it as Rosenblum's hypothesis. Our study permits to give a complete proof of Hypothesis 1. Moreover, we find the precise formula for finding the function from the space $H^p(\mathbb{D}, w)$ given its boundary value which belongs to $H^p(\mathbb{T}, w)$, where $w \in \mathcal{R}$. It should be mentioned that the class $\mathcal{R}$ is large enough (see (4)). These relations need further study.

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