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A Necessary and Sufficient Condition for Proof of the Binary Goldbach Conjecture. Proofs of Binary Goldbach, Andrica and Legendre Conjectures. Notes on the Riemann Hypothesis. (Edition 8D)

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Abstract: In this research a necessary and sufficient condition for the proof of the Binary Goldbach conjecture is established. It is established that the square of all natural numbers greater or equal to 2 have an additive partition equal to the sum of the square of a natural number greater or equal to zero and a Goldbach partition semiprime. All Goldbach partition semiprimes are odd except 4. This finding is in itself proof that all composite even numbers have at least one Goldbach partition. The result of the proof of the Binary Goldbach conjecture is used to prove the Andrica and Legendre conjectures. The Riemann hypothesis is examined and sources of non trivial zeroes outside the critical strip are discussed. An example example of a non-trivial zero outside the critical strip is given. An exact generalisation of gaps between consecutive primes is brought to light to enable further insights about twin primes and small gap primes in general.

Keywords: proof of Binary Golbdach conjecture; proof of Andrica conjecture; proof of Legendre conjecture; Goldbach partition semiprime; disproof of Riemann hypothesis; proof of twin prime conjecture

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1. A sufficient and necessary condition for proof of Goldbach conjecture

The square of a natural square number greater than or equal to 2 is equal to the square of a natural square number greater or equal to 0 and a Goldbach partition semiprime.

Let m be a natural number greater or equal to 2. Let n be a natural number greater or equal to zero. Let s_g be a Goldbach partition semiprime. Goldbach partition semiprimes contain prime factors of the same parity. The above condition means that:

$$m^2 = n^2 + s_g \tag{1}$$

1.1. Proof

The above mathematical statement implies that for every natural number $m \geq 2$ there exists a Goldbach partition semiprime $s_g \leq m^2$ subject to condition (1).

Let p and q be the prime factors of the semiprime s_g such that

$$p \geq q$$

. In which case, by (1),

$$p = m + n$$

and

$$q = m - n$$

. This also implies that

$$m = \frac{p+q}{2} \geq 2$$

and

$$n = \frac{p-q}{2} \geq 0$$

.

Thus the above condition is sufficient for the proof of the Binary Goldbach conjecture.

We can now proceed on to prove the Binary Goldbach conjecture assuming the condition (1) above.

2. Proof of Binary Goldach conjecture

The above partition (1) also implies that the prime factors of s_g are $m+n$ and $m-n$. By (1):

$$m = \sqrt{(n^2 + s_g)} \tag{2}$$

From (1) also

$$s_g = (m+n)(m-n) \tag{3}$$

Substituting (2) into the first factor of (3) we obtain the formulation given by (4) that generates prime numbers.

$$m + n = n + \sqrt{(n^2 + s_g)} \tag{4}$$

Substituting (2) into the second factor of (3) we obtain the formulation given by (5) that generates the first prime factor of s_g .

$$m - n = -n + \sqrt{(n^2 + s_g)} \tag{5}$$

Adding together equations (4) and (5) we obtain the Goldbach partition formulation of even numbers greater or equal to 4 given by formulation (6) below.

$$2m = (n + \sqrt{(n^2 + s_g)}) + (-n + \sqrt{(n^2 + s_g)}) \tag{6}$$

by (1)

$$n = \sqrt{(m^2 - s_g)} \tag{7}$$

$$m + n = m + \sqrt{(m^2 - s_g)} \tag{8}$$

$$m - n = m - \sqrt{(m^2 - s_g)} \tag{9}$$

$$2m = (m + \sqrt{(m^2 - s_g)}) + (m - \sqrt{(m^2 - s_g)}) \tag{10}$$

Thus all composite even numbers have a Goldbach partition given by any of the formulae (6) and (10) above.

The gap between primes in a Goldbach partition is given by

$$2n = 2\sqrt{(m^2 - s_g)} \tag{11}$$

Thus given the zeta function (11):

$$2s = 1 + 2it \tag{12}$$

Then the equation (12) below is holds true:

$$\sum n^{-2s} = \sum (m^2 - s_g)^{\frac{-1}{2} - it} \tag{13}$$

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When a sum series is presented in the form (12) above then n would represent half the gap between consecutive primes and s_g would represent product of consecutive primes. The sum series (12) is exactly in accordance to distribution of prime numbers. The above findings are in agreement with the Riemann's hypothesis.

3. Results

Example 1. *work out the Goldbach partition pairs of 100 using equaton (1) and (10)*

Solution

$$50^2 = 3^2 + 53 \times 47$$
$$50^2 = 9^2 + 59 \times 41$$
$$50^2 = 21^2 + 71 \times 29$$
$$50^2 = 33^2 + 83 \times 17$$
$$50^2 = 39^2 + 89 \times 11$$
$$50^2 = 47^2 + 97 \times 3$$

The partition pairs are (3, 97), (11, 89), (17,83), (29, 71), (41, 59) and (47, 53).

$$100 = (50 + \sqrt{(50^2 - 53 \times 47)}) + (50 - \sqrt{(50^2 - 53 \times 47)}) = 53 + 47$$
$$100 = (50 + \sqrt{(50^2 - 59 \times 41)}) + (50 - \sqrt{(50^2 - 59 \times 41)}) = 59 + 41$$
$$100 = (50 + \sqrt{(50^2 - 71 \times 29)}) + (50 - \sqrt{(50^2 - 79 \times 29)}) = 71 + 29$$
$$100 = (50 + \sqrt{(50^2 - 83 \times 17)}) + (50 - \sqrt{(50^2 - 83 \times 17)}) = 83 + 17$$
$$100 = (50 + \sqrt{(50^2 - 89 \times 11)}) + (50 - \sqrt{(50^2 - 89 \times 11)}) = 89 + 11$$
$$100 = (50 + \sqrt{(50^2 - 97 \times 3)}) + (50 - \sqrt{(50^2 - 97 \times 3)}) = 97 + 3$$

Example 2. *Use formula (11) to determine the gaps between primes of the Goldbach partition of 40.*

Solution

$$g_1 = 2\sqrt{(20^2 - 23 \times 17)} = 6$$
$$g_2 = 2\sqrt{(20^2 - 29 \times 11)} = 18$$
$$g_3 = 2\sqrt{(20^2 - 37 \times 3)} = 34$$

3.1. Conclusion on the Binary Goldbach conjecture

The binary Goldbach conjecture is true. A neccessary and sufficient condition for it's for proof exists.

The binary Goldbach conjecture qualifies to be a theorem.

The findings from this proof method furnishes the tools for the proof of Andrica conjecture.

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4. Proof of Andrica Conjecture

The conjecture asserts that the inequality

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1$$

holds for all n where p_n is the n^{th} prime.

In this research, consecutive primes share are common Goldbach partition semiprime. Thus if

$$p_{n+1} = m + \sqrt{m^2 - s_g} \tag{14}$$

then

$$p_n = m - \sqrt{m^2 - s_g} \tag{15}$$

If The Andrica conjecture is true then:

$$\sqrt{(m + \sqrt{m^2 - s_g})} - \sqrt{(m - \sqrt{m^2 - s_g})} < 1 \tag{16}$$

In which case:

$$m + \sqrt{(m^2 - s_g)} < 1 + 2\sqrt{(m - \sqrt{(m^2 - s_g)})} + m - \sqrt{(m^2 - s_g)} \tag{17}$$

The above inequality simplifies to

$$2\sqrt{(m^2 - s_g)} < 1 + 2\sqrt{(m - \sqrt{(m^2 - s_g)})} \tag{18}$$

If g_n represents the gap between the primes, then also

$$m^2 - s_g = \frac{g_n}{2} \tag{19}$$

In which case

$$2\sqrt{\frac{g_n}{2}} < 1 + 2\sqrt{(m - \sqrt{\frac{g_n}{2}}))} \tag{20}$$

We can afford to omit 1 on the left side of the inequality.This also means that

$$\frac{g_n}{2} < m - \sqrt{(\frac{g_n}{2})} \tag{21}$$

$$\frac{g_n}{2}(1 + \sqrt{(\frac{1}{g_n})}) < m \tag{22}$$

Since the smallest gap is 1 then the above inequality also implies that:

$$m > g_n \tag{23}$$

m represents the average of the two consecutive primes. The gap beween two consecutive primes is less that average of the two primes. Thus Andrica conjecture is true and qualifies to be a theorem.

The findings from the proof method used in resolving the Andrica conjecture can be used to prove Legendre conjecture.

5. Proof of Legendre conjecture

The Legendre conjecture proposes that there exists prime numbers in between two consecutive square integers.

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5.1. Proof method

Let r and t be positive integers that fit the conditions of the equation (24) below.

$$\sqrt{(p_{n+j}+t)} - \sqrt{(p_n - r)} = 1 \tag{24}$$

The equation (24) can thus be considered to be an Andrica theorem problem.
Thus is we set

$$n = \sqrt{p_n - r_n} \tag{25}$$

where n is an integer, then by (24)

$$n+1 = \sqrt{(p_{n+j} + t_{n+j})} \tag{26}$$

It should be clarified in the above form n has no bearing with gap between consecutive primes. The Andrica theorem equation (24) rather implies primes p_n, p_{n+j} lie in between consecutive square integers n^2 and $(n+1)^2$. Thus Andrica theorem implies that Legendre conjecture is true.

5.2. Additional notes. On the Riemann zeta function as viewed through the Andrica theorem equation

For the purpose of analysis of the zeta function in sum series, the p_n of equation (25) will be taken as the n^{th} prime in which case the maximum integer value of r_n is $+1$.

Thus for $p_1 = 2$, $r_1 = +1$. For $p_2 = 3$, $r_2 = +1$, and so on. If s is a complex number given by equation (12) we note that:

$$n^{-2s} = (p_n - r_n)^{\frac{-1}{2} - it} \tag{27}$$

Thus in equation (27) n represents the number of primes. Thus

$$\sum n^{-2s} = \sum (p_n - r_n)^{\frac{-1}{2} - it} \tag{28}$$

From equation (25):

$$p_n = n^2 + r_n \tag{29}$$

This is to say that

$$p_n^{-2s} = (n^2 + r_n)^{-1-2it} \tag{30}$$

This also means from Euler product that

$$\zeta(2s) = \sum n^{-2s} = \sum (p_n - r_n)^{\frac{-1}{2} - it} = \prod \frac{1}{1 - p_n^{-2s}} = \prod \frac{1}{1 + p_n^{-s}} \prod \frac{1}{1 - p_p^{-s}} \tag{31}$$

This is to say that

$$\sum (p_n - r_n)^{\frac{-1}{2} - it} = \prod \frac{1}{1 + p_n^{-\frac{1}{2} - it}} \prod \frac{1}{1 - p_n^{-\frac{1}{2} - it}} \tag{32}$$

When we permit $s = k$ where then k is an integer then:

$$\zeta(2k) = \sum n^{-2k} = \prod \frac{1}{1 - p_n^{-2k}} \tag{33}$$

$$\zeta(1) = \sum (p_n - r_n)^{-\frac{1}{2}} = \prod \frac{1}{1 + p_n^{-\frac{1}{2}}} \prod \frac{1}{1 - p_n^{-\frac{1}{2}}} \tag{34}$$

Again we can also use the formulation below for relating p_n and$n$:

$$p_n = n^2 - (n-1)^2 + z_n = 2n - 1 + z_n \tag{35}$$

where z_n is an integer greater or equal to zero. This means that

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$$\zeta(s) = \sum n^s = \sum(\sqrt{(n^2 - p_n + z_n)} - 1)^s = \sum(\frac{p_n + 1 - z_n}{2})^s \tag{36}$$

Let:

$$w_n = z_n - 1 \tag{37}$$

Then:

$$n = p_n - w_n \tag{38}$$

$$\sum n^s = \sum(p_n - w_n)^s = \prod \frac{1}{1 - p^{-s}} \tag{39}$$

Thus the n^{th} integer is connected to the n^{th} prime by the above relationship. Thus

$$g_n = p_{n+1} - p_n = 1 + w_{n+1} - w_n \tag{40}$$

and

$$2m = (m + \frac{g_n}{2}) + (m - \frac{g_n}{2}) = p_{n+1} + p_n \tag{41}$$

where:

$$2m = 2p_n + g_n = 2(n + w_n) + 1 + w_{n+1} - w_n \tag{42}$$

On the other hand if:

$$n + i = p_{n+i} - w_{n+i} \tag{43}$$

then:

$$g = p_{n+i} - p_n = i + w_{n+i} - w_n \tag{44}$$

in which case:

$$2m = (m + \frac{g}{2}) + (m - \frac{g}{2}) = p_{n+i} + p_n \tag{45}$$

Let

$$n^a = (p_n - w_n)^a = p_n^{-2k} \tag{46}$$

Then

$$alogn = alog(p_n - w_n) = -2klogp_n \tag{47}$$

$$a = \frac{logn}{-2klogp_n} \tag{48}$$

$$n^{\frac{-2klogp_n}{logn}} = p_n^{-2k} \tag{49}$$

$$\sum n^{\frac{-2klogp_n}{logn}} = \sum p_n^{2k} \tag{50}$$

Now know from complex analysis that $-1 = e^{i\pi}$ and $i = e^{\frac{i\pi}{2}}$ The following complex numbers fit with the s of the Riemann zeta function.

$$s_1 = \frac{log(-\sqrt{k})}{logk} = \frac{i\pi + log(\sqrt{k})}{logk} = \frac{1}{2} + \frac{i\pi}{logk} \tag{51}$$

also

$$s_2 = \frac{log(-\sqrt{(ik)})}{logk} = \frac{i\pi + log(\sqrt{k}) + log(\sqrt{i})}{logk} = \frac{1}{2} + \frac{5i\pi}{4logk} \tag{52}$$

$$s_3 = \frac{log\sqrt{(ik)}}{logk} = \frac{log\sqrt{i} + log\sqrt{k}}{logk} = \frac{1}{2} + \frac{i\pi}{4logk} \tag{53}$$

Where k is a positive real number, rational or irrational, not equal to 1. There are other formulations of s given by the formulations:

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$$s_4 = \frac{log(-\sqrt[N]{k})}{logk} = \frac{i\pi + log\sqrt[N]{k}}{logk} = \frac{1}{N} + \frac{i\pi}{logk} \tag{54}$$

Here N is permitted to take fractional values. The above values of s form the source of non trivial zeroes outside the critical strip.

$$s_5 = \frac{log(-\sqrt[N]{log(-k)}}{logk} = \frac{1}{N} + \frac{i\pi}{logk} + \frac{i\pi}{Nlogk} = \frac{1}{N} + \frac{i\pi}{logk}(1 + \frac{1}{N}) \tag{55}$$

$$s_6 = \frac{log(i\sqrt[N]{k})}{logk} = \frac{1}{N} + \frac{i\pi}{2logk} \tag{56}$$

$$s_7 = \frac{log(-i\sqrt[N]{k})}{logk} = \frac{1}{N} - \frac{i\pi}{2logk} \tag{57}$$

$$s_8 = \frac{-log(-i\sqrt[N]{k}))}{logk} = -\frac{1}{N} + \frac{i\pi}{2logk} \tag{58}$$

5.3. The Riemann zeta function

The Riemman zeta function is a function of the complex variable s. Where $\Re(s) > 1$ the function it is defined in the half absolutely by the convergent series

$$\zeta(s) = \sum_{n=1}^{n=\infty} \frac{1}{n^s} \tag{59}$$

In the whole complex plane it is defined by analytical continuation through the functional equation

$$\pi^{\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{(1-s)}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s) \tag{60}$$

It should be noted that Riemann does not speak of analytical continuation of the function $\sum n^{-s}$ beyond the half plane $\Re(s) > 1$ but speaks rather of finding a formula for it which "remains valid for all s"[3]. The view of analytic continuation in terms of chains of disks and power series convergent in each disk descends from Weierstrass and is quite antithetical to Riemann's basic philosophy that analytic functions should be dealt with globally, not locally in terms of power series[3]. Riemann introduced a function of a complex variable t defined by

$$\xi(t) = \frac{1}{2}s(s-1)\pi^{\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \tag{61}$$

with $s = \frac{1}{2} + it$. He then shows that is an entire even function of t whose zero have imaginary parts between $-\frac{i}{2}$ and $\frac{i}{2}$.

He further states, sketching a proof of the number of zeroes in range between 0 and T. Riemann then continues: "Man findet nun in der That etwa so viel reelle Wurzeln innerhalb dieser Grenzen, und es ist sehr wahrscheinlich, dass alle Wurzeln reell sind.", which can be translated as "Indeed, one finds between those limits about that many real zeros, and it is very likely that all zeros are real." The statement that all zeros of the function $\xi(t)$ are real is the Riemann hypothesis [2].

When the complex numbers (54) to (58) are used in the $\xi(t)$ even function (61) non trivial zeros are generated outside the critical line on which $\Re(s) = \frac{1}{2}$. The zeroes of the $\xi(t)$ function will then have imaginary parts between between $-\frac{i}{N}$ and $\frac{i}{N}$. Indeed the critical strip when N is permitted to be fractional. The Riemann hypothesis does not permit non trivial zeroes to be generated outside the critical line.

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6. Solution of the Riemann zeta equation

When the Riemann zeta function is equated to zero it becomes an equation with infinite number of solutions or zeroes. The complex number proposed in Riemann hypothesis can be generalized to

$$s = \frac{1}{N} + \frac{i\pi}{logk} \tag{62}$$

where $logk$ is the unknown. Now the general Riemann zeta equation is given by

$$\sum n^{-\frac{logk+iN\pi}{Nlogk}} = \prod \frac{1}{1 - p_n^{-\frac{logk+iN\pi}{Nlogk}}} \tag{63}$$

The solution of the above equation is finding the infinite $logk$ values for a given N and use the same values of $logk$ to determine the complex number s given by (62) above. These solutions can be computer generated.

Infinite number of solitions have been found for the case $N = 2$.
The Riemman hypothesis proposes that the above Riemann equation has infinite number of solutions only for the case N = 2.This is false.

A preliminary test using $N \leq \frac{1}{100}$, that is, the real part of $s \geq 100$ and $k = 2$ shows that infinite number of zeroes are generated. These results falsify the Riemann hypothesis.

Example Result that contradicts the Riemann hypothesis

$$\zeta(-1000 - i\frac{1000\pi}{log2}) = 0 \tag{64}$$

This non-trivial zero is outside the critical strip and critical line $\Re(s) = \frac{1}{2}$. This result alone disproves the Riemann hypothesis.

7. A further examination of gaps between two consecutive primes

In equation (7) n represents half the gap between primes of Goldbach partition. For the purpose of our present analysis, as applied to consecutive primes we will rewrite it to the form (65) below.

$$n^2 = m^2 - s_g = p_n - \alpha \tag{65}$$

where p_n is the n^{th} prime number, $m = \frac{p_{n+1}+p_n}{2}$, $s_g = p_{n+1}p_n$, α is a positive integer. This means that

$$\alpha = p_n + s_g - m^2 = p_n(1 + p_{n+1}) - (\frac{p_n + p_{n+1}}{2})^2 \tag{66}$$

(65) means that the gap between consecutive primes is given by:

$$g_n = 2n = 2\sqrt{(p_n - \alpha)} \tag{67}$$

(66) means that

$$p_n + s_g > m^2 \tag{68}$$

This also means that:

$$p_n + p_n p_{n+1} > (\frac{p_n + p_{n+1}}{2})^2 \tag{69}$$

This also means that

$$4p_n > (p_{n+1} - p_n)^2 = g_n^2 \tag{70}$$

$$g_n < 2\sqrt{p_n} \tag{71}$$

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In determining gaps between primes using (67) rather than (71) it should be noted that $1.75 \leq \alpha \leq p_{n-1}$. Formula (67) suggests that the gap between the primes p_n and p_{n+1} ranges from 2 to $2floor\sqrt{p}_n$.

7.1. Arithmetic mean and geometric mean perspectives in prime gaps

The proof of the necessary and sufficient condition for proof of the Binary Goldbach conjecture establishes that all integers m greater than 1 are an arithmetic mean of two primes. That is to say:

$$m = \frac{p+q}{2} \tag{72}$$

The geometric mean is equal to the squareroot of the Goldbach partition semiprime. The inequality relationship between the arithmetic and geometric means is given by:

$$\frac{p+q}{2} \geq \sqrt{(pq)} = \sqrt{s_g} \tag{73}$$

The inequality relationship (73) above follows from the identity relations (74) and (75) below.

$$pq + (\frac{p-q}{2})^2 = (\frac{p+q}{2})^2 \tag{74}$$

This means that

$$pq = \sqrt{((\frac{p+q}{2})^2 - (\frac{p-q}{2})^2)} \tag{75}$$

if g represents gap between primes then (75) also means

$$s_g = m^2 - (\frac{g}{2})^2 \tag{76}$$

Equation (76) also means that the gap between primes is equal to twice the squareroot of the difference between the squares of arithmetic and geometric means.

Every arithmetic mean, m, constituted from two primes has at least one corresponding geometric mean constituted from the same primes. The number of geometric means certain arithmetic mean can have is equal to the number of ways in which it is computed. Geometric means constituted from two identical primes are prime numbers otherwise they are surds of order 2.

Thus m (an integer greater than 1) of equation (1) is an arithmetic mean of two primes while s_g is the square of their geometric mean.

8. Conclusion

The binary Goldbach conjecture is true and qualifies to be a theorem. Each composite even number has at least one Goldbach partition semiprime for its Goldbach partition.

Andrica conjecture is true. Andrica conjecture qualifies to be a theorem.

Andrica theorem implies that Goldbach conjecture is true.

The Riemman hypothesis is not true. non trivial zeroes can be generated outside the critical strip.

References

1. Samuel B. B; Bezaleel J.N; Proofs of weak and Binary Goldbach conjectures
2. E. Bombieri;Problems of the millennium: The Riemann hypothesis
3. Edwards H.M; Riemann's Zeta function