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Jishe Feng *

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Proof the Collatz Conjecture by a New View of Natural Numbers

FENG Jishe

School of Mathematics and Information Engineering, Longdong University, Qingyang, Gansu, 745000, China; gsfjs6567@126.com.

Abstract: The two main strategies used in this study are the binary representation and the decomposition of a natural number into many compound functions of odd function and even function. The Collatz conjecture regarding odd even numbers in number theory can be examined and discussed using them in this way: The sequence created by the finite iterations of the Collatz function becomes the ultimately periodic sequence if any natural number is the beginning value, proving the conjecture that has been held for 85 years.

Keywords: binary representation; composite function; Collatz conjecture; ultimately periodic sequence; chaos

MSC(2010): 03D20; 11B25; 11B83

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1. Introduction

In the study of number theory, odd and even numbers are a fundamental pair of ideas. Natural number sets come in two different varieties. Numerological theory frequently examines the connections between various numbers. There are numerous conjectures that attempt to generalize the law of different sorts of natural numbers discovered in a restricted range to the entire infinite set of natural numbers. This article will examine the famous Collatz conjecture, which states that for each natural number n, if it is even, divide by 2, if it is odd, multiply by 3, add 1, and so on, the result must finally reach 1. It is also referred to as the $3n+1$ conjecture and was put forth in 1937 by Lothar Collatz, also known as the $3n+1$ problem. The mathematician Paul Erdos once said of this conjecture: "Mathematics may not be ready for such problems"[1,2].

The inconsistencies between the finite and the infinite, as well as the relation between various kinds, present difficulties in the study of number theory problems. We are talking about the connection between two different mathematical ideas: the iterated sequence is a ultimately periodic sequence whether the initial value is odd or even.

The finite and the infinite can be connected by the useful mathematical construct known as a function, and the resulting outcomes will also be finite and infinite. A special function that has particular significance in discrete mathematics is the piecewise function. Compound functions and piecewise functions combined is a highly clever mathematical trick. Particularly in number theory, which is the most fundamental idea and mathematical expression, the sequence of numbers is a close connection between functions and finite and infinite. Numerous conjectures are obtained through restricted iteration of an iterative algorithm, which is a widely used method in number theory, but people typically lack the means to demonstrate the accuracy and reasoned nature of conjectures. A fresh technique or new knowledge is frequently used to support a hypothesis. For the Collatz conjecture, we can describe it as a function:

$$T(n) = \begin{cases} 3n+1, & \text{if } n \text{ is odd number,} \\ \frac{n}{2} & \text{if } n \text{ is even number.} \end{cases} \tag{1.1}$$

The following sequence is obtained via the composite function (iteration): $a = \{n, T(n), T(T(n)), T(T(T(n))), \cdots\} = \{n, T(n), T^2(n), T^3(n), \cdots\}$. Consequently, the Collatz

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conjecture can be stated as follows: The sequence a always leads to the integer 1, regardless of where you start with the natural number n, namely $T^m(n) = 1$. The series a is an infinite sequence of ultimately period [9,10]: the preperiod $\eta(n)$ varies with the initial value n, but the ultimately period is always $\{4, 2, 1\}$.

2. Composition of odd and even functions and binary representation of natural numbers

If a natural number can be divided by 2, it is said to be an even number; otherwise, it is said to be odd number. The Peano's Axiom states that 1 is the smallest natural number. The set of natural numbers $N = \{1, 2, 3, \cdots\}$ may be separated into odd and even sets, we will utilize the standard definition of natural numbers in this work.

$\{$**natural number**$\} = \{$**odd number**$\} \bigcup \{$**even number**$\}$.

In the set of natural numbers where 1 is the smallest odd number and 2 is the smallest even number, we can use the expression $n = 2k - 1$ to indicate that it is an odd, and the expression $n = 2k$ to indicate that it is an even, where k is any natural number.

We introduce two functions $O(x) = 2x + 1$ to express odd numbers greater than 1, and $E(x) = 2x$ to express even numbers.

Definition 1 A natural number n is obtained by composition of the odd function $O(x) = 2x + 1$ and the even function $E(x) = 2x$ several times, namely $n = f(f(\cdots f(1))) = f^k(1)$, the function f is either odd function $O(x)$ or even function $E(x)$.

For example, $7 = 2*3+1 = 2*(2*1+1+1) = O(O(1)), 189 = 2*2*(2*4+1) = 2*(2*(2*(2*1))+1) = E(O(E(E(1))))$. In order to more clearly express the odd-even composition process of a natural number, we use **binary representation of a natural number** n, for example: $7 = (111)_2$, $18 = (10010)_2$.

Similarly, we can have a piecewise function

$$f^{-1}(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd number,} \\ \frac{n}{2}, & \text{if } n \text{ is even number.} \end{cases} \tag{2.2}$$

If $n = f(f(\cdots f(1))) = f^k(1)$, then the inverse function is $f^{-k}(n) = 1$.

For generalization, we give the definitions of three natural numbers:

Definition 2(i) By applying the odd function $O(x)$ m compositions, a natural number, namely $O^m(1) = 2^m - 1 = 2^{m-1} + 2^{m-2} + \cdots + 2 + 1 = (11\cdots1)_2$, is obtained. such as $3 = (11)_2, 7 = (111)_2, 15 = (1111)_2, 31 = (11111)_2, 63 = (111111)_2, \cdots$, which we call it as **pure odd number**;

(ii) By applying the even function $E(x)$ m compositions, a natural number, namely $E^m(1) = 2^m = (10\cdots0)_2$, is obtained, such as, $2 = (10)_2, 4 = (100)_2, 8 = (1000)_2, 16 = (10000)_2, 32 = (100000)_2, 64 = (1000000)_2, \cdots$, which we call it as **pure even number**;

(iii) The natural number obtained by the composition of odd function $O(x)$ and even function $E(x)$, we call it **mixed number**. Such as, $18 = (10010)_2, 28 = (11100)_2, 67 = (1000011)_2, 309 = (100110101)_2$.

In particular, the natural numbers obtained by the finite alternately composition of the odd function $O(x)$ and the even function $E(x)$, namely, $[E(O(1))]^m = (101\cdots101)_2$. Such as $5 = (101)_2, 21 = (10101)_2, 85 = (1010101)_2, 341 = (101010101)_2, 1365 = (10101010101)_2, 5461 = (1010101010101)_2, \cdots$, which we call **hard number**.

Definition 3 The binary string of a natural number is a representation of its odd-even composite function, where the 1 in the $i(i > 0)$-bit from right to left is the $i(i > 0)$ sub-odd function, and 0 is the corresponding sub-even function.

For a natural number n, if its binary string has k bits, then the degree of composite function is $k - 1$. The binary string of a pure odd number is made of all 1; and the binary string of a pure even number is all 0 aside from one 1 in left; the binary string of a mixed number is made of many 0 and 1. Figure 1 shows the decomposition of the composite function (inverse of the composite function) of 60.

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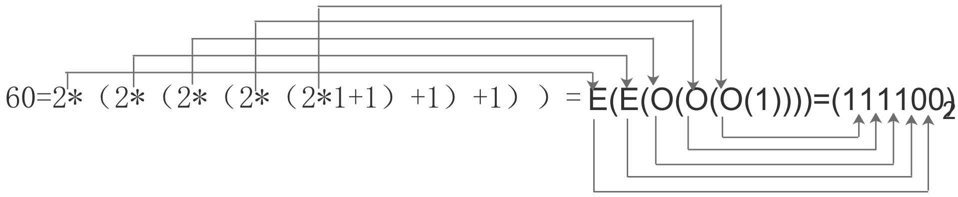

Figure 1. Natural number 60 is obtained from 1 through the composition of five even and odd functions, its binary string is $(111100)_2$.

In this way, we can classify the set of natural numbers in another way,

Property 4 The set of natural numbers can be divided into three different sets:

$\{$ natural number $\}$ = $\{$ pure odd number $\} \bigcup \{$ pure even number $\} \bigcup \{$ mixed number $\}$

The pure odd number is an odd number, the pure even number is an even number, and mixed numbers can be either an odd number or an even number when compared to the conventional classification. The last bit of the binary string, which is either 0 or 1, indicates whether it is even or odd. The entire binary string implicitly indicates one of the following three types: all bits are 0s with the exception of one 1 in the left implicitly pure even, all bits are 1s implicitly pure odd, and there are 0 and 1 in any number of bits of the binary string implicitly mixed.

Example 1 (1)60, 97 are mixed numbers.

(2)64,1180591620717411303424 are pure even numbers.

(3)63,1180591620717411303423 are pure odd numbers.

$60 = 2*30 = 2*(2*15) = 2*(2*(2*7+1)) = 2*(2*(2*(2*(2*3)+1)+1)) = 2*(2*(2*(2*(2*(2*1+1)+1)+1))) = E(E(O(O(O(1)))) = (111100)_2$;

$97 = 2*48+1 = 2*(2*24)+1 = 2*(2*(2*12))+1 = 2*(2*(2*(2*6)))+1 = 2*(2*(2*(2*(2*3))))+1 = 2*(2*(2*(2*(2*(2*1+1)))))+1 = O(E(E(E(E(O(1)))))) = (1100001)_2$.

$64 = 2^6 = (1000000)_2, 1180591620717411303424 = 2^{70} = (10000\cdots0)$ are pure even numbers:

$63 = 2^6 - 1 = (111111)_2, 1180591620717411303423 = 2^{70} - 1 = (11\cdots1)_2$ are pure odd numbers.

This composite function has special significance, for example,

Example 2 Let $[x]$ be x the integer part, that is, $[x] = max\{y \in Z : y \leq x\}$, $\{x\}$ be fractional part, namely $\{x\} = x - [x]$. Assuming $g(x) = \frac{1}{[x]-\{x\}+1}$, is rational set Q^+ and rational set Q can be expressed as respectively

$Q^+ = \{g(0), g^2(0), g^3(0), \cdots\}$,

$Q = \{0, g(0), -g(0), g^2(0), -g^2(0), g^3(0), -g^3(0), \cdots\}$.

See the references[5] for proof.

3. Tabular and algebraic expressions

For the Collatz conjecture, if expressed by the function $T(n)$, we find that $T^k(n) = 1$.

For the sake of discussion, we combine the compound function (iterative relation) to get the reduced Collatz function

$$RT(n) = \begin{cases} \frac{3n+1}{2^m}, & \text{if } n \text{ is odd number,} \\ \frac{n}{2^r} & \text{if } n \text{ is even number.} \end{cases} \quad (3.3)$$

The result is an odd number. We introduce tabular form and algebraic expression to express the reduced Collatz function $RT(n)$.

For example, for $n = 67$ we use the formula (1.1) and iteration, get the following table. Where the last column is its algebraic expression:.

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67→	202 →	101			$\frac{3^8}{2^{19}}\cdot 67$		$\frac{3^7}{2^{19}}$
101→	304→	152→	76→	38→	19		$\frac{3^6}{2^{18}}$
19→	58→	29					$\frac{3^5}{2^{14}}$
29→	88→	44→	22→	11			$\frac{3^4}{2^{13}}$
11→	34→	17					$\frac{3^3}{2^{10}}$
17→	52→	26→	13				$\frac{3^2}{2^9}$
13→	40→	20→	10→	5			$\frac{3}{2^7}$
5→	16→	8→	4→	2→	1		$\frac{1}{2^4}$
1							

Representing the numbers in the table by its binary string to get

1000011→	11001010→	1100101			$\frac{3^8}{2^{19}}\cdot 67$		$\frac{3^7}{2^{19}}$
1100101→	100110000→	10011000→	1001100→	100110→	10011		$\frac{3^6}{2^{18}}$
10011→	111010→	11101					$\frac{3^5}{2^{14}}$
11101→	1011000→	101100→	10110→	1011			$\frac{3^4}{2^{13}}$
1011→	100010→	10001					$\frac{3^3}{2^{10}}$
10001→	110100→	11010→	1101				$\frac{3^2}{2^9}$
1101→	101000→	10100→	1010→	101			$\frac{3}{2^7}$
101→	10000→	1000→	100→	10→	1		$\frac{1}{2^4}$
1							

The above table is simplified with the help of the reduced Collatz function, and the simplified binary string table obtained is as follows, and we will use this form as the default table in the rest of this article.

1000011→	11001010→	1100101	$\frac{3^8}{2^{19}}\cdot 67$	$\frac{3^7}{2^{19}}$
1100101→	100110000→	10011		$\frac{3^6}{2^{18}}$
10011→	111010→	11101		$\frac{3^5}{2^{14}}$
11101→	1011000→	1011		$\frac{3^4}{2^{13}}$
1011→	100010→	10001		$\frac{3^3}{2^{10}}$
10001→	110100→	1101		$\frac{3^2}{2^9}$
1101→	101000→	101		$\frac{3}{2^7}$
101→	10000→	1		$\frac{1}{2^4}$
1				

In the future, we will use this table as a research tool to write the algebraic expression of the last column in the table:

$$T^{27}(67) = T(8,19,67) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{13}} + \frac{3^5}{2^{14}} + \frac{3^6}{2^{18}} + \frac{3^7}{2^{19}} + \frac{3^8}{2^{19}}\cdot 67 = 1$$

In general, the algebraic expressions are obtained: Starting from the last row of the table and going up to the binary corresponding to the initial value n of the first row, the numerator is $3^k, k = 0,1,2,3,\cdots$, The denominator is $2^{r_k}, r_0 = m_0, r_k = r_{k-1} + m_k, k = 1,2,3,\cdots, m_k$ here for the before k lines at the end of the second column of binary string number 0. The details are expressed in the last line of the corresponding table, and we write out the algebraic expression of them as follows:

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$$\begin{aligned} T^{106}(31) &= \frac{1}{2^4}+\frac{3}{2^9}+\frac{3^2}{2^{10}}+\frac{3^3}{2^{11}}+\frac{3^4}{2^{14}}+\frac{3^5}{2^{18}}+\frac{3^6}{2^{20}}+\frac{3^7}{2^{22}}+\frac{3^8}{2^{26}}+\frac{3^9}{2^{27}} \\ &\quad+\frac{3^{10}}{2^{28}}+\frac{3^{11}}{2^{29}}+\frac{3^{12}}{2^{32}}+\frac{3^{13}}{2^{33}}+\frac{3^{14}}{2^{34}}+\frac{3^{15}}{2^{35}}+\frac{3^{16}}{2^{36}}+\frac{3^{17}}{2^{37}}+\frac{3^{18}}{2^{39}} \\ &\quad+\frac{3^{19}}{2^{40}}+\frac{3^{20}}{2^{42}}+\frac{3^{21}}{2^{43}}+\frac{3^{22}}{2^{44}}+\frac{3^{23}}{2^{47}}+\frac{3^{24}}{2^{49}}+\frac{3^{25}}{2^{50}}+\frac{3^{26}}{2^{51}}+\frac{3^{27}}{2^{52}} \\ &\quad+\frac{3^{28}}{2^{54}}+\frac{3^{29}}{2^{55}}+\frac{3^{30}}{2^{56}}+\frac{3^{31}}{2^{58}}+\frac{3^{32}}{2^{59}}+\frac{3^{33}}{2^{61}}+\frac{3^{34}}{2^{63}}+\frac{3^{35}}{2^{64}}+\frac{3^{36}}{2^{65}} \\ &\quad+\frac{3^{37}}{2^{66}}+\frac{3^{38}}{2^{67}}+\frac{3^{39}}{2^{67}}\cdot 31 \\ &= 1 \end{aligned}$$

$$\begin{aligned} T^{107}(63) &= \frac{1}{2^4}+\frac{3}{2^9}+\frac{3^2}{2^{10}}+\frac{3^3}{2^{11}}+\frac{3^4}{2^{14}}+\frac{3^5}{2^{18}}+\frac{3^6}{2^{20}}+\frac{3^7}{2^{22}}+\frac{3^8}{2^{26}}+\frac{3^9}{2^{27}} \\ &\quad+\frac{3^{10}}{2^{28}}+\frac{3^{11}}{2^{29}}+\frac{3^{12}}{2^{32}}+\frac{3^{13}}{2^{33}}+\frac{3^{14}}{2^{34}}+\frac{3^{15}}{2^{35}}+\frac{3^{16}}{2^{36}}+\frac{3^{17}}{2^{37}}+\frac{3^{18}}{2^{39}} \\ &\quad+\frac{3^{19}}{2^{40}}+\frac{3^{20}}{2^{42}}+\frac{3^{21}}{2^{43}}+\frac{3^{22}}{2^{44}}+\frac{3^{23}}{2^{47}}+\frac{3^{24}}{2^{49}}+\frac{3^{25}}{2^{50}}+\frac{3^{26}}{2^{51}}+\frac{3^{27}}{2^{52}} \\ &\quad+\frac{3^{28}}{2^{54}}+\frac{3^{29}}{2^{55}}+\frac{3^{30}}{2^{56}}+\frac{3^{31}}{2^{58}}+\frac{3^{32}}{2^{59}}+\frac{3^{33}}{2^{63}}+\frac{3^{34}}{2^{64}}+\frac{3^{35}}{2^{65}}+\frac{3^{36}}{2^{66}} \\ &\quad+\frac{3^{37}}{2^{67}}+\frac{3^{38}}{2^{68}}+\frac{3^{39}}{2^{68}}\cdot 63 \\ &= 1 \end{aligned}$$

$$\begin{aligned} T^{118}(97) &= \frac{1}{2^4}+\frac{3}{2^9}+\frac{3^2}{2^{10}}+\frac{3^3}{2^{11}}+\frac{3^4}{2^{14}}+\frac{3^5}{2^{18}}+\frac{3^6}{2^{20}}+\frac{3^7}{2^{22}}+\frac{3^8}{2^{26}}+\frac{3^9}{2^{27}} \\ &\quad+\frac{3^{10}}{2^{28}}+\frac{3^{11}}{2^{29}}+\frac{3^{12}}{2^{32}}+\frac{3^{13}}{2^{33}}+\frac{3^{14}}{2^{34}}+\frac{3^{15}}{2^{35}}+\frac{3^{16}}{2^{36}}+\frac{3^{17}}{2^{37}}+\frac{3^{18}}{2^{39}} \\ &\quad+\frac{3^{19}}{2^{40}}+\frac{3^{20}}{2^{42}}+\frac{3^{21}}{2^{43}}+\frac{3^{22}}{2^{44}}+\frac{3^{23}}{2^{47}}+\frac{3^{24}}{2^{49}}+\frac{3^{25}}{2^{50}}+\frac{3^{26}}{2^{51}}+\frac{3^{27}}{2^{52}} \\ &\quad+\frac{3^{28}}{2^{54}}+\frac{3^{29}}{2^{55}}+\frac{3^{30}}{2^{56}}+\frac{3^{31}}{2^{58}}+\frac{3^{32}}{2^{59}}+\frac{3^{33}}{2^{61}}+\frac{3^{34}}{2^{63}}+\frac{3^{35}}{2^{64}}+\frac{3^{36}}{2^{65}} \\ &\quad+\frac{3^{37}}{2^{66}}+\frac{3^{38}}{2^{69}}+\frac{3^{39}}{2^{70}}+\frac{3^{40}}{2^{71}}+\frac{3^{41}}{2^{73}}+\frac{3^{42}}{2^{75}}+\frac{3^{43}}{2^{75}}\cdot 97 \\ &= 1 \end{aligned}$$

$$\begin{aligned} T^{91}(10027) &= \frac{1}{2^4}+\frac{3}{2^9}+\frac{3^2}{2^{10}}+\frac{3^3}{2^{11}}+\frac{3^4}{2^{14}}+\frac{3^5}{2^{18}}+\frac{3^6}{2^{20}}+\frac{3^7}{2^{22}}+\frac{3^8}{2^{26}}+\frac{3^9}{2^{27}} \\ &\quad+\frac{3^{10}}{2^{28}}+\frac{3^{11}}{2^{29}}+\frac{3^{12}}{2^{32}}+\frac{3^{13}}{2^{33}}+\frac{3^{14}}{2^{34}}+\frac{3^{15}}{2^{35}}+\frac{3^{16}}{2^{36}}+\frac{3^{17}}{2^{37}}+\frac{3^{18}}{2^{39}} \\ &\quad+\frac{3^{19}}{2^{42}}+\frac{3^{20}}{2^{43}}+\frac{3^{21}}{2^{44}}+\frac{3^{22}}{2^{45}}+\frac{3^{23}}{2^{48}}+\frac{3^{24}}{2^{49}}+\frac{3^{25}}{2^{53}}+\frac{3^{26}}{2^{56}}+\frac{3^{27}}{2^{58}} \\ &\quad+\frac{3^{28}}{2^{60}}+\frac{3^{29}}{2^{61}}+\frac{3^{30}}{2^{61}}\cdot 10027 \\ &= 1 \end{aligned}$$

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4. The characteristics of sequence of binary string of the Collatz function iteration

The Collatz function is expressed in binary form as

$$T(n) = \begin{cases} (11)_2 \cdot (1 \times \cdots \times 1)_2 + 1 = (1 \times \times \times 10 \cdots 0)_2, & \text{if } n \text{ is odd number,} \\ \frac{(1 \times \cdots \times 10 \cdots 00)_2}{(10)_2} = (1 \times \cdots \times 10 \cdots 0)_2, & \text{if } n \text{ is even number.} \end{cases} \tag{4.4}$$

The characteristics of the first and last parts are represented by the Figure 2.

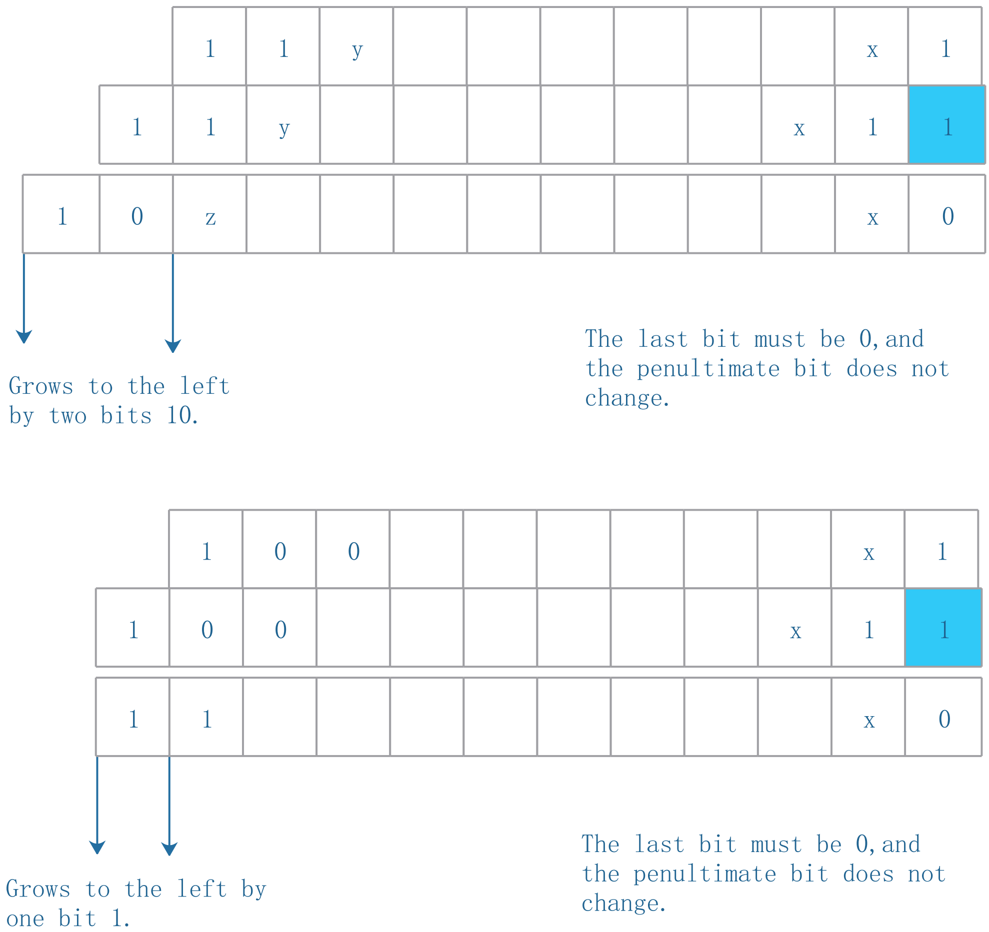

Figure 2. Binary representation of the Collatz function, add up at most two digits from the left side of the binary string, subtract at least one digit from the right side of the binary string.

By observing the binary strings of iterated Collatz functions with initial values of 31,63 (see the tabulation procedure in reference [7]) and 97,10027 as the follows, we get some laws of pure even, pure odd and mixed numbers with initial values. In the following, we will analyze and discuss the table of the iterative process of the Collatz functions from the two dimensions of row and column of the tables.

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			$\frac{3^{43}}{2^{75}} \cdot 97$
$97 = (1100001)_2 \rightarrow$	$(100100100)_2 \rightarrow$	$(1001001)_2$	$\frac{3^{42}}{2^{75}}$
$73 = (1001001)_2 \rightarrow$	$(11011100)_2 \rightarrow$	$(110111)_2$	$\frac{3^{41}}{2^{73}}$
$55 = (110111)_2 \rightarrow$	$(10100110)_2 \rightarrow$	$(1010011)_2$	$\frac{3^{40}}{2^{71}}$
$83 = (1010011)_2 \rightarrow$	$(11111010)_2 \rightarrow$	$(1111101)_2$	$\frac{3^{39}}{2^{70}}$
$125 = (1111101)_2 \rightarrow$	$(101111000)_2 \rightarrow$	$(101111)_2$	$\frac{3^{38}}{2^{69}}$
$47 = (101111)_2 \rightarrow$	$(10001110)_2 \rightarrow$	$(1000111)_2$	$\frac{3^{37}}{2^{66}}$
$71 = (1000111)_2 \rightarrow$	$(11010110)_2 \rightarrow$	$(1101011)_2$	$\frac{3^{36}}{2^{65}}$
$107 = (1101011)_2 \rightarrow$	$(101000010)_2 \rightarrow$	$(10100001)_2$	$\frac{3^{35}}{2^{64}}$
$161 = (10100001)_2 \rightarrow$	$(111100100)_2 \rightarrow$	$(1111001)_2$	$\frac{3^{34}}{2^{63}}$
$121 = (1111001)_2 \rightarrow$	$(101101100)_2 \rightarrow$	$(1011011)_2$	$\frac{3^{33}}{2^{61}}$
$91 = (1011011)_2 \rightarrow$	$(100010010)_2 \rightarrow$	$(10001001)_2$	$\frac{3^{32}}{2^{59}}$
$137 = (10001001)_2 \rightarrow$	$(110011100)_2 \rightarrow$	$(1100111)_2$	$\frac{3^{31}}{2^{58}}$
$103 = (1100111)_2 \rightarrow$	$(100110110)_2 \rightarrow$	$(10011011)_2$	$\frac{3^{30}}{2^{56}}$
$155 = (10011011)_2 \rightarrow$	$(111010010)_2 \rightarrow$	$(11101001)_2$	$\frac{3^{29}}{2^{55}}$
$233 = (11101001)_2 \rightarrow$	$(1010111100)_2 \rightarrow$	$(10101111)_2$	$\frac{3^{28}}{2^{54}}$
$175 = (10101111)_2 \rightarrow$	$(1000001110)_2 \rightarrow$	$(100000111)_2$	$\frac{3^{27}}{2^{52}}$
$263 = (100000111)_2 \rightarrow$	$(1100010110)_2 \rightarrow$	$(110001011)_2$	$\frac{3^{26}}{2^{51}}$
$395 = (110001011)_2 \rightarrow$	$(10010100010)_2 \rightarrow$	$(1001010001)_2$	$\frac{3^{25}}{2^{50}}$
$593 = (1001010001)_2 \rightarrow$	$(11011110100)_2 \rightarrow$	$(110111101)_2$	$\frac{3^{24}}{2^{49}}$
$445 = (110111101)_2 \rightarrow$	$(10100111000)_2 \rightarrow$	$(10100111)_2$	$\frac{3^{23}}{2^{47}}$
$167 = (10100111)_2 \rightarrow$	$(111110110)_2 \rightarrow$	$(11111011)_2$	$\frac{3^{22}}{2^{44}}$
$251 = (11111011)_2 \rightarrow$	$(1011110010)_2 \rightarrow$	$(101111001)_2$	$\frac{3^{21}}{2^{43}}$
$377 = (101111001)_2 \rightarrow$	$(10001101100)_2 \rightarrow$	$(100011011)_2$	$\frac{3^{20}}{2^{42}}$
$283 = (100011011)_2 \rightarrow$	$(1101010010)_2 \rightarrow$	$(110101001)_2$	$\frac{3^{19}}{2^{40}}$
$425 = (110101001)_2 \rightarrow$	$(10011111100)_2 \rightarrow$	$(100111111)_2$	$\frac{3^{18}}{2^{39}}$
$319 = (100111111)_2 \rightarrow$	$(1110111110)_2 \rightarrow$	$(111011111)_2$	$\frac{3^{17}}{2^{37}}$
$479 = (111011111)_2 \rightarrow$	$(10110011110)_2 \rightarrow$	$(1011001111)_2$	$\frac{3^{16}}{2^{36}}$
$719 = (1011001111)_2 \rightarrow$	$(100001101110)_2 \rightarrow$	$(10000110111)_2$	$\frac{3^{15}}{2^{35}}$
$1079 = (10000110111)_2 \rightarrow$	$(110010100110)_2 \rightarrow$	$(11001010011)_2$	$\frac{3^{14}}{2^{34}}$
$1619 = (11001010011)_2 \rightarrow$	$(1001011111010)_2 \rightarrow$	$(100101111101)_2$	$\frac{3^{13}}{2^{33}}$
$2429 = (100101111101)_2 \rightarrow$	$(1110001111000)_2 \rightarrow$	$(1110001111)_2$	$\frac{3^{12}}{2^{32}}$
$911 = (1110001111)_2 \rightarrow$	$(101010101110)_2 \rightarrow$	$(10101010111)_2$	$\frac{3^{11}}{2^{29}}$
$1367 = (10101010111)_2$	$(1000000000110)_2$	$(100000000011)_2$	$\frac{3^{10}}{2^{28}}$
$2051 = (100000000011)_2 \rightarrow$	$(1100000001010)_2 \rightarrow$	$(110000000101)_2$	$\frac{3^{9}}{2^{27}}$
$3077 = (110000000101)_2 \rightarrow$	$(10010000010000)_2 \rightarrow$	$(1001000001)_2$	$\frac{3^{8}}{2^{26}}$
$577 = (1001000001)_2 \rightarrow$	$(11011000100)_2 \rightarrow$	$(110110001)_2$	$\frac{3^{7}}{2^{22}}$
$433 = (110110001)_2 \rightarrow$	$(10100010100)_2 \rightarrow$	$(101000101)_2$	$\frac{3^{6}}{2^{20}}$
$325 = (101000101)_2 \rightarrow$	$(1111010000)_2 \rightarrow$	$(111101)_2$	$\frac{3^{5}}{2^{18}}$
$61 = (111101)_2 \rightarrow$	$(10111000)_2 \rightarrow$	$(10111)_2$	$\frac{3^{4}}{2^{14}}$
$23 = (10111)_2 \rightarrow$	$(1000110)_2 \rightarrow$	$(100011)_2$	$\frac{3^{3}}{2^{11}}$
$35 = (100011)_2 \rightarrow$	$(1101010)_2 \rightarrow$	$(110101)_2$	$\frac{3^{2}}{2^{10}}$
$53 = (110101)_2$	$(10100000)_2$	$(101)_2$	$\frac{3}{2^{9}}$
$5 = (101)_2$	$(10000)_2$	$(1)_2$	$\frac{1}{2^{4}}$

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			$\frac{3^{30}}{2^{61}}\cdot 10027$
$10027=(10011100101011)_2 \rightarrow$	$(111010110000010)_2 \rightarrow$	$(11101011000001)_2$	$\frac{3^{29}}{2^{61}}$
$15041=(11101011000001)_2 \rightarrow$	$(1011000001000100)_2 \rightarrow$	$(10110000010001)_2$	$\frac{3^{28}}{2^{60}}$
$11281=(10110000010001)_2 \rightarrow$	$(1000010000110100)_2 \rightarrow$	$(10000100001101)_2$	$\frac{3^{27}}{2^{58}}$
$8461=(10000100001101)_2 \rightarrow$	$(110001100101000)_2 \rightarrow$	$(110001100101)_2$	$\frac{3^{26}}{2^{56}}$
$3173=(110001100101)_2 \rightarrow$	$(10010100110000)_2 \rightarrow$	$(1001010011)_2$	$\frac{3^{25}}{2^{53}}$
$595=(1001010011)_2 \rightarrow$	$(11011111010)_2 \rightarrow$	$(1101111101)_2$	$\frac{3^{24}}{2^{49}}$
$893=(1101111101)_2 \rightarrow$	$(101001111000)_2 \rightarrow$	$(101001111)_2$	$\frac{3^{23}}{2^{48}}$
$335=(101001111)_2 \rightarrow$	$(1111101110)_2 \rightarrow$	$(111110111)_2$	$\frac{3^{22}}{2^{45}}$
$503=(111110111)_2 \rightarrow$	$(10111100110)_2 \rightarrow$	$(1011110011)_2$	$\frac{3^{21}}{2^{44}}$
$755=(1011110011)_2 \rightarrow$	$(100011011010)_2 \rightarrow$	$(10001101101)_2$	$\frac{3^{20}}{2^{43}}$
$1133=(10001101101)_2 \rightarrow$	$(110101001000)_2 \rightarrow$	$(110101001)_2$	$\frac{3^{19}}{2^{42}}$
$425=(110101001)_2 \rightarrow$	$(10011111100)_2 \rightarrow$	$(100111111)_2$	$\frac{3^{18}}{2^{39}}$
$319=(100111111)_2 \rightarrow$	$(1110111110)_2 \rightarrow$	$(111011111)_2$	$\frac{3^{17}}{2^{37}}$
$479=(111011111)_2 \rightarrow$	$(10110011110)_2 \rightarrow$	$(1011001111)_2$	$\frac{3^{16}}{2^{36}}$
$719=(1011001111)_2 \rightarrow$	$(100001101110)_2 \rightarrow$	$(10000110111)_2$	$\frac{3^{15}}{2^{35}}$
$1079=(10000110111)_2 \rightarrow$	$(110010100110)_2 \rightarrow$	$(11001010011)_2$	$\frac{3^{14}}{2^{34}}$
$1619=(11001010011)_2 \rightarrow$	$(1001011111010)_2 \rightarrow$	$(100101111101)_2$	$\frac{3^{13}}{2^{33}}$
$2429=(100101111101)_2 \rightarrow$	$(1110001111000)_2 \rightarrow$	$(1110001111)_2$	$\frac{3^{12}}{2^{32}}$
$911=(1110001111)_2 \rightarrow$	$(101010101110)_2 \rightarrow$	$(10101010111)_2$	$\frac{3^{11}}{2^{29}}$
$1367=(10101010111)_2 \rightarrow$	$(1000000000110)_2 \rightarrow$	$(100000000011)_2$	$\frac{3^{10}}{2^{28}}$
$2051=(100000000011)_2 \rightarrow$	$(1100000001010)_2 \rightarrow$	$(110000000101)_2$	$\frac{3^{9}}{2^{27}}$
$3077=(110000000101)_2 \rightarrow$	$(10010000010000)_2 \rightarrow$	$(1001000001)_2$	$\frac{3^{8}}{2^{26}}$
$577=(1001000001)_2 \rightarrow$	$(11011000100)_2 \rightarrow$	$(110110001)_2$	$\frac{3^{7}}{2^{22}}$
$433=(110110001)_2 \rightarrow$	$(10100010100)_2 \rightarrow$	$(101000101)_2$	$\frac{3^{6}}{2^{20}}$
$325=(101000101)_2 \rightarrow$	$(1111010000)_2 \rightarrow$	$(111101)_2$	$\frac{3^{5}}{2^{18}}$
$61=(111101)_2 \rightarrow$	$(10111000)_2 \rightarrow$	$(10111)_2$	$\frac{3^{4}}{2^{14}}$
$23=(10111)_2 \rightarrow$	$(1000110)_2 \rightarrow$	$(100011)_2$	$\frac{3^{3}}{2^{11}}$
$35=(100011)_2 \rightarrow$	$(1101010)_2 \rightarrow$	$(110101)_2$	$\frac{3^{2}}{2^{10}}$
$53=(110101)_2 \rightarrow$	$(10100000)_2 \rightarrow$	$(101)_2$	$\frac{3}{2^{9}}$
$5=(101)_2 \rightarrow$	$(10000)_2 \rightarrow$	$(1)_2$	$\frac{1}{2^{4}}$

4.1. Row characteristic

1) In each row, the first column is always odd (empty when the initial value is even), and the last bit of its binary string must be 1,

The second column must be even, its binary string must end with at least one 0, the number of subsequent even numbers must be as many as the number of zeros at the end of the second column's binary string, and the number of zeros at the end of each even number to the right is one less than the previous one, until all zeros are deleted to become the last odd number in the row.

The last column must be odd, and the last bit of its binary string must be 1.

2) When there are only three numbers in a row, that is, only one even number, the last odd number must be greater than the first odd number (the first column); When there are more than three numbers in a row, that is, more than two even numbers, the last odd number must be smaller than the first odd number.

3) The preceding binary string is identical from the second column to the last column in one line, except for the all 0 at the end.

4.2. Column characteristic

From top to bottom, the binary string in the first column of two adjacent rows has the following two characteristics:

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1) If the number of bits of 1 in the last substring of the previous row is greater than 1, the number of bits of 1 in the last substring of the next row is reduced by one, and the momentum of this reduction of one bit remains unchanged until it finally becomes only one; The corresponding number is greater than the number in the previous row;

2) If the last substring of the binary string in a row contains only one bit of 1, then the last substring of the binary string in the next row contains either one bit of 1 or many bits of 1, and the corresponding number is smaller than the number in the previous row;

3) Within each line, the number of bits in the first binary string is increased by 1 bit with the number in the second binary string, but for hard numbers, the number is increased by 2 bits. The reduction is at least 1 bit, and when it is a hard number of m bits, the number is reduced by $m+1$.

4.3. Comprehensive characteristic

1) The substring at the end of the binary (right) is observed that when the substring is 1, the units digit of the corresponding decimal number can be any one of $1,3,5,7,9$. For instance, $161 = (10100001)_2$, $433 = (110110001)_2$, $325 = (101000101)_2$, $577 = (1001000001)_2$, $2429 = (100101111101)_2$.

2) The number ending in decimal is 9 and the corresponding binary number can end in any digits of 1, for instance, $319 = (100111111)_2$, $479 = (111011111)_2$,$719 = (1011001111)_2$,$1079 = (10000110111)_2$,$1619 = (11001010011)_2$,$2429 = (100101111101)_2$.

3) When the last substring of binary is reduced by one bit from many, the corresponding decimal number's units are always reciprocated within the three groups of numbers: $1,7$, or $3,5$, and unit bit is always 9.

4.4. 4.4 Discussion according three sets

The following is a discussion of the natural numbers according to our classification, that is, when the initial value of the iteration process of the Collatz function is pure even, pure odd and mixed numbers, it always reaches the smallest natural number 1, thus proving the establishment of Collatz conjecture.

(1)For a pure even $n = 2^k = (10\cdots 0)_2$, it requires only an iteration of the k times Collatz function to reach the smallest natural number 1, that is, the conjecture holds.

(2)For a special class of mixed numbers, if a_k in the iteration sequence a of the Collatz function is a special class of mixed numbers - the hard number $\frac{4^k-1}{3} = (101\cdots 101)_2$, then its sequent iteration result is

$$a_k = \frac{4^k - 1}{3} = (101\cdots 101\cdots 101)_2, a_{k+1} = 4^k = 2^{2k} = (10\cdots 0)_2, \cdots, a_{3k+1} = 1.$$

This means that the Collatz conjecture is valid at this point. For ordinary mixed numbers, this is a complicated process, which is carried out in conjunction with the discussion of pure odd numbers in follows.

(3) For a pure odd number $n = 2^k - 1 = (11\cdots 1)_2$, if the Collatz function is iterated in binary form, that is, the second column in the preceding table becomes a mixed number, which can be observed by several examples. Research papers on this topic are [6,7,8]. By checking the change law of the last substring in the binary string and the change law of the total length of the binary string, we show that the Collatz function always reaches the minimum natural number $\mathbf{1}$ in the finite iteration value, thus proving the correctness of Collatz conjecture.

In view of the general, a pure odd $n = 2^r - 1$, there is a mathematical formula: $2^r - 1 = 2^{r-1} + 2^{r-2} + 2^{r-3} + \cdots + 2 + 1$. Two numbers r and m in the following tables are the values we verify for the Collatz conjecture by iterating $T^m(2^r - 1) = 1, r = 1, 2, 3, \cdots, 203$ in the Maple program below:

restart;

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```
i := 2^r − 1;
convert(i, binary, decimal);
jj := 0;
while i ≠ 1 do
i := piecewise(irem(i, 2) = 0, (1/2)i, 3i + 1);
convert(i, binary, decimal);
 jj := jj + 1
enddo
```

By verifying the first **203** pure odd number $2^r - 1$, we can get the following rule:

1) Odd number r greater than 1, the corresponding test Collatz function value has the following relation:

if $T^m(2^r - 1) = 1$, then $T^{m+1}(2^{r+1} - 1) = 1$.

2) $T^{856}(2^{57} - 1) = T^{856}(2^{65} - 1) = 1$

3) $T^{857}(2^{58} - 1) = T^{857}(2^{66} - 1) = 1$

4) $T^{1451}(2^{86} - 1) = T^{1451}(2^{117} - 1) = 1$

5)$T^{1455}(2^{90} - 1) = T^{1455}(2^{121} - 1) = 1$

6)$T^{1456}(2^{91} - 1) = T^{1456}(2^{122} - 1) = 1$

7)$T^{1457}(2^{92} - 1) = T^{1457}(2^{123} - 1) = 1$

8)$T^{1458}(2^{93} - 1) = T^{1458}(2^{124} - 1) = 1$

r	m	r	m	r	m	r	m	r	m	r	m	r	m
1	0	30	449	59	858	88	1360	117	1451	146	2010	175	2039
2	7	31	450	60	859	89	1454	118	1452	147	2011	176	2040
3	16	32	451	61	860	90	1455	119	1484	148	2012	177	2041
4	17	33	527	62	861	91	1456	120	1485	149	2013	178	2042
5	106	34	528	63	862	92	1457	121	1455	150	2014	179	2061
6	107	35	529	64	863	93	1458	122	1456	151	2015	180	2062
7	46	36	530	65	856	94	1459	123	1457	152	2016	181	2063
8	47	37	531	66	857	95	1460	124	1458	153	2017	182	2064
9	61	38	532	67	729	96	1461	125	1658	154	2018	183	2047
10	62	39	533	68	730	97	1462	126	1659	155	2019	184	2048
11	156	40	534	69	930	98	1463	127	1660	156	2020	185	2049
12	157	41	535	70	931	99	1464	128	1661	157	2021	186	2050
13	158	42	536	71	932	100	1465	129	1662	158	2022	187	2069
14	159	43	586	72	933	101	1466	130	1663	159	2023	188	2070
15	129	44	587	73	934	102	1467	131	1995	160	2024	189	2071
16	130	45	588	74	935	103	1437	132	1996	161	2025	190	2072
17	224	46	589	75	1073	104	1438	133	1604	162	2026	191	2073
18	225	47	590	76	1074	105	1439	134	1605	163	2027	192	2074
19	177	48	591	77	938	106	1440	135	1606	164	2028	193	2972
20	178	49	592	78	939	107	1441	136	1607	165	2029	194	2973
21	303	50	593	79	940	108	1442	137	1608	166	2030	195	2974
22	304	51	594	80	941	109	1474	138	1609	167	2031	196	2975
23	473	52	595	81	1446	110	1475	139	2003	168	2032	197	2728
24	474	53	852	82	1447	111	1476	140	2004	169	2033	198	2729
25	444	54	853	83	1448	112	1477	141	1961	170	2034	199	2730
26	445	55	598	84	1449	113	1646	142	1962	171	2035	200	2731
27	384	56	599	85	1450	114	1647	143	2007	172	2036	201	2980
28	385	57	856	86	1451	115	1648	144	2008	173	2037	202	2981
29	448	58	857	87	1359	116	1649	145	2009	174	2038	203	2085

5. Comparison of two piecewise functions and the proof of Collatz conjecture

Comparing the Collatz function $T(x)$ and the function $f^{-1}(x)$, if their domain is defined as the set of natural numbers, we find that they have the following relation:

1) The function $f^{-1}(x)$ is strictly monotonically decreasing,

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2) When x is purely even, the function $T(x)$ is only one case of the functions, is strictly monotonically decreasing;

3) When x is a pure or mixed odd number, the function $T(x)$ is wavy, which is increasing, followed by one or more decreasing processes, that is, "increase – decrease – increase", or "increase – decrease $\cdots$ decrease – increase". For example, Figure 3 and Figure 4 are the plots of the iterated sequence of Collatz functions with initial values of pure odd $255 = 2^8 - 1 = (11111111)_2$ and mixed odd number $97 = (1100001)_2$, respectively.

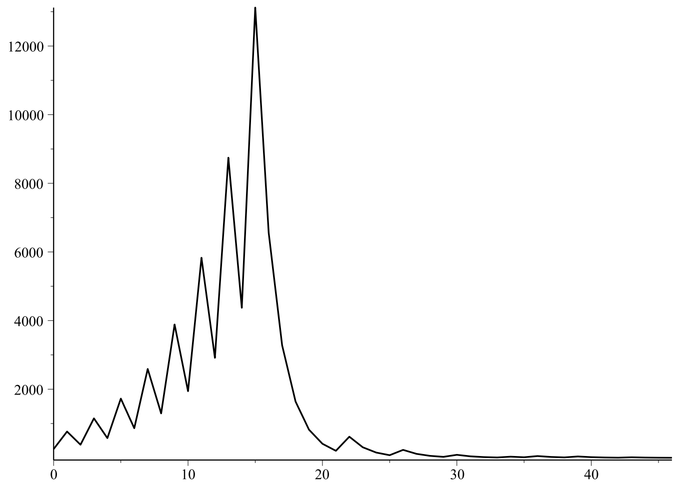

Figure 3. Point plot of a sequence of 47 iterations of the Collatz function for pure odd 255.

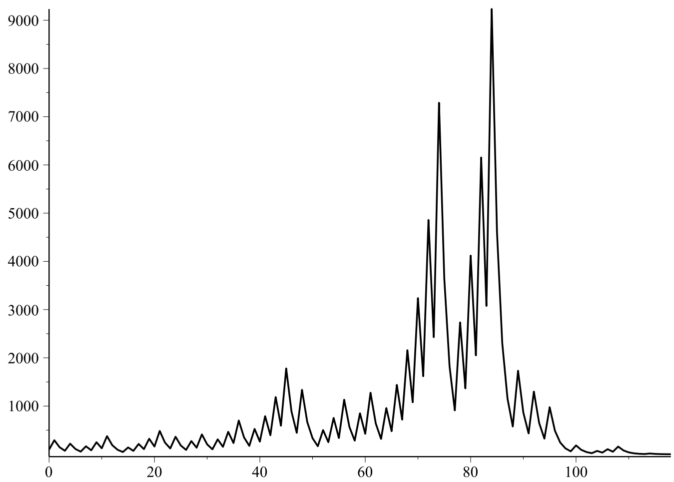

Figure 4. Point plot of a sequence of 118 iterations of the Collatz function for mixed number 97.

Due to the fact that an odd number can be either pure or mixed, when x is odd, the Collatz function $3x + 1$ can be parted into two parts. i.e., $3x + 1 = 2x + (x + 1)$.

(i) when x is a pure odd number, i.e., $x = 2^r - 1$, The binary string $2x = 2^{r+1} - 2$ is even, with just one 0 at the end, and the other part $x + 1 = 2^r$ is pure even. As a result, $3x + 1 = 2x + (x + 1) = 2^{r+1} + 2^r - 2$ is a mixed even number with only one 0 in the last bit and $r - 1$ bits 1 in the second-to-last substring, corresponding $\frac{3x+1}{2} = 2^r + 2^{r-1} - 1 > x$, it means function T(x) is increase.

(ii) When x is a mixed odd number, it just has one 1 in the last binary substring, the value $\frac{3x+1}{2^r} < x, (r > 1)$ is smaller odd than x, it means function T(x) is decrease. The last binary substring of number $\frac{3x+1}{2^r}, (r > 1)$ has two kinds, which :

(a) There many 1 in the last binary substring. For example, $893 = (1101111101)_2$, $\frac{3\times 893+1}{2^3} = 335 = (101001111)_2 = 335, 335 < 893$.

(b) Only 1 in the last binary substring. For example, $\mathbf{17} = (\mathbf{10001})_2$, $\frac{3\times 17+1}{2^2} = (\mathbf{1101})_2 = \mathbf{13}$, $\mathbf{13} < \mathbf{17}$.

When the end of the binary substring is 1, using $\times$ denotes either 1 or 0, we discuss the changes of the last substring three digits and four digits in the procedure of the Collatz function sequences:

$\times\mathbf{001} \rightarrow \times\times\mathbf{100} \rightarrow \times\times\mathbf{1}, \quad \times\mathbf{101} \rightarrow \times\mathbf{000} \rightarrow \times$

$\times\mathbf{0101} \rightarrow \times\times\mathbf{0000} \rightarrow \times\times, \quad \times\mathbf{1101} \rightarrow \times\times\mathbf{1000} \rightarrow \times\times\mathbf{1}$

$\mathbf{3x+1}$ can adjust the structure of its binary substring, when the end of the binary substring is 1, the value of Collatz function $T(x) = \frac{3x+1}{2^r}, r > 1$ decrease, thus the number of binary string digits decrease at least 2. This process continues several times, and eventually you can reach the minimum value of 1. The Collatz function shows that the Collatz conjecture holds.

Another Proof We give the statement "period three implies chaos"[3] another interpretation: for any positive integer n, the sequence of the Collatz is an ultimately periodic sequence, its preperiod $\eta(n)$ is a related-to n positive, and the least period $\{4,2,1\}$, $\rho(n) = 3$.

We use the inversion of "period three implies chaos"[3] is the Collatz sequence of any initial value n. when the Collatz Conjecture is correct, then the reverse order of sequence obtained for any natural initial n, the sequence of the Collatz function is an ultimately periodic of period 3, $\{1,4,2\}$.

6. Conclusion

From previous proof of the conjecture, it becomes a theorem.

Theorem For any natural number n, if it is even, divide by 2, if it is odd, multiply by 3, add 1, and so on, the result must finally reach 1. Give another statement: for any positive integer n, the sequence of the Collatz function is an ultimately periodic sequence, its preperiod $\eta(n)$ is a related-to n positive, and the least period $\{4,2,1\}$, $\rho(n) = 3$.

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