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Article

On Fuzzy Near Best Approximation

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Abstract: Given a fuzzy normed space (X, N), we will introduce the notion of fuzzy near best approximation within a relative distance $\rho \geq 0$. Some basic properties are characterized and also many examples for illustration are presented.

Keywords: Fuzzy near best approximation; Direct sum; Tensor product; Fuzzy norm

MSC: 46A32; 41A50; 41A17; 41A65

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1. INTRODUCTION AND PRELIMINARIES

Let X be a normed linear space and Y be a subset of X. If $x \in X$, then the distance of x from Y is denoted by $d(x, Y)$ that is

$$d\,(x, Y) = \inf\{\|x-y\| \;\; s.t.\; y \in Y\}\,.$$

An element $y \in Y$ is said to be a best approximation to $x \in X$ from Y if $\|x-y\| = d(x, Y)$. The set of all best approximations to $x \in X$ from Y is denoted by $P_Y(x)$. If for any $x \in X$, $P_Y(x) \neq \emptyset$, then we say that Y is proximinal in X. Also if for any $x \in X$, $P_Y(x)$ is singleton, therefore Y is a Chebyshev subset of X. A sequence $\{y_n\}_n \subseteq Y$ is called a minimizing sequence for $x \in X$ if $\lim_{n\to\infty} \|x-y_n\| = d\,(x, Y)$ [4].

An element $y^n \in Y$ is said to be a near best approximation to x within a relative distance $\rho \geq 0$ if,

$$\|x-y^n\| \leq (1+\rho)\left\|x-x^b\right\| = (1+\rho)d\,(x, Y)$$

where x^b is a best approximation to x from Y[5]. The set of all near best approximations to $x \in X$ from Y is denoted by $P^n_Y(x)$.
Let $\{X_i\}_{i\in I}$ be a family of linear spaces. Then the algebraic direct sum of the spaces X_i, i.e.,

$$\sum_{i\in I} X_i = \left\{x = (x_i)_{i\in I} \middle| x_i = 0 \quad \text{for all but finitely many} \quad i \in I\right\}$$

with the pointwise vector-space operations as follows, is a linear space,

$$x + y = (x_i + y_i)_{i\in I}$$

and

$$\alpha x = (\alpha x_i)_{i\in I}$$

for all $x, y \in \sum_{i\in I} X_i$ and $\alpha \in \mathbb{C}$ or $\mathbb{R}$ [1].
Also let X and Y be linear spaces over $\mathbb{C}$ or $\mathbb{R}$. Then the algebraic tensor product of X and Y is denoted by $X \otimes Y$. If X' and Y' are the dual spaces of X and Y respectively, then for all $x \in X$ and $y \in Y$, the map $x \otimes y : X' \times Y' \longrightarrow \mathbb{C}$ (or $\mathbb{R}$) defined by

$$(x \otimes y)\,(f, g) = f\,(x)\,g\,(y)\,, f \in X', g \in Y'$$

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is a bilinear map. For the basic properties concerning the tensor product of linear spaces, we refer the reader to [2].

Definition 1.1. [7]Let X be a linear space. A function $N : X \times \mathbb{R} \longrightarrow [0,1]$ is said to be fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$:

1 -$N(x,t) = 0$ for $t \leq 0$.
2 -$N(x,t) = 1$ for every $t \in \mathbb{R}^+$ if and only if $x = 0$.
3 -$N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ for every $c \neq 0$ and $t \in \mathbb{R}$.
4 -$N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$ for every $s, t \in \mathbb{R}$.
5 -$N(x,.)$ is non-decreasing on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.

Note that by part 3 of Definition 1.1, $N(-x,t) = N(x,t)$ for all $x \in X$ and $t \in \mathbb{R}$.

Definition 1.2. [6]Let Y be a nonempty subset of a fuzzy normed space (X, N). For $x \in X$ and $t \in \mathbb{R}$, let

$$d(Y,x,t) = \sup\{N(y-x,t), \quad y \in Y\}.$$

An element $y_0 \in Y$ is said to be a fuzzy best approximation to x from Y if

$$N(y_0 - x, t) = d(Y,x,t),$$

for all $t \in \mathbb{R}$. The set of all fuzzy best approximations to x from Y is denoted by $P_Y^f(x)$.

Definition 1.3. Let X be a linear space and Y be a subset of X. Also let $N : X \times \mathbb{R} \longrightarrow [0,1]$ be a fuzzy norm on X. An element $y_0 \in Y$ is said to be a fuzzy near best approximation to x from Y within a relative distance $\rho \geq 0$ if,

$$N(x - y_0, t) \geq N\left(x - Y, \frac{t}{1+\rho}\right)$$

for all $t \in \mathbb{R}$, where

$$N\left(x - Y, \frac{t}{1+\rho}\right) = \sup\left\{N\left(x - y, \frac{t}{1+\rho}\right), y \in Y\right\}.$$

The set of all fuzzy near best approximations to x from Y within the relative distance ρ is denoted by $P_Y^{fn(\rho)}(x)$.

Remark 1.4. Trivially the notion of fuzzy best approximation is nothing else than fuzzy near best approximation within the relative distance $\rho = 0$.

Proposition 1.5. Let X be a linear space and Y be a nonempty subset of X. Also let $x, y \in X, z \in Y, \rho \geq 0$ and $\alpha \in \mathbb{R}$. Then

1- If $P_Y^{fn(\rho)}(x) \neq \emptyset$, then $P_{\alpha Y}^{fn(\rho)}(\alpha x) = \alpha P_Y^{fn(\rho)}(x)$
2- If $\alpha \neq 0$, then $P_Y^{fn(\rho)}(\alpha x) = \alpha P_{\frac{Y}{\alpha}}^{fn(\rho)}(x)$
3- $P_Y^{fn(\rho)}(x+z) = P_{Y-z}^{fn(\rho)}(x) + z$
4- $P_{Y+y}^{fn(\rho)}(x+y) = P_Y^{fn(\rho)}(x) + y$.

Proof. We'll prove the first part. The rest of the parts are easily verified. In the case where $\alpha = 0$, the equality trivially holds. For $\alpha \neq 0$, let $z_0 \in P_{\alpha Y}^{fn(\rho)}(\alpha x)$. Then

$$N(\alpha x - z_0, t) \geq N\left(\alpha x - \alpha Y, \frac{t}{1+\rho}\right), \quad (\forall t \in \mathbb{R}).$$

Therefore

$$N\left(x - \frac{z_0}{\alpha}, \frac{t}{|\alpha|}\right) \geq N\left(x - Y, \frac{t}{|\alpha|(1+\rho)}\right), \quad (\forall t \in \mathbb{R}).$$

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Replacing t by $|\alpha|\, t$, we have

$$N\left(x-\frac{z_0}{\alpha},t\right) \geq N\left(x-Y,\frac{t}{1+\rho}\right), \quad (\forall t \in \mathbb{R}).$$

So $\frac{z_0}{\alpha} \in P_Y^{fn(\rho)}(x)$. Hence $z_0 \in \alpha P_Y^{fn(\rho)}(x)$.

Conversely, let $z_0 \in \alpha P_Y^{fn(\rho)}(x)$. Then $\frac{z_0}{\alpha} \in P_Y^{fn(\rho)}(x)$. Therefore

$$N\left(x-\frac{z_0}{\alpha},t\right) \geq N\left(x-Y,\frac{t}{1+\rho}\right), \quad (\forall t \in \mathbb{R}).$$

So

$$N(\alpha x - z_0, |\alpha|\, t) \geq N\left(x-Y,\frac{t}{1+\rho}\right), \quad (\forall t \in \mathbb{R}).$$

Replacing t by $\frac{t}{|\alpha|}$, we have

$$\begin{aligned} N(\alpha x - z_0, t) &\geq N\left(x-Y,\frac{t}{|\alpha|\,(1+\rho)}\right) \\ &= N\left(\alpha x-\alpha Y,\frac{t}{1+\rho}\right), \quad (\forall t \in \mathbb{R}). \end{aligned}$$

Then $z_0 \in P_{\alpha Y}^{fn(\rho)}(\alpha x)$. So we can conclude that $P_{\alpha Y}^{fn(\rho)}(\alpha x) = \alpha P_Y^{fn(\rho)}(x)$. □

Proposition 1.6. Let (X, N) be a fuzzy normed linear space and Y be a subset of X. Then every fuzzy best approximation to $x \in X$ from Y is a fuzzy near best approximation to x from Y within every relative distance $\rho \geq 0$.

Proof. Let $y_0 \in P_Y^f(x)$. So $N(x-y_0,t) = N(x-Y,t)$ for all $t \in \mathbb{R}$. We shall show that $N(x-y_0,t) \geq N\left(x-Y,\frac{t}{1+\rho}\right)$ for all $t \in \mathbb{R}$ and every $\rho \geq 0$. Let $y \in Y$ and $t \in \mathbb{R}$. So by Definition 1.1 part 5,

$$N\left(x-y,\frac{t}{1+\rho}\right) \leq N(x-y,t) \leq N(x-Y,t) = N(x-y_0,t).$$

It follows that $N\left(x-y,\frac{t}{1+\rho}\right) \leq N(x-y_0,t)$ for all $t \in \mathbb{R}$ and $y \in Y$. Hence

$$\sup\left\{N\left(x-y,\frac{t}{1+\rho}\right), \quad y \in Y\right\} \leq N(x-y_0,t)$$

for all $t \in \mathbb{R}$. Therefore

$$N\left(x-Y,\frac{t}{1+\rho}\right) \leq N(x-y_0,t)$$

for all $t \in \mathbb{R}$, providing $y_0 \in P_Y^{fn(\rho)}(x)$. □

Proposition 1.7. Let (X, N) be a fuzzy normed linear space, Y be a subset of X, $x \in X$, and $\rho \geq 0$. If Y is convex, then $P_Y^f(x)$ and $P_Y^{fn(\rho)}(x)$ are convex.

Proof. Let $y_1, y_2 \in P_Y^{fn(\rho)}(x)$ and $0 < \alpha < 1$. So by Definition 1.1, for all $t \in \mathbb{R}$ we have

$$\begin{aligned}
&N\left(x-\left(\alpha y_1+(1-\alpha)\, y_2\right), t\right) \\
&=N\left(\alpha x+(1-\alpha)\, x-\left(\alpha y_1+(1-\alpha)\, y_2\right), \alpha t+(1-\alpha)\, t\right) \\
&=N\left(\alpha\left(x-y_1\right)+(1-\alpha)\left(x-y_2\right), \alpha t+(1-\alpha)\, t\right) \\
&\geq \min \left\{N\left(\alpha\left(x-y_1\right), \alpha t\right), N\left((1-\alpha)\left(x-y_2\right),(1-\alpha)\, t\right)\right\} \\
&=\min \left\{N\left(x-y_1, t\right), N\left(x-y_2, t\right)\right\} \\
&\geq N\left(x-Y, \frac{t}{1+\rho}\right).
\end{aligned}$$

Hence $\alpha y_1+(1-\alpha)\, y_2 \in P_Y^{fn(\rho)}(x)$. □

The next example shows that the notion of fuzzy near best approximation is different from the notion of fuzzy best approximation.

Example 1.8. Suppose that $X = \mathbb{R}$, $Y = [1, 1.5]$, $x = 0$ and $\rho = 0.5$. Also let

$$N(x,t) = \begin{cases} 0 & t \leq |x| \\ 1 & t > |x| \end{cases}$$

be a fuzzy norm on X. Then, $P_Y(0) = P_Y^f(0) = \{1\}$ and $P_Y^{fn(0.5)}(0) = Y$. Generally for $0 \leq \rho \leq 0.5$, $P_Y^{fn(\rho)}(0) = [1, 1+\rho]$. Indeed, if $0 < t \leq 1$, then $N(1,t) = 0$ and for all $y \in Y$, $t \leq y$. So $N(y,t) = 0$ for all $y \in Y$. It follows that $N(Y,t) = 0$. Hence $N(1,t) = N(Y,t)$ for all $0 < t \leq 1$. If $t > 1$, then $N(1,t) = 1$ and so $N(Y,t) = 1$. Therefore $N(1,t) = N(Y,t)$ for all $t \in \mathbb{R}$. This shows that $1 \in P_Y^f(0)$. If $1 < y_0 \leq 1.5$, then for $t = 1 + \frac{y_0 - 1}{2}$, $N(y_0, t) = 0$ and $N(1,t) = 1$. It follows that $0 = N(y_0, t) \neq N(Y,t) = 1$ for $t = 1 + \frac{y_0-1}{2}$. So $y_0 \notin P_Y^f(0)$. Hence $P_Y^f(0) = \{1\}$.
Now we will show that $P_Y^{fn(0.5)}(0) = [1, 1.5] = Y$. Let $y_0 \in [1, 1.5]$. If $t \leq y_0$, then $N(y_0, t) = 0$ and $\frac{2}{3}t \leq \frac{2}{3}y_0 \leq \frac{2}{3}\left(\frac{3}{2}\right) = 1$. So $N\left(y, \frac{2}{3}t\right) = 0$ for all $y \in [1, 1.5]$. It follows that $N\left(Y, \frac{2}{3}t\right) = 0$. Hence $N(y_0, t) \geq N\left(Y, \frac{2}{3}t\right)$ for all $t \leq y_0$. If $t > y_0$, then $N(y_0, t) = 1 \geq N\left(Y, \frac{2}{3}t\right)$. Therefore $N(y_0, t) \geq N\left(Y, \frac{2}{3}t\right)$ for all $t \in \mathbb{R}$. Hence $y_0 \in P_Y^{fn(0.5)}(0)$. This shows that $P_Y^{fn(0.5)}(0) = [1, 1.5] = Y$.

Example 1.9. Suppose that $X = \mathbb{R}$, $Y = (0, 1)$, $x = 0$ and $\rho \geq 0$. Also let

$$N(x,t) = \begin{cases} 0 & t \leq |x| \\ 1 & t > |x| \end{cases}$$

be a fuzzy norm on X. Then $P_Y(0) = P_Y^f(0) = P_Y^{fn(\rho)}(0) = \emptyset$. Indeed, since $P_Y(0) \subseteq P_Y^f(0) \subseteq P_Y^{fn(\rho)}(0)$, it's enough to prove that $P_Y^{fn(\rho)}(0) = \emptyset$. Let $y_0 \in Y$ be an arbitrary element. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{y_0}{1+\rho}$. So if $t = y_0$, then

$$0 = N(y_0, t) < N\left(\frac{1}{n}, \frac{t}{1+\rho}\right) = 1.$$

It follows that

$$0 = N(y_0, t) < N\left(Y, \frac{t}{1+\rho}\right) = 1.$$

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Hence $y_0 \notin P_Y^{fn(\rho)}(0)$ and so $P_Y^{fn(\rho)}(0) = \emptyset$.

Proposition 1.10. Let X be a linear space, Y be a subset of X and $N : X \times \mathbb{R} \longrightarrow [0,1]$ be a fuzzy norm on X. For any $\rho \geq 0$, $x \in Y$, $P_Y^{fn(\rho)}(x) = \{x\}$.

Proof. As $1 = N(0,t) = N(x-x,t) \geq N\left(x - Y, \frac{t}{1+\rho}\right)$ for all $t > 0$, we can conclude that $N(x-x,t) \geq N\left(x - Y, \frac{t}{1+\rho}\right)$ for all $t \in \mathbb{R}$. Hence $x \in P_Y^{fn(\rho)}(x)$.
Now let $y_0 \in P_Y^{fn(\rho)}(x)$. Then for all $t > 0$

$$\begin{aligned} N(x-y_0,t) &\geq N\left(x - Y, \frac{t}{1+\rho}\right) \\ &\geq N\left(x - x, \frac{t}{1+\rho}\right) \\ &= 1. \end{aligned}$$

Therefore $N(x-y_0,t) = 1$ for all $t > 0$. Hence $x - y_0 = 0$. Then $y_0 = x$. □

Theorem 1.11. Let X be a linear space, Y be a subset of X and $N : X \times \mathbb{R} \longrightarrow [0,1]$ be a fuzzy norm on X. For $x \in X$, if $\rho_1 \leq \rho_2$, then $P_Y^{fn(\rho_1)}(x) \subseteq P_Y^{fn(\rho_2)}(x)$.

Proof. If $y_0 \in P_Y^{fn(\rho_1)}(x)$, then

$$N(x-y_0,t) \geq N\left(x - Y, \frac{t}{1+\rho_1}\right).$$

Since for all $t \in \mathbb{R}$ and $y \in Y$

$$N\left(x - y, \frac{t}{1+\rho_1}\right) \geq N\left(x - y, \frac{t}{1+\rho_2}\right),$$

$$N\left(x - Y, \frac{t}{1+\rho_1}\right) \geq N\left(x - Y, \frac{t}{1+\rho_2}\right).$$

Therefore

$$N(x-y_0,t) \geq N\left(x - Y, \frac{t}{1+\rho_2}\right).$$

It follows that $y_0 \in P_Y^{fn(\rho_2)}(x)$. □

Theorem 1.12. Let X be a linear space, Y be a subset of X and $N : X \times \mathbb{R} \longrightarrow [0,1]$ be a fuzzy norm on X. If $x \in X$ and $\rho_1 \geq \rho_2 \geq \rho_3 \geq \ldots$ such that $\rho_m \longrightarrow \rho$ as $m \longrightarrow \infty$, then $P_Y^{fn(\rho)}(x) \subseteq \bigcap_{m=1}^{\infty} P_Y^{fn(\rho_m)}(x)$. In particular, if $N(z,.)$ is lower semicontinuous at every $z \in x - Y$, then $P_Y^{fn(\rho)}(x) = \bigcap_{m=1}^{\infty} P_Y^{fn(\rho_m)}(x)$.

Proof. By Theorem 1.11, since $\rho_m \geq \rho$ for all $m \in \mathbb{N}$,

$$\begin{aligned} &P_Y^{fn(\rho)}(x) \subseteq P_Y^{fn(\rho_1)}(x) \\ &P_Y^{fn(\rho)}(x) \subseteq P_Y^{fn(\rho_2)}(x) \\ &\vdots \\ &P_Y^{fn(\rho)}(x) \subseteq P_Y^{fn(\rho_m)}(x), m \in \mathbb{N}. \end{aligned}$$

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Then $P_Y^{fn(\rho)}(x) \subseteq \bigcap_{m=1}^{\infty} P_Y^{fn(\rho_m)}(x)$.
If $y_0 \in \bigcap_{m=1}^{\infty} P_Y^{fn(\rho_m)}(x)$ and $N(z,.)$ is lower semicontinuous for every $z \in x - Y$, then for all $m \in \mathbb{N}$ and for all $t \in \mathbb{R}$ we have

$$\begin{aligned} N(x - y_0, t) \geq & N\left(x - Y, \frac{t}{1+\rho_m}\right) \\ \geq & N\left(x - y, \frac{t}{1+\rho_m}\right), y \in Y. \end{aligned}$$

Since $N(x-y,.)$ is lower semicontinuous for all $x - y$ and $\frac{t}{1+\rho_m} \leq \frac{t}{1+\rho}$ for every $m \in \mathbb{N}$,

$$\begin{aligned} N(x - y_0, t) \geq & \lim_{m \longrightarrow \infty} N\left(x - y, \frac{t}{1+\rho_m}\right) \\ = & N\left(x - y, \frac{t}{1+\rho}\right), y \in Y. \end{aligned}$$

So $\bigcap_{m=1}^{\infty} P_Y^{fn(\rho_m)}(x) \subset P_Y^{fn(\rho)}(x)$.

$$\begin{aligned} N(x - y_0, t) \geq & \sup\left\{N\left(x - y, \frac{t}{1+\rho}\right), y \in Y\right\} \\ = & N\left(x - Y, \frac{t}{1+\rho}\right). \end{aligned}$$

Hence $\bigcap_{m=1}^{\infty} P_Y^{fn(\rho_m)}(x) \subseteq P_Y^{fn(\rho)}(x)$. □

We use the following lemma in the proof of Proposition 2.1 and Theorem 2.5.

Lemma 1.13. Let $0 \leq a_i \leq 1$ and $0 \leq b_i \leq 1$ for all $1 \leq i \leq m$. Then

$$\begin{aligned} & \min\left\{\min\{a_i, b_i\} \middle| 1 \leq i \leq m\right\} \\ = & \min\left\{\min\left\{a_i \middle| 1 \leq i \leq m\right\}, \min\left\{b_i \middle| 1 \leq i \leq m\right\}\right\}. \end{aligned}$$

Proof. It's obvious. □

2. Fuzzy Near Best Approximation On Direct Sum And Tensor Product Of Linear Spaces

Proposition 2.1. Let $\{(X_i, N_i)\}_{i \in I}$ be a family of fuzzy normed spaces. Then $(\sum_{i \in I} X_i, N)$ is a fuzzy normed space, where $N : (\sum_{i \in I} X_i) \times \mathbb{R} \longrightarrow [0,1]$ is defined by

$$N\left((x_i)_{i \in I}, t\right) = \inf\left\{N_i(x_i, t) \middle| i \in I\right\}.$$

Proof. To prove the above proposition, we only prove conditions 4 and 5 of Definition 1.1 and we leave the rest to the reader. To prove the fourth part, suppose that $(x_i)_{i \in I}, (y_i)_{i \in I} \in \sum_{i \in I} X_i$ and $s, t \in \mathbb{R}$. So $x_i = 0$ and $y_i = 0$ for all but finitely many $i_k \in I, 1 \leq k \leq m$. Clearly if $s + t \leq 0$, then the inequality

$$N\left((x_i)_{i \in I} + (y_i)_{i \in I}, s + t\right) \geq \min\left\{N\left((x_i)_{i \in I}, s\right), N\left((y_i)_{i \in I}, t\right)\right\} \tag{2.1}$$

holds. Also if $s + t > 0$, $s \leq 0$ or $t \leq 0$, then obviously inequality 2.1 holds.

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Let $s+t>0, s>0$ and $t>0$. Then

$$
\begin{aligned}
& N\left((x_i)_{i\in I}+(y_i)_{i\in I}, s+t\right) \\
& = N\left((x_i+y_i)_{i\in I}, s+t\right) \\
& \inf\left\{N_i\left(x_i+y_i, s+t\right) \middle| i\in I\right\} \\
& \geq \inf\left\{\min\left\{N_i\left(x_i,s\right), N_i\left(y_i,t\right)\right\} \middle| i\in I\right\} \\
& = \inf\left\{\min\left\{N_{i_k}\left(x_{i_k},s\right), N_{i_k}\left(y_{i_k},t\right)\right\}, 1 \middle| 1\leq k\leq m\right\} \\
& = \min\left\{\min\left\{N_{i_k}\left(x_{i_k},s\right), N_{i_k}\left(y_{i_k},t\right)\right\}, 1 \middle| 1\leq k\leq m\right\} \\
& = \min\left\{\min\left\{N_{i_k}\left(x_{i_k},s\right), N_{i_k}\left(y_{i_k},t\right)\right\} \middle| 1\leq k\leq m\right\} \\
& = \min\left\{\min\left\{N_{i_k}\left(x_{i_k},s\right) \middle| 1\leq k\leq m\right\}, \min\left\{N_{i_k}\left(y_{i_k},t\right) \middle| 1\leq k\leq m\right\}\right\} \\
& = \min\left\{\min\left\{N_i\left(x_i,s\right) \middle| i\in I\right\}, \min\left\{N_i\left(y_i,t\right) \middle| i\in I\right\}\right\} \\
& = \min\left\{\inf\left\{N_i\left(x_i,s\right) \middle| i\in I\right\}, \inf\left\{N_i\left(y_i,t\right) \middle| i\in I\right\}\right\} \\
& = \min\left\{N\left((x_i)_{i\in I}, s\right), N\left((y_i)_{i\in I}, t\right)\right\}.
\end{aligned}
$$

To prove the fifth part, suppose that $(x_i)_{i\in I} \in \sum_{i\in I} X_i$. So $x_i = 0$ for all but finitely many $i_k \in I$, $1\leq k\leq m$. Hence

$$
\begin{aligned}
& \lim_{t\longrightarrow\infty} N\left((x_i)_{i\in I}, t\right) \\
& = \lim_{t\longrightarrow\infty}\left\{\inf\left\{N_i\left(x_i,t\right) \middle| i\in I\right\}\right\} \\
& = \lim_{t\longrightarrow\infty}\left\{\inf\left\{N_{i_k}\left(x_{i_k},t\right), 1 \middle| 1\leq k\leq m\right\}\right\} \\
& = \lim_{t\longrightarrow\infty}\left\{\min\left\{N_{i_k}\left(x_{i_k},t\right), 1 \middle| 1\leq k\leq m\right\}\right\} \\
& = \lim_{t\longrightarrow\infty}\left\{\min\left\{N_{i_k}\left(x_{i_k},t\right) \middle| 1\leq k\leq m\right\}\right\} \\
& = \min\left\{\lim_{t\longrightarrow\infty} N_{i_k}\left(x_{i_k},t\right) \middle| 1\leq k\leq m\right\} \\
& = \min\{1\} \\
& = 1.
\end{aligned}
$$

Since $N_i(x_i, .)$ is increasing for all $x_i \in X_i$, $N\left((x_i)_{i\in I}, .\right)$ is increasing for all $(x_i)_{i\in I}$. □

Corollary 2.2. Let $N_i : X_i \times \mathbb{R} \longrightarrow [0,1]$ be a fuzzy norm on X_i for $i = 1,2,\ldots,n$. Then $N : X_1 \times X_2 \times \cdots \times X_n \times \mathbb{R} \longrightarrow [0,1]$ defined by

$$N\left((x_1,x_2,\ldots,x_n), t\right) = \min\left\{N_1(x_1,t), N_2(x_2,t), \ldots, N_n(x_n,t)\right\}$$

is a fuzzy norm on $X_1 \times X_2 \times \cdots \times X_n$.

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Proposition 2.3. Let $\{(X_i, N_i)\}_{i=1}^{n}$ be a finite family of fuzzy normed spaces and $N : X_1 \times X_2 \times \cdots \times X_n \times \mathbb{R} \longrightarrow [0,1]$ is defined by

$$N\left(\left(x_1, x_2, \ldots, x_n\right), t\right) = \min\left\{N_1\left(x_1, t\right), N_2\left(x_2, t\right), \ldots, N_n\left(x_n, t\right)\right\}.$$

Also let $Y_i \subseteq Xi$, $x_i \in X_i$ and $\rho_i \geq 0$ for all $1 \leq i \leq n$. Then

$$P_{Y_1}^{fn(\rho_1)}\left(x_1\right) \times P_{Y_2}^{fn(\rho_2)}\left(x_2\right) \times \cdots \times P_{Y_n}^{fn(\rho_n)}\left(x_n\right) \subseteq P_{Y_1 \times Y_2 \times \cdots \times Y_n}^{fn(\max\{\rho_i\})}\left(x_1, x_2, \ldots, x_n\right). \tag{2.2}$$

Proof. Let $(y_1, y_2, \ldots, y_n) \in P_{Y_1}^{fn(\rho_1)}(x_1) \times P_{Y_2}^{fn(\rho_2)}(x_2) \times \cdots \times P_{Y_n}^{fn(\rho_n)}(x_n)$. Then $y_1 \in P_{Y_1}^{fn(\rho_1)}(x_1)$, $y_2 \in P_{Y_2}^{fn(\rho_2)}(x_2), \ldots, y_n \in P_{Y_n}^{fn(\rho_n)}(x_n)$. Therefore

$$\begin{aligned}
&N_1\left(x_1 - y_1, t\right) \geq N_1\left(x_1 - z_1, \frac{t}{1+\rho_1}\right) \geq N_1\left(x_1 - z_1, \frac{t}{1+\max_{1\leq i\leq n}\{\rho_i\}}\right) \\
&N_2\left(x_2 - y_2, t\right) \geq N_2\left(x_2 - z_2, \frac{t}{1+\rho_2}\right) \geq N_2\left(x_2 - z_2, \frac{t}{1+\max_{1\leq i\leq n}\{\rho_i\}}\right) \\
&\vdots \\
&N_n\left(x_n - y_n, t\right) \geq N_n\left(x_n - z_n, \frac{t}{1+\rho_n}\right) \geq N_n\left(x_n - z_n, \frac{t}{1+\max_{1\leq i\leq n}\{\rho_i\}}\right)
\end{aligned}$$

for every $z_i \in Y_i$ and $t \in \mathbb{R}$. Then

$$\begin{aligned}
&\min\left\{N_1\left(x_1 - y_1, t\right), N_2\left(x_2 - y_2, t\right), \ldots, N_n\left(x_n - y_n, t\right)\right\} \\
&\geq \min\left\{N_1^{(1)}, N_2^{(2)}, \ldots, N_n^{(n)}\right\},
\end{aligned}$$

where $N_k^{(k)} = N_k\left(x_k - z_k, \frac{t}{1+\max_{1\leq i\leq n}\{\rho_i\}}\right)$, $1 \leq k \leq n$. Therefore

$$\begin{aligned}
&N\left(\left(x_1, x_2, \ldots, x_n\right) - \left(y_1, y_2, \ldots, y_n\right), t\right) \\
&\geq \sup_{(z_1, z_2, \ldots, z_n) \in Y_1 \times Y_2 \times \cdots \times Y_n} \left\{N\left(\left(x_1, x_2, \ldots, x_n\right) - \left(z_1, z_2, \ldots, z_n\right), \frac{t}{1+\max_{1\leq i\leq n}\{\rho_i\}}\right)\right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
&N\left(\left(x_1, x_2, \ldots, x_n\right) - \left(y_1, y_2, \ldots, y_n\right), t\right) \\
&\geq N\left(\left(x_1, x_2, \ldots, x_n\right) - Y_1 \times Y_2 \times \cdots \times Y_n, \frac{t}{1+\max_{1\leq i\leq n}\{\rho_i\}}\right).
\end{aligned}$$

□

In the next example, we will show that the converse of inclusion 2.2 is not true in general.

Example 2.4. Let $X_1 = X_2 = \mathbb{R}$, $Y_1 = Y_2 = [1,3]$, and $N_i : X_i \times \mathbb{R} \longrightarrow [0,1]$ is defined by

$$N_i\left(\alpha, t\right) = \begin{cases} 0 & t \leq |\alpha| \\ 1 & t > |\alpha| \end{cases}$$

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for $i = 1, 2$. Assume that $N((x, y), t) = \min(N_1(x, t), N_2(y, t))$, $\rho = 1$, $x_1 = 0$ and $x_2 = \frac{1}{2}$. Then we have

$$P_{Y_1}^{fn(1)}(0) = [1, 2]$$
$$P_{Y_2}^{fn(1)}\left(\frac{1}{2}\right) = [1, \frac{3}{2}]$$
$$P_{Y_1}^{fn(1)}(0) \times P_{Y_2}^{fn(1)}\left(\frac{1}{2}\right) = [1, 2] \times [1, \frac{3}{2}].$$

It is easy to see that

$$N((x, y), t) = \begin{cases} 0 & t \leq \max(|x|, |y|) \\ 1 & t > \max(|x|, |y|) \end{cases}$$

for all $x, y, t \in \mathbb{R}$. Also a sufficient effort can be applied to show that

$$P_{Y_1 \times Y_2}^{fn(1)}\left(0, \frac{1}{2}\right) = [1, 2] \times [1, \frac{5}{2}].$$

Theorem 2.5. Let (X, N_1) and (Y, N_2) be fuzzy normed spaces. Also let B_X and B_Y be the bases of X and Y respectively. Define $N : (X \otimes Y) \times \mathbb{R} \longrightarrow [0, 1]$ by

$$N\left(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j, t\right) = \min\left\{N_1(\alpha_j x_j, t), N_2(\alpha_j y_j, t) \,\middle|\, 1 \leq j \leq n\right\},$$

where $x_j \in B_X, y_j \in B_Y, n \in \mathbb{N}, \alpha_j \in \mathbb{C}$ and $t \in \mathbb{R}$. Then $(X \otimes Y, N)$ is a fuzzy normed space.

Proof. (1) : At the first we will prove that N is well defined. Let $(z, s), (w, t) \in (X \otimes Y) \times \mathbb{R}$. So $z = w$ and $s = t$. Hence there exists an $n \in \mathbb{N}$, $\{x_j\}_{j=1}^n \subseteq B_X$, $\{y_j\}_{j=1}^n \subseteq B_Y$ and $\alpha_j, \beta_j \in \mathbb{C}$ such that $z = \sum_{j=1}^n \alpha_j x_j \otimes y_j$ and $w = \sum_{j=1}^n \beta_j x_j \otimes y_j$. It follows that $\alpha_j = \beta_j$ for all $1 \leq j \leq n$. Therefore

$$\begin{aligned} &\min\left\{N_1(\alpha_j x_j, s), N_2(\alpha_j y_j, s) \,\middle|\, 1 \leq j \leq n\right\} \\ &= \min\left\{N_1(\beta_j x_j, t), N_2(\beta_j y_j, t) \,\middle|\, 1 \leq j \leq n\right\}, \end{aligned}$$

providing $N(z, s) = N(w, t)$.
In the sequel we will prove the parts 2, 3, 4 and 5 of Definition 1.1.
(2) : Let $z = \sum_{j=1}^n \alpha_j x_j \otimes y_j$ and $N(z, t) = 1$ for all $t > 0$. So

$$\min\left\{N_1(\alpha_j x_j, t), N_2(\alpha_j y_j, t) \,\middle|\, 1 \leq j \leq n\right\} = 1$$

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for all $t > 0$. It follows that $N_1(\alpha_j x_j, t) = N_2(\alpha_j y_j, t) = 1$ for all $1 \le j \le n$ and for all $t > 0$. Hence $\alpha_j x_j = 0$ and $\alpha_j y_j = 0$ for all $1 \le j \le n$. Therefore $\alpha_j = 0$ for all $1 \le j \le n$. This shows that $z = 0 \otimes 0$. Also for all $t > 0$, since $0 \otimes 0 = 0x \otimes y$ for all $x \in B_X$ and $y \in B_Y$,

$$\begin{aligned}
&N(0 \otimes 0, t) \\
&= N(0x \otimes y, t) \\
&= \min\{N_1(0, t), N_2(0, t)\} \\
&= \min\{1\} \\
&= 1.
\end{aligned}$$

(3) : Let $c \neq 0$ and $\sum_{j=1}^{n} \alpha_j x_j \otimes y_j \in X \otimes Y$. So

$$\begin{aligned}
&N\left(c \sum_{j=1}^{n} \alpha_j x_j \otimes y_j, t\right) \\
&= N\left(\sum_{j=1}^{n} c\alpha_j x_j \otimes y_j, t\right) \\
&= \min\left\{N_1(c\alpha_j x_j, t), N_2(c\alpha_j y_j, t) \,\middle|\, 1 \le j \le n\right\} \\
&= \min\left\{N_1\left(\alpha_j x_j, \frac{t}{|c|}\right), N_2\left(\alpha_j y_j, \frac{t}{|c|}\right) \,\middle|\, 1 \le j \le n\right\} \\
&= N\left(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j, \frac{t}{|c|}\right),
\end{aligned}$$

for all $t \in \mathbb{R}$.

(4) : Let $z, w \in X \otimes Y$ and $s, t \in \mathbb{R}$. So $z = \sum_{j=1}^{n} \alpha_j x_j \otimes y_j$ and $w = \sum_{j=1}^{n} \beta_j x_j \otimes y_j$, where $n \in \mathbb{N}, \alpha_j, \beta_j \in \mathbb{C}$, $\{x_j\}_{j=1}^{n} \subseteq B_X$ and $\{y_j\}_{j=1}^{n} \subseteq B_Y$. Hence

$$\begin{aligned}
&N\left(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j + \sum_{j=1}^{n} \beta_j x_j \otimes y_j, s + t\right) \\
&= N\left(\sum_{j=1}^{n} (\alpha_j + \beta_j) x_j \otimes y_j, s + t\right) \\
&= \min\left\{N_1((\alpha_j + \beta_j) x_j, s + t), N_2((\alpha_j + \beta_j) y_j, s + t) \,\middle|\, 1 \le j \le n\right\} \\
&= \min\left\{N_1(\alpha_j x_j + \beta_j x_j, s + t), N_2(\alpha_j y_j + \beta_j y_j, s + t) \,\middle|\, 1 \le j \le n\right\} \\
&\ge \min\left\{\min(N_1(\alpha_j x_j, s), N_1(\beta_j x_j, t)), \min(N_2(\alpha_j y_j, s), N_2(\beta_j y_j, t)) \,\middle|\, 1 \le j \le n\right\} \\
&= \min\left\{\min(A_j, B_j), \min(C_j, D_j) \,\middle|\, 1 \le j \le n\right\} \\
&= \min\left\{\min\left\{A_j, C_j \,\middle|\, 1 \le j \le n\right\}, \min\left\{B_j, D_j \,\middle|\, 1 \le j \le n\right\}\right\} \\
&= \min\left\{N\left(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j, s\right), N\left(\sum_{j=1}^{n} \beta_j x_j \otimes y_j, t\right)\right\},
\end{aligned}$$

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where

$$A_j = N_1(\alpha_j x_j, s),$$
$$B_j = N_1(\beta_j x_j, t),$$
$$C_j = N_2(\alpha_j y_j, s),$$
$$D_j = N_2(\beta_j y_j, t).$$

(5) : Let $s \leq t$ and $z = \sum_{j=1}^{n} \alpha_j x_j \otimes y_j$. Clearly $N_1(\alpha_j x_j, s) \leq N_1(\alpha_j x_j, t)$ and $N_2(\alpha_j y_j, s) \leq N_2(\alpha_j y_j, t)$ for all $1 \leq j \leq n$. So

$$\min\left\{N_1(\alpha_j x_j, s), N_2(\alpha_j y_j, s) \,\middle|\, 1 \leq j \leq n\right\}$$
$$\leq \min\left\{N_1(\alpha_j x_j, t), N_2(\alpha_j y_j, t) \,\middle|\, 1 \leq j \leq n\right\}.$$

It follows that

$$N\left(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j, s\right) \leq N\left(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j, t\right).$$

Hence $N(z, .) : \mathbb{R} \longrightarrow [0, 1]$ is increasing for all $z \in X \otimes Y$. Also

$$\begin{aligned}
&\lim_{t \longrightarrow \infty} N(z, t) \\
&= \lim_{t \longrightarrow \infty} \min\left\{N_1(\alpha_j x_j, t), N_2(\alpha_j y_j, t) \,\middle|\, 1 \leq j \leq n\right\} \\
&= \min\left\{\lim_{t \longrightarrow \infty} N_1(\alpha_j x_j, t), \lim_{t \longrightarrow \infty} N_2(\alpha_j y_j, t) \,\middle|\, 1 \leq j \leq n\right\} \\
&= \min\{1\} \\
&= 1.
\end{aligned}$$

□

Theorem 2.6. Let X and Y be linear spaces and $N : (X \otimes Y) \times \mathbb{R} \longrightarrow [0, 1]$ be a fuzzy norm. Then for all $x \in X \setminus \{0\}$ and $y \in Y \setminus \{0\}$, the maps $N_x : Y \times \mathbb{R} \longrightarrow [0, 1]$ and $N_y : X \times \mathbb{R} \longrightarrow [0, 1]$, where $N_x(z, t) = N(x \otimes z, t)$ and $N_y(w, t) = N(w \otimes y, t)$, are fuzzy norms on Y and X respectively.

Proof. We only prove that $N_x : Y \times \mathbb{R} \longrightarrow [0, 1]$ is a fuzzy norm for all $x \in X \setminus \{0\}$.
(1) : Let $z \in Y$ and $t \leq 0$. So $N_x(z, t) = N(x \otimes z, t) = 0$.
(2) : If $z = 0$, then $N_x(0, t) = N(x \otimes 0, t) = N(0 \otimes 0, t) = 1$ for all $t > 0$. Also if $N_x(z, t) = 1$ for all $t > 0$, then $N(x \otimes z, t) = 1$ for all $t > 0$. It follows that $x \otimes z = 0 \otimes 0$. Since $x \neq 0$, there exists $f \in X'$ such that $f(x) \neq 0$. Let $g \in Y'$ be an arbitrary element. As $x \otimes z$ is a bilinear map on $X' \times Y'$, $(x \otimes z)(f, g) = (0 \otimes 0)(f, g)$. So $f(x) g(z) = 0$ for all $g \in Y'$. Since $f(x) \neq 0$, $g(z) = 0$ for all $g \in Y'$.

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It follows that $z = 0$.
(3) : Let $c \neq 0$ and $z \in Y$. So

$$\begin{aligned} & N_x(cz, t) \\ = & N(x \otimes cz, t) \\ = & N(cx \otimes z, t) \\ = & N\left(x \otimes z, \frac{t}{|c|}\right) \\ = & N_x\left(z, \frac{t}{|c|}\right) \end{aligned}$$

for all $z \in Y$ and $t \in \mathbb{R}$.
(4) : Let $z_1, z_2 \in Y$ and $s, t \in \mathbb{R}$. So

$$\begin{aligned} & N_x(z_1 + z_2, s + t) \\ = & N(x \otimes (z_1 + z_2), s + t) \\ = & N(x \otimes z_1 + x \otimes z_2, s + t) \\ \geq & \min(N(x \otimes z_1, s), N(x \otimes z_2, t)) \\ = & \min(N_x(z_1, s), N_x(z_2, t)). \end{aligned}$$

(5) : Let $s \leq t$ and $z \in Y$. So $N(x \otimes z, s) \leq N(x \otimes z, t)$. It follows that $N_x(z, s) \leq N_x(z, t)$. Hence $N_x(z, .) : \mathbb{R} \longrightarrow [0, 1]$ is increasing and

$$\begin{aligned} & \lim_{t \longrightarrow \infty} N_x(z, t) \\ = & \lim_{t \longrightarrow \infty} N(x \otimes z, t) \\ = & 1 \end{aligned}$$

for all $z \in Y$.
Therefore $N_x : Y \times \mathbb{R} \longrightarrow [0, 1]$ is a fuzzy norm. □

Example 2.7. Let $X = Y = \mathbb{R}^2$ be linear spaces over $\mathbb{R}$ with the bases $B_X = B_Y = \{e_1 = (1, 0), e_2 = (0, 1)\}$. Also let $N_1 = N_2 : \mathbb{R}^2 \times \mathbb{R} \longrightarrow [0, 1]$ are defined by

$$N_i\left(\sum_{j=1}^{2} \alpha_j e_j, t\right) = \begin{cases} 0 & t \leq \max(|\alpha_1|, |\alpha_2|) \\ 1 & t > \max(|\alpha_1|, |\alpha_2|) \end{cases}$$

for $i = 1, 2$. Clearly N_1 and N_2 are fuzzy norms on $\mathbb{R}^2$. According to Theorem 2.5, if $N : (X \otimes Y) \times \mathbb{R} \longrightarrow [0, 1]$ is defined by

$$\begin{aligned} & N(\alpha_1 e_1 \otimes e_1 + \alpha_2 e_1 \otimes e_2 + \alpha_3 e_2 \otimes e_1 + \alpha_4 e_2 \otimes e_2, t) \\ & = \min\begin{pmatrix} N_1(\alpha_1 e_1, t), & N_1(\alpha_2 e_1, t), & N_1(\alpha_3 e_2, t), & N_1(\alpha_4 e_2, t) \\ N_2(\alpha_1 e_1, t), & N_2(\alpha_2 e_2, t), & N_2(\alpha_3 e_1, t), & N_2(\alpha_4 e_2, t) \end{pmatrix}, \end{aligned}$$

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then the equalities

$$\begin{aligned} N_2\left(\alpha_1 e_1, t\right) &= N_1\left(\alpha_1 e_1, t\right), \\ N_2\left(\alpha_2 e_2, t\right) &= N_1\left(\alpha_2 e_1, t\right), \\ N_2\left(\alpha_3 e_1, t\right) &= N_1\left(\alpha_3 e_2, t\right), \\ N_2\left(\alpha_4 e_2, t\right) &= N_1\left(\alpha_4 e_2, t\right) \end{aligned}$$

imply that

$$\begin{aligned} & N\left(\alpha_1 e_1 \otimes e_1 + \alpha_2 e_1 \otimes e_2 + \alpha_3 e_2 \otimes e_1 + \alpha_4 e_2 \otimes e_2\right) \\ & = \min\left\{N_1\left(\alpha_1 e_1, t\right), N_1\left(\alpha_2 e_1, t\right), N_1\left(\alpha_3 e_2, t\right), N_1\left(\alpha_4 e_2, t\right)\right\} \\ & = \begin{cases} 0 & t \leq \max\left(\left|\alpha_i\right|, 1 \leq i \leq 4\right) \\ 1 & t > \max\left(\left|\alpha_i\right|, 1 \leq i \leq 4\right). \end{cases} \end{aligned}$$

Let $x_0 = -\frac{1}{5}e_1, y_0 = -2e_2, K_1 = B_X, K_2 = B_Y, K = B_{X\otimes Y} = B_X \otimes B_Y$ and $\rho = \frac{1}{10}$. Also let $P_{K_1}^{fn\left(\frac{1}{10}\right)}\left(x_0\right)$ and $P_{K_2}^{fn\left(\frac{1}{10}\right)}\left(y_0\right)$ be the set of all fuzzy near best approximations to x_0 and y_0 within the relative distance $\rho = \frac{1}{10}$ with respect to the fuzzy norms N_1 and N_2 respectively. If $P_{K}^{fn\left(\frac{1}{10}\right)}\left(x_0 \otimes y_0\right)$ be the set of all fuzzy near best approximations to $x_0 \otimes y_0$ within the relative distance $\rho = \frac{1}{10}$ with respect to the fuzzy norm N, then a straightforward calculation reveals that

$$\begin{aligned} P_{K_1}^{fn\left(\frac{1}{10}\right)}\left(x_0\right) &= \left\{e_2\right\} \\ P_{K_2}^{fn\left(\frac{1}{10}\right)}\left(y_0\right) &= \left\{e_1\right\} \\ P_{K}^{fn\left(\frac{1}{10}\right)}\left(x_0 \otimes y_0\right) &= \left\{e_1 \otimes e_2\right\} \neq \left\{e_2 \otimes e_1\right\}. \end{aligned}$$

This example shows that there is no a relation between $P_{K}^{fn\left(\frac{1}{10}\right)}\left(x_0 \otimes y_0\right)$ and $P_{K_1}^{fn\left(\frac{1}{10}\right)}\left(x_0\right) \otimes P_{K_2}^{fn\left(\frac{1}{10}\right)}\left(y_0\right)$.

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