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Article

Topological Gauge Theory of Josephson Junction Arrays: The Discovery of Superinsulation

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Abstract: We review the topological gauge theory description of Josephson junction arrays (JJA), fabricated systems which exhibit the superconductor to insulator transition (SIT). This description revealed the topological nature of the phases appearing at the SIT and lead to the discovery of a new state of matter, the superinsulator, characterized by an infinite resistance, even at finite temperature, due to linear confinement of electric charges. This discovery is particularly relevant for the physics of superconducting films with emergent granularity, which are modeled by JJA, and share the same phase diagram.

Keywords: Josephson junction arrays; topological field theory; topological phases of matter; superinsulators

1. Introduction

Josephson junction arrays (JJA) [1] are fabricated materials which exhibit the superconductor to insulator transition (SIT) [2,3], a paradigmatic example of a quantum phase transition. Planar quadratic JJA are square lattices formed by superconducting island (typical size O(μm)), fabricated on a substrate, with lattice spacing l (O(100nm)), Josephson coupling E_J and capacitances C between nearest neighbour islands. Each islands is also characterized by the phase of the local order parameter, and by a ground capacitance C_0. When $C \gg C_0$, JJA have only two relevant energy scales, E_J, the energy scale associated with the tunneling of Cooper pairs between the islands, and $E_C = e^2/2C$, the charging energy, with e the electron charge. The parameters E_J and E_C can be traded for a massive parameter $\omega_P = \sqrt{2E_C E_J}$, which represents the plasma frequency of the array, and a dimensionless parameter $g = \sqrt{\frac{\pi^2 E_J}{2E_C}}$, which, as we will see, plays a crucial role in the phase structure of the theory. In experimental realizations of JJA, the transition between the different phases is achieved by varying E_J and taking E_C essentially fixed. Varying E_C is much more difficult.

In the classical limit, $E_J \gg E_C$, global phase coherence is realized and the array behaves as a superconductor. Above a critical temperature the arrays undergoes a Bere- zinskii-Kosterlitz-Thouless (BKT) transition [4] and superconductivity is destroyed. When, instead, $E_c \gg E_J$, the tunneling between the islands is suppressed and the arrays shows an insulating behaviour. As it has been shown in [5,6], however, this is not an insutaltors, but a new state of matter called superinsulator.

Superinsulators are emergent condensed matter states dual to superconductors, and exhibit an infinite resistance at finite temperatures. They were first theoretically predicted in [5], while the final form of the theory describing this new state of matter was established in [7]. Superinsulators have, than, been experimentally found in [8–14] in thin superconducting films which, close to the SIT, have an emergent granularity and behave as a self organized JJA.

The infinite resistance which characterize superinsulators is due to linear charge confinement [7] of both Cooper pairs and electrons in a magnetic monopole plasma, which squeezes electric field lines into electric flux tubes connecting charge-anticharge pairs, in analogy with the Meissner effects in superconductors [13]. Superinsulators realize, thus, an Abelian version of the dual superconductivity mechanism advocated by 't Hooft to explain quark confinement [15]. In this picture chromo-electric

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strings will form with quarks at their ends and, when quarks are pulled apart, it is energetically favourable to pull out of the vacuum additional quark-antiquark pairs and to form several short strings. Free colour charge can never be observed on distances larger than $1/\Lambda_{QCD}$ and quarks are, thus, confined.

In [5] we derived a topological gauge theory description of JJA. For planar JJA the relevant term, dominant at large distances is the mixed Chern-Simons term [16], which is nothing else than the (2+1)-dimensional version of the BF term [17], which is the relevant term at long distances in higher dimensions. This description has made it possible to derive the quantum phase diagram of JJA and to understand the nature of the various phases. This result is particularly important since, as we pointed out before, JJA are a model for thin superconducting films near the SIT, and the gauge theory which describes JJA is also an effective theory for these materials at the SIT.

We have identified three possible phases: when a parameter η, which depends on the details of the array, is $\eta < 1$, there is a direct transition between the superinsulating phase and the superconducting phase. The superconducting phase is a new type of superconductivity, which we called type III superconductivity [18], characterized by the fact that the gap is opened trough a topological mechanism and not trough the Higgs menhanism and is not described by the standard Landau-Ginzurg theory. Superconductivity is destroyed by proliferation of vortices, implying a BKT transition in two dimensions and a Vogel-Fulcher-Tamman transition in three dimensions, and not by the breaking of Cooper pairs [18,19]. This superconductivity model can be relevant for high-T_C superconductivity [20]. When $\eta > 1$ an intermediate phase opens up between the superinsulating and the superconducting phase [21,22]. This metallic phase in two spatial dimensions, called Bose metal (BM) since charge carriers are Cooper pairs, arising at the SIT and experimentally found in superconducting films [23–29], has for long time challenged the understanding of electronic fluids [30,31]. In fact it is believed that a metallic phase in two-dimensions cannot exist due to localization. Moreover it was not clear why certain films had this phase and others not. In [21] we have been able to identify this phase as a bosonic topological insulators and to establish, trough the parameter η, which depends on the film characteristics, when this phase appears.

In section 2 we will review the topological gauge theory formulation of JJA. We will, in particular, discuss the role of the kinetic term for vortices in the description of JJA. In section 3 we will derive the quantum phase strucure of JJA and, in section 4, we will describe the properties of the new phase of matter, theoretically predicted thanks to the topological gauge theory description of JJA, the superinsulator.

2. Topological gauge theory of JJA

We will consider JJA defined on a square lattice, so we start by defining our lattice notation. We will consider a 3-dimensional Euclidean lattice with sites denoted by $\{x\}$ and directions, indicated by Greek letters, going from 0 to 2. The lattice spacing is l_0 in the 0 direction (time) and l in the 1 and 2 directions (space).The forward and backward finite difference and shift operators are defined as (we denoted the lattice spacing as l_μ only to distinguish between the 0 and 1 and 2 directions)

$$\begin{aligned} \Delta_\mu f(x) &= f(x + l_\mu \hat{\mu}) - f(x) \ , & S_\mu f(x) &= f(x + l_\mu \hat{\mu}) \ , \\ \hat{\Delta}_\mu f(x) &= f(x) - f(x - l_\mu \hat{\mu}) \ , & \hat{S}_\mu f(x) &= f(x - l_\mu \hat{\mu}) \ , \end{aligned} \tag{1}$$

where $\hat{\mu}$ denotes a unit vector in direction μ. Summation by parts on the lattice interchanges both the two finite differences (with a minus sign) and the two shift operators. Using these definition we introduce the operators:

$$K_{\mu\nu} = S_\mu \epsilon_{\mu\alpha\nu} \Delta_\alpha \ , \qquad \hat{K}_{\mu\nu} = \epsilon_{\mu\alpha\nu} \hat{\Delta}_\alpha \hat{S}_\nu \ , \tag{2}$$

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where no summation is implied over the equal indices μ and ν. These two operators are interchanged by summation by parts with no minus sign. They allow us to define a gauge invariant version of the Chern-Simons operator $\epsilon_{\mu\alpha\nu}\partial_\alpha$ on the lattice [5]. Their products gives the lattice Maxwell operator:

$$K_{\mu\alpha}\hat{K}_{\alpha\nu} = \hat{K}_{\mu\alpha}K_{\alpha\nu} = -\delta_{\mu\nu}\Delta + \Delta_\mu\hat{\Delta}_\nu \,, \tag{3}$$

where $\Delta = \hat{\Delta}_\mu\Delta_\mu$ is the 3D Laplace operator.

Our starting point is the Hamiltonian for a planar JJA with nearest neighbours Josephson couplings E_J, ground capacitances C_0 and nearest neighbours capacitances C [1,32]:

$$\begin{aligned} H &= \sum_{\mathbf{x}} \frac{C_0}{2} V_{\mathbf{x}}^2 + \sum_{<\mathbf{xy}>} \frac{C}{2} \left(V_{\mathbf{y}} - V_{\mathbf{x}}\right)^2 + E_J \left(1 - \cos\ \left(\varphi_{\mathbf{y}} - \varphi_{\mathbf{x}}\right)\right) \,, \\ H &= \sum_{\mathbf{x}} \frac{1}{2} V_{\mathbf{x}} \left(C_0 - C\Delta\right) V_{\mathbf{x}} + \sum_{\mathbf{x},i} E_J \left(1 - \cos\ \left(\Delta_i \varphi_{\mathbf{x}}\right)\right) \,, \end{aligned} \tag{4}$$

where $< \mathbf{xy} >$ indicates nearest neighbours and $\Delta \equiv \hat{\Delta}_i\Delta_i$ is the two-dimensional finite difference Laplacian. In what follows we will use natural units $c = 1, \hbar = 1, \varepsilon_0 = 1$.

Each island is characterized by an electric potential $V_{\mathbf{x}}$ and by the phase of the order parameter $\varphi_{\mathbf{x}}$, which are quantum-mechanically conjugated to the charges $\mathcal{E}_{\mathbf{x}}$ on the islands. The charges are quantized in integer multiples of 2e (Cooper pairs), $\mathcal{E}_{\mathbf{x}} = 2e q_{\mathbf{x}}$, $q_{\mathbf{x}} \in Z$.

Using the discrete version of Poisson's equation:

$$\left(C_0 - C\Delta\right) V_{\mathbf{x}} = \mathcal{E}_{\mathbf{x}} \,, \tag{5}$$

and introducing the charging energy $E_C \equiv e^2/2C$, we can rewrite the Hamiltonian eq. (4) as:

$$H = \sum_{\mathbf{x}} 4E_C\, q_{\mathbf{x}} \frac{1}{C_0/C - \Delta} q_{\mathbf{x}} + \sum_{\mathbf{x},i} E_J \left(1 - \cos\ \left(\Delta_i\varphi_{\mathbf{x}}\right)\right) \,. \tag{6}$$

The integer charges $q_{\mathbf{x}}$ in eq. (6) interact via a two-dimensional Yukawa potential with mass $\sqrt{C_0/C}/\ell$ which, in the limit $C \gg C_0$, becomes a 2-dimensional Coulomb potential.

The next step in our derivation of the gauge theory description is the construction of the phase-space path-integral representation [5] of the JJA. To this end we introduce a fictitious temperature $\beta = 1/T$ and write the partition function of the JJA as:

$$\begin{aligned} Z &= \sum_{\{q\}} \int_{-\pi}^{+\pi} \mathcal{D}\varphi \, \exp(-S) \,, \\ S &= \int_0^\beta dt \sum_{\mathbf{x}} i\, q_{\mathbf{x}}\, \dot{\varphi}_{\mathbf{x}} + 4E_C\, q_{\mathbf{x}} \frac{1}{C_0/C - \Delta} q_{\mathbf{x}} + \sum_{\mathbf{x},i} E_J \left(1 - \cos\ \left(\Delta_i\varphi_{\mathbf{x}}\right)\right) \,, \end{aligned} \tag{7}$$

which, in the limit $C_0 \gg C$, experimentally accessible, becomes

$$\begin{aligned} Z &= \sum_{\{q\}} \int_{-\pi}^{+\pi} \mathcal{D}\varphi \, \exp(-S) \,, \\ S &= \int_0^\beta dt \sum_{\mathbf{x}} i\, q_{\mathbf{x}}\, \dot{\varphi}_{\mathbf{x}} + 4E_C\, q_{\mathbf{x}} \frac{1}{-\Delta} q_{\mathbf{x}} \\ &\quad + \sum_{\mathbf{x},i} E_J \left(1 - \cos\ \left(\Delta_i\varphi_{\mathbf{x}}\right)\right) \,. \end{aligned} \tag{8}$$

Continuous time must be treated for a discrete time with interval steps l_0, since our degrees of freedom changes only by integer steps, introducing, thus, also a forward and backward finite time differences Δ_0 and $\hat{\Delta}_0$. Using the Villain representation [33], we can get rid of the cosine at the price of introducing a set of integer link variables a_i. By introducing real charge currents j_i, we can express the quadratic term $(\Delta_i\varphi + 2\pi a_i)^2$, which comes from the Villain approximation, as a Gaussian integrals over these variables, thereby obtaining:

$$
\begin{aligned}
Z &= \sum_{\{a_i\},\{j_0\}} \int \mathcal{D}j_i \int_{-\pi}^{+\pi} \mathcal{D}\varphi \, \exp(-S) \, , \\
S &= \sum_x i\, j_0 \Delta_0 \varphi + i j_i \left(\Delta_i \varphi + 2\pi a_i\right) + 4\ell_0 E_C \, j_0 \frac{1}{-\Delta} j_0 + \frac{1}{2\ell_0 E_J} {j_i}^2 \, ,
\end{aligned}
\tag{9}
$$

where we renamed j_0 the integer charges $q_{\mathbf{x}}$.

The longitudinal part of a_i is not physical, and can be reabsorbed in a redefinition of φ. As we will see, the transverse degrees of freedom encode the vortex degrees of freedom. In fact, it is important to notice that eq. (9) contains a kinetic term only for the charges, but not for vortices. While for overdamped junction this omission has no consequences, in the general case, this is not correct. As shown in [32], the kinetic term for vortices is generated by integration over charge fluctuations and must, thus, be included at tree level. Moreover in our case the Coulomb interaction is long-range, and the dissipation is reduced [34], making the vortex kinetic term relevant. The kinetic term for the vortices represents tunneling events between adjacent plaquettes of the lattice. These events are the generalization of quantum phase slips, which plays a crucial role in Josephson chain [35], to two dimensions. They can be seen as half-lines of simultaneous phase slips of opposite chirality, which end in the island between the two adjacent plaquettes. Ballistic vortex motion, represented by phase slips corresponding to one vortex tunneling from one plaquette to the other, has actually been exeperimentally observed in [36]. The kinetic term for vortices will involve the time derivative of a_i, $(\Delta_0 a_i)^2$. Let us start by introducing a fictitious electric field $\varphi_i = K_{i\mu} a_\mu$ and a real Lagrange multiplier a_0. Taking the coefficient of the kinetic term as $\pi^2/4\ell_0 E_C$, we write the vortex kinetic term and the charge Coulomb interaction as:

$$
\begin{aligned}
Z &= \sum_{\{a_i\},\{j_0\}} \int \mathcal{D}a_0 \mathcal{D}j_i \int_{-\pi}^{+\pi} \mathcal{D}\varphi \, \exp(-S) \, , \\
S &= \sum_x i\, j_0 \left(\Delta_0 \varphi + 2\pi a_0\right) + i j_i \left(\Delta_i \varphi + 2\pi a_i\right) + \frac{1}{2\ell_0 E_J} {j_i}^2 + \frac{\pi^2}{4\ell_0 E_C} \varphi_i^2 \, ,
\end{aligned}
\tag{10}
$$

where now the Coulomb interaction between the charges follows from the Gauss law constraint associated with the Lagrange multiplier a_0. In this case we have chosen a particular value of the vortex mass for which the JJA is dual under interchanges of charges and vortices and of $\pi^2 E_J \leftrightarrow 2E_C$, or alternatively, $g \leftrightarrow 1/g$. This is the self-dual approximation introduced in [5]. It is important to notice, however, that a different vortex mass will, in general, renormalize the value of E_C, and as consequence change g, the parameter which, as we will see drive the quantum phase transition. A bigger or smaller vortex mobility will, thus, influence the possible phases which are observed.

The next step is to notice that the integration over φ in eq. (10) gives the constraint $\Delta_\mu j_\mu = 0$, the current j_μ is, thus, conserved and can be represented in terms of a fictitious gauge field b_μ

$$
j_0 = K_{0i} b_i \;\; , \;\; j_i = K_{i0} b_0 + K_{ij} b_j , \tag{11}
$$

In eq. (11) b_0 is a real variable, and b_i and a_i are integers, which can be made real using Poisson's formula:

$$
\sum_{n_\mu} f\left(n_\mu\right) = \sum_{k_\mu} \int dn_\mu f\left(n_\mu\right) \mathrm{e}^{i2\pi n_\mu k_\mu} \, , \tag{12}
$$

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so that all components of the gauge fields a_μ and b_μ are real, at the price of introducing integer link variables Q_i and M_i,

$$Z = \sum_{\{Q_i\}} \sum_{\{M_i\}} \int \mathcal{D}a_\mu \mathcal{D}b_\mu \int_{-\pi}^{+\pi} \mathcal{D}\varphi \exp(-S) \, ,$$

$$S = \sum_x i2\pi \, a_\mu K_{\mu\nu} b_\nu + \frac{1}{2\ell_0 E_J} {j_i}^2 + \frac{\pi^2}{4\ell_0 E_C} \phi_i^2 + i2\pi a_i Q_i + i2\pi b_i M_i$$
$$+ b_i \left(\hat{K}_{i0} \Delta_0 \varphi + \hat{K}_{ij} \Delta_j \varphi \right) + b_0 \hat{K}_{0i} \Delta_i \varphi \, . \quad (13)$$

The last step of our derivation consist in reabsorbing the quantity $\left(\hat{K}_{i0} \Delta_0 \varphi + \hat{K}_{ij} \Delta_j \varphi \right)$ in a redefinition of the integers M_i, and in introducing a set of integers M_0 trough the definition $\hat{K}_{0i} \Delta_i \varphi = 2\pi M_0$. This step is justified by noticing that $\hat{K}_{\mu\nu} \Delta_\nu \varphi$ are the circulations of the array phases around the plaquettes orthogonal to the direction μ in 3D Euclidean space-time and are thus quantized as 2π integers. At this point, using again Poisson's formula, we can express the integral over φ as a sum over the integer M_0, giving the final expression for the topological gauge theory describing JJA:

$$Z = \sum_{\{Q_i\}} \sum_{\{M_\mu\}} \int \mathcal{D}a_\mu \mathcal{D}b_\mu \exp(-S) \, ,$$

$$S = \sum_x i2\pi \, a_\mu K_{\mu\nu} b_\nu + \frac{1}{2\ell_0 E_J} {j_i}^2 + \frac{\pi^2}{4\ell_0 E_C} \phi_i^2 + i2\pi a_i Q_i + i2\pi b_\mu M_\mu \, . \quad (14)$$

The first term in the action is the lattice version of the mixed Chern-Simons term, which, being linear in derivatives, is dominant at large distances. The second and third terms are the electric fields for the fictitious gauge fields a_μ and b_μ respectively. The effect of the CS term is to give a mass to these gauge fields without the Higgs mechanism [16]. The mass that these fields acquires is nothing else than the plasma frequency of the array: $m_{top} = \omega_p = \sqrt{8 E_C E_J}$. The two kinetic terms have a coupling constant with dimensions of mass and are thus naively irrelevant. They cannot, however, be neglected since the correct topological limit $m_{top} \to \infty$ has to be derived from the full theory including the kinetic terms in order to describe physical systems, otherwise the states will not be normalizable. The topological limit $E_J \to \infty$, $E_C \to \infty$ is not well defined without specifying the value of their ratio g in this limit, and, as we will show, the phase diagram depends crucially on this.

The dual field strengths of the fictitious gauge fields, $j_\mu = K_{\mu\nu} b_\nu$ and $\varphi_\mu = K_{\mu\nu} a_\nu$, represent charge and vortex fluctuations, respectively. The integers fields Q_i and M_i are the electric and magnetic topological excitations respectively. Together with the vortex number M_0, the latter form a 3-current M_μ which is conserved due to gauge invariance in the b_μ gauge sector, $\hat{\Delta}_\mu M_\mu = 0$. Perfect duality is broken by the absence of the integer variable Q_0. As we will see, for the M_μ variables, the quantity $\Delta_0 M_0 = \pm 1$ corresponds to vortices that appear and disappears on the arrays, possible because topological charges are not conserved. Noether charges coming from symmetry are, instead, conserved and we do not have the variable Q_0. Contrary to charges, vortices are, thus, topological excitations, characterized by a topological quantum number. The configuration space of the theory of vortices decomposes into so-called superselection sectors, characterized by the integer total vortex number, which are connected via instantons, non-perturbative configurations representing quantum tunneling events between topological vacua [37]. As a consequence, charges are conserved but vortices are not and can "appear" and"disappear" via quantum tunneling events forming the instantons.

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By rescaling the emergent gauge fields, expressed in their canonical dimensions, by $1/2\pi$, and using lattice derivatives instead of finite differences, we can rewrite the action in eq. (14) as

$$S = \sum_x i\frac{l_0 l^2}{2\pi} a_\mu K_{\mu\nu} b_\nu + \frac{l_0 l^2}{8\pi^2 E_J} j_i{}^2 + \frac{l_0 l^2}{16 E_C}\phi_i^2 + i l a_i Q_i + i l_0 b_0 M_0 + l b_i M_i \ . \tag{15}$$

This equation is the limit for $\epsilon = 1, \mu \to \infty$ and $v = 1/\sqrt{\epsilon\mu} = 1/\sqrt{\mu} \ll 1$ of the action:

$$\begin{aligned} S = \sum_x \quad & i\frac{\ell^2 l_0}{2\pi} a_\mu k_{\mu\nu} b_\nu + \frac{\ell^2 l_0 v^2}{8\pi^2 E_J} j_0 j_0 + \frac{l_0 l^2}{8\pi^2 E_J} j_i{}^2 + \\ & \frac{\ell^2 l_0 v^2}{16 E_C}\varphi_0\varphi_0 + + \frac{l_0 l^2}{16\ E_C}\phi_i^2 + i l a_i Q_i + i l_0 b_0 M_0 + l b_i M_i \ . \end{aligned} \tag{16}$$

We want now to compute the induced action for the Q_i and M_μ obtained by integrating over the emergent gauge fields. To this end we will add the term $i l_0 a_0 Q_0$, and the sum over Q_0 in the partition function. This term will make the action completely self-dual with respect to electric and magnetic degrees of freedom. This will allow us to rewrite eq. (16) as:

$$\begin{aligned} S = \sum_x \quad & i\frac{\ell^2 l_0 v}{2\pi} a_\mu k_{\mu\nu} b_\nu + \frac{\ell^2 l_0 v}{8\pi^2 E_J} j_\mu j_\mu + \frac{\ell^2 l_0 v}{16 E_C}\varphi_\mu\varphi_\mu + \\ & + i l a_\mu Q_\mu + i l b_\mu M_\mu \ , \end{aligned} \tag{17}$$

where we rescaled $l_0 = l/v$, $d_0 \to d_0$, $j_i(\varphi_i) \to \frac{1}{v} j_i(\varphi_i)$. In eq. (17) gauge invariance imposes the constraint $\hat{d}_\mu Q_\mu = \hat{d}_\mu M_\mu = 0$, but, since $\hat{d}_i Q_i = 0$ we also have $\hat{d}_0 Q_0$, implying that there are no "electric instantons". To recover the exact result for JJA we should put $Q_0 = 0$ at the end of the calculation. Making the theory perfectly self-dual, however, will not change the nature of the possible phases, we thus use this approximation.

Integrating out the emergent gauge fields we get [5,6]:

$$\begin{aligned} S_{\rm TOP} \quad & = \sum_x v\frac{8E_C}{\ell} Q_\mu \frac{1}{v^4 m^2 - d_0\hat{d}_0 - v^2\nabla_2^2} Q_\mu + \\ & + v\frac{4\pi^2 E_J}{\ell} M_\mu \frac{1}{v^4 \tilde{m}^2 - d_0\hat{d}_0 - v^2\nabla_2^2} M_\mu + \\ & + i\frac{2\pi v^6 m^2}{\ell} Q_\mu \frac{K_{\mu\nu}}{(d_0\hat{d}_0 + v^2\nabla_2^2)(v^4 m^2 - d_0\hat{d}_0 - v^2\nabla_2^2)} M_\nu \ . \end{aligned} \tag{18}$$

The last imaginary term in the action is a lattice version of the topological linking of electric and magnetic strings, which, due to the Dirac condition, in the limit $vm \to \infty$ becomes an integer. This term is an integer also if we set $Q_0 = 0$. We can, thus, drop it [5].

3. Quantum Phase Structure

The $T = 0$ quantum phase structure of JJA is determined the behaviour of the integer fields Q_μ and M_μ. These can be decomposed into a transverse component, $\hat{d}_\mu Q^T_\mu = 0$, $\hat{d}_\mu M^T_\mu = 0$, representing closed electric and magnetic loops or infinitely long strings, and a longitudinal component, $\hat{k}_{\mu\nu} Q^L_\nu = 0$, $\hat{k}_{\mu\nu} M^L_\nu = 0$ representing open electric and magnetic strings ending on electric and magnetic monopoles $q = \hat{d}_\mu Q^L_\mu$ and $m = \hat{d}_\mu M^L_\mu$. Notice, however, that, in eq. (15) only the integers Q_i appear implying that $\hat{d}_i Q_i = 0$.

To determine when infinitely long electric or magnetic strings proliferate, one has to look at the energy-entropy balance determined by the parameters of the model. Let us start from eq. (18), where we ignore the imaginary term, as explained. Near the transition, we expect to have very long strings and large loops. These configuration are, in general, very random and we expect that forces between

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links in the same loop and in other ones cancel out. We, thus, retain only the self-interaction terms in eq. (18) [38]. We assign to a closed string made of N links and integer quantum numbers Q and M_M on all the lattice links forming the string and zero elsewhere an energy (equivalent to Euclidean action in statistical field theory)

$$S_{\rm top} = \pi m\ell G(m\ell) \left[\frac{2E_C}{\pi^2 E_J} Q^2 + \frac{\pi^2 E_J}{2E_C} M^2 \right] N \,, \tag{19}$$

where $G(m\ell)$ is the diagonal element of the lattice kernel $G(m\ell, x-y)$ representing the inverse of the operator $\ell^2(m^2 - \nabla^2)$. The kernel $G(m\ell, x-y)$ is defined by the equation

$$\ell^2 \left(m^2 - \nabla^2 \right) G(ml, x-y) = \delta_{x-y,0} \,. \tag{20}$$

Defining the Fourier transform $G(m\ell, x-y) = \int_{-\pi}^{\pi} d^3k \, G(m\ell, k) \exp(ik\cdot x)$ we obtain

$$\int_{-\pi}^{\pi} d^3k \, G(m\ell, k) \, \ell^2 \left(m^2 - \nabla^2 \right) e^{ik\cdot x} = \frac{1}{(2\pi)^3} \int_{-\pi}^{\pi} d^3k \, e^{ik\cdot x} \,. \tag{21}$$

Applying, finally, the finite difference operator $\ell^2 \left(m^2 - \nabla^2 \right)$ to the exponential in the Fourier transform gives the final result

$$G(ml) \equiv G(ml, 0) = \frac{1}{(2\pi)^3} \int_{-\pi}^{\pi} d^3k \frac{1}{(m\ell)^2 + \sum_{i=0}^{2} 4 \sin\left(\frac{k^i}{2}\right)^2} \,. \tag{22}$$

The string entropy, however, is also proportional to their length, being given by μN with $\mu \approx \ln(5)$ since at each step the non-backtracking strings can choose among 5 possible directions on how to continue. One can thus assign the free energy

$$F = \pi m\ell G\left(m\ell\right) \left[\frac{1}{g} \, Q^2 + g \, M^2 - \frac{1}{\eta} \right] N \,, \tag{23}$$

to a string of length $L = \ell N$ carrying electric and magnetic quantum numbers Q and M, respectively. Here we have introduced the dimensionless parameter

$$\eta = \frac{\pi m\ell G(m\ell)}{\mu} \,, \tag{24}$$

which, together with the ratio $g = \sqrt{\frac{\pi^2 E_J}{2E_C}}$ fully determines the quantum phase structure, as we now show.

The ground state of the quantum model is found by minimizing its free energy as a function of N. When the energy term in eq. (23) dominates, the free energy is positive and consequently minimized by short closed loop configurations. When, instead, the entropy dominates, the free energy is negative and minimized by large strings, long closed loops and instantons that break the original $\mathbb{R}$ gauge symmetry down to $\mathbb{Z}$. The condition for condensation of long strings with integer quantum numbers Q and M is thus given by

$$\eta \frac{1}{g} Q^2 + \eta g M^2 < 1 \,. \tag{25}$$

If two or more condensations are allowed, one has to choose the one with the lowest free energy.

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This condition describes the interior of an ellipse with semiaxes

$$r_Q = \sqrt{g\frac{1}{\eta}}\,,$$
$$r_M = \sqrt{\frac{1}{g}\frac{1}{\eta}}\,, \tag{26}$$

on a square lattice of integer electric and magnetic charges. The phase diagram is consequently found by simply recording which integer charges lie within the ellipse when the semi-axes are varied,

$$\eta \quad < 1 \rightarrow \begin{cases} \sqrt{\frac{2E_C}{\pi^2 E_J}} < 1\,, \text{electric condensation} &= \text{type III superconductor}\,, \\ \sqrt{\frac{2E_C}{\pi^2 E_J}} > 1\,, \text{magnetic condensation} &= \text{superinsulator}\,, \end{cases}$$
$$\eta \quad > 1 \rightarrow \begin{cases} \sqrt{\frac{2E_C}{\pi^2 E_J}} < \frac{1}{\eta}\,, \text{electric condensation} &= \text{type III superconductor}\,, \\ \frac{1}{\eta} < \sqrt{\frac{2E_C}{\pi^2 E_J}} < \eta\,, \text{no condensation} &= \text{bosonic topological insulator/Bose metal}\,, \\ \sqrt{\frac{2E_C}{\pi^2 E_J}} > \eta\,, \text{magnetic condensation} &= \text{superinsulator}\,. \end{cases}$$

For $\eta > 1$ the SIT in JJA occurs via an intermediate phase in which both Q_μ and M_μ are diluted and the action for JJA reduces to the action of a bosonic topological insulator [39]:

$$S = \sum_x i2\pi\, a_\mu K_{\mu\nu} b_\nu\,. \tag{27}$$

The existence of this phase was first predicted in [5]. The SIT point [40] is known to be an universal quantum phase transition point with resistance R equal to the quantum of resistance R_Q. This topological state is often referred to as a Bose metal because it hosts symmetry-protected metallic edge states [30]. It forms due to the same competition of two quantum orders, the charge condensate and the vortex condensate that leads to the universal SIT point. This intermediate phase is separated by two quantum BKT transition from the superinsulating phase for $g < 1$ and from the superconducting phase for $g > 1$ [21]. For $g \geq 1$, $R \leq R_Q$, the saturation of the resistance has been experimentally observed in [41] in Al/InAs JJA using a gate voltage, which suppresses charges tunneling, thereby confirming the validity of our model. In terms of the array parameters, this means varying E_J keeping E_C fixed to increase the value of g.

The superconducting phase is the phase in which the electric topological defects proliferate, and it is this new type of superconductivity that we called type III [18], characterized by a topological gap and superconductivity lost trough a BKT transition both at $T = 0$ and at finite T, as discussed in the introduction.

For $\eta < 1$, instead, when $\sqrt{2E_C/\pi^2 E_J}$ lies between η and $1/\eta$ there is coexistence of electric and magnetic excitations indicating a direct first order phase transition between the superinsulator and the superconductor .

3.1. *Superinsulators*

The superinsulating phase is the phase in which the magnetic monopoles defects M_μ are dense. To understand the nature of this phase we will compute the induced effective action for the electromagnetic gauge potential A_μ minimally coupled to the electric

$$S \rightarrow S + i\sum_{\mathbf{x}} l_0 l^2 A_\mu j_\mu = S + i\sum_{\mathbf{x},\mathbf{i}} \left(l^2 b_0 F_0 + l_0 l b_i F_i\right)\,, \tag{28}$$

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where $F_\mu = \hat{k}_{\mu\nu} A_\nu$. The induced action is:

$$\mathrm{e}^{-S_{\mathrm{eff}}(A_\mu)} = \sum_{Q_\mu, M_\mu} \int_{a_\mu, b_\mu} \mathcal{D}a_\mu \mathcal{D}b_\mu \, \mathrm{e}^{-S(a_\mu, b_\mu, Q_\mu, M_\mu, A_\mu)} \, , \tag{29}$$

and the corresponding induced current is:

$$j^\mu = \frac{1}{\ell^3} \frac{\delta}{\delta A_\mu} S_{\mathrm{eff}}(A_\mu) \; . \tag{30}$$

By noticing that the coupling with the external electromagnetic field can be taken into account by shifting M_0 and M_i in $\sum_{\mathbf{x}} (l_0 b_0 M_0 + i b_i M_i)$ to:

$$M_0 \to M_0' = M_0 + \frac{l^2}{2\pi} F_0 \;, M_i \to M_i' = M_i + \frac{l l_0}{2\pi} F_i \;, \tag{31}$$

we obtain from eq. (18):

$$\begin{aligned} S_{\mathrm{top}}(Q_\mu, M_\mu, A_\mu) &= \sum_{x,i} \ell_0 \ell^2 \, g \frac{\mu\eta\ell}{4\pi^2} \left[v \left(F_0 + \frac{2\pi}{\ell^2} M_0 \right)^2 + \frac{1}{v} \left(F_i + \frac{2\pi}{\ell_0 \ell} M_i \right)^2 \right] \\ &+ \sum_x \frac{1}{g} \mu\eta \, Q_\mu Q_\mu + \sum_{x,i} i \frac{\mu\eta m v \ell}{2\pi} \left(\ell_0 A_0 Q_0 + \ell A_i Q_i \right) \, , \end{aligned} \tag{32}$$

where, also in this case, we retained only self-interactions. This corresponds to the limit $\ell_0 \omega_P = l_0 m \gg 1$.

In the superinsulating phase, the electric topological excitations Q_i are suppressed because of their large energy, so that $Q_i = 0$ and we restore $Q_0 = 0$. We obtain, thus, from eq. (32):

$$\begin{aligned} &\sum_{\{M_i\}} \int_{-\pi}^{+\pi} \mathcal{D}A_\mu \, \mathrm{e}^{S(A_\mu, M_i)} \, , \\ &S(A_\mu, M_i) = \frac{g}{4\pi \ell_0 \omega_P} \sum_x (F_i - 2\pi M_i)^2 \, , \end{aligned} \tag{33}$$

where we used $m = \omega_P$ and finite lattice differences. In eq. (33) the gauge fields F_i are periodic under the shift $F_i \to F_i + 2\pi M_i$ and are compact, they are angular variables defined on the interval $[-\pi, +\pi]$. What we obtain is the deep non-relativistic limit of Polyakov's compact QED action [42,43], in which only electric fields survives. It is the compacteness of the gauge fields that allows the presence of magnetic monopoles (instantons). As in the relativistic case, as we will show, the presence of monopoles gives linear confinement of probe charges, which become bound by electric flux tubes. There is, however, a crucial difference with the relativistic case. To see this, we decompose M_i into its transverse and longitudinal components [42,43]:

$$\begin{aligned} &M_i = {M^T}_i + {M^L}_i \, , {M^T}_i = \epsilon_{ij} \Delta_j n + \epsilon_{ij} \Delta_j \xi \, , n \in Z \\ &{M^L}_i = \Delta_i \lambda \, , \Delta\lambda = \hat{\Delta}_i \Delta_i \lambda = m \, , \end{aligned} \tag{34}$$

where m are integer magnetic monopoles. The sum over $\{M_i\}$ can be traded for the sum over $\{n\}$ and $\{q\}$. The integers $\{n\}$ are used to shift the integration domain for the gauge field A_μ to $[-\infty, +\infty]$, and the real variables $\{\xi\}$ can be absorbed into the gauge field, giving an integral over the non-compact gauge field A_μ and a sum over the monopoles degrees of freedom in the partition function:

$$Z = Z_0 Z_{\mathrm{inst}} = Z_0 \sum_{\{m\}} e^{-\frac{\pi g}{\ell_0 \omega_P} \sum_x m_x \frac{1}{-\Delta} m_x} \tag{35}$$

where Z_0 is the Gaussian integral over A_μ. While in the relativistic case monopoles interact with a potential $1/x$ and are, thus, always in a plasma phase and always confine charges, in JJA, monopoles interact with the inverse of the spatial Laplacian, which gives a $(e^2_{\rm eff}/2\pi)\log|\mathbf{x}|$ potential in 2d with $e^2_{\rm eff} = \pi^2 l_0 \omega_P / g$. Monopoles undergo, thus, a quantum BKT transition with g playing the role of a inverse temperature: for low values of g instantons are free and confine probes charges; instantons undergoes a confinng transition, which is actually the SIT, where probe charges are liberated.

To see how instantons modify the Coulomb potential and cause linear confinement of probe charges, we compute the expectation value of the Wilson loop operator $W(C)$, whith C a closed loop in 3D Euclidean space-time restricted to the plane formed by the Euclidean time and one of the space coordinates, which gives the interaction potential between to external probe charges of strength $\pm q_{\rm ext}$:

$$\langle W(C)\rangle = \frac{1}{Z_{A_\mu,M_i}} \sum_{\{M_i\}} \int_{-\pi}^{+\pi} \mathcal{D}A_\mu \, \mathrm{e}^{-\frac{g}{4\pi\ell_0\omega_P} \sum_x (F_i - 2\pi M_i)^2} \mathrm{e}^{i q_{\rm ext} \sum_C A_\mu} , \tag{36}$$

where we absorbed a factor l in A_μ. A linear interaction between the probe charges will give rise to an area-law: [37,42]

$$\langle W(C)\rangle = e^{-\sigma A} , \tag{37}$$

where A is the area of the surface S enclosed by the loop C and σ is called the string tension [37,42], which we will compute in what follows. Using the lattice Stoke theorem, one rewrites Eq. (36) as

$$\langle W(C)\rangle = \frac{1}{Z_{A_\mu,M_i}} \sum_{\{M_i\}} \int_{-\pi}^{+\pi} \mathcal{D}A_\mu \, \mathrm{e}^{-\frac{g}{4\pi\ell_0\omega_P} \sum_x (F_i - 2\pi M_i)^2} \mathrm{e}^{i q_{\rm ext} \sum_S S_i (F_i - 2\pi M_i)} , \tag{38}$$

where the quantities S_i are unit vectors perpendicular to the plaquettes forming the surface S encircled by the loop C and vanish on all other plaquettes. We have also multiplied the Wilson loop operator by 1 in the form $\exp(-i2\pi q_{\rm ext} M_i)$ on all plaquettes forming S.

Repeating the steps that lead to eq. (35) we obtain:

$$\langle W(C)\rangle = \frac{1}{Z_m} \sum_{\{m\}} e^{-\frac{\pi g}{\ell_0\omega_P} \sum_x m_x \frac{1}{-\Delta} m_x} e^{i2\pi q_{\rm ext} \sum_S \hat{\Delta}_i S_i \frac{1}{-\Delta} m_x} , \tag{39}$$

where we negleted Z_0, since the integral over non-compact gauge field A_μ gives then the Gaussian fluctuations around the instantons m, which do not contribute to confinement [6].

We will now perform the sum over the instantons and introduce an auxiliary scalar field χ to write the quadratic term in the instantons in eq. (39) as a Gaussian integral over χ. At low g, in the deep superinsulationg regime we will use the dilute instanton approximation, summing only over single monopoles configuration $m = \pm 1$, obtainin thus [42]:

$$\langle W(C)\rangle = \frac{1}{Z_\chi} \int \mathcal{D}\chi \, e^{-\frac{\ell_0\omega_P}{4\pi g} \sum_x \Delta_i\chi\Delta_i\chi + \frac{8\pi g}{\ell_0\omega_P} z(1-\cos(\chi + q_{\rm ext}\eta))} , \tag{40}$$

where the angle $\eta = 2\pi\hat{\Delta}_i S_i/(-\Delta_2)$ represents a dipole sheet on the Wilson surface S and the monopole fugacity z is determined by the self-interaction as

$$z = e^{-\frac{\pi g}{\ell_0\omega_P} G(0)} , \tag{41}$$

with $G(0)$ being the inverse of the 2D Laplacian at coinciding arguments. Eq. (40) can be rewritten as:

$$\langle W(C)\rangle = \frac{1}{Z_\chi} \int \mathcal{D}\chi \, e^{-\frac{\ell_0\omega_P}{2\pi g} \sum_x \frac{1}{2}\Delta_i(\chi - q_{\rm ext}\eta)\Delta_i(\chi - q_{\rm ext}\eta) + \mu^2(1-\cos(\chi))} , \tag{42}$$

where we wrote: $\mu^2 = 4\pi g z/\ell_0\omega_P$ and shift $\chi \to \chi - q_{\rm ext}$.

To compute the string tension σ we have to evaluate the integral eq. (42). We will do it both for Cooper pairs, with $q_{\rm ext} = 1$ and single electrons, corresponding to $q_{\rm ext} = 1/2$ in our case. We will evaluate eq. (42) using the saddle point approximation:

$$\Delta_2 \chi_{\rm cl} = q_{\rm ext} \Delta_2 \eta + \mu^2 \sin \chi_{\rm cl} \ , \tag{43}$$

valid for small g where the integral is dominated by the classical solution to the equation of motion and which reduces to a one-dimensional equation when we consider time and one spatial coordinate, e.g. in the (t, x_2) plane far from the boundaries of S:

$$\hat{\Delta}_{x1} \Delta_{x1} \chi_{\rm cl} = -2\pi q_{\rm ext} \hat{\Delta}_{x1} S_1 + \mu^2 \sin \chi_{\rm cl} \ . \tag{44}$$

To solve this equation we will use the continuum limit [42] with boundary conditions $\chi_{\rm cl} \to 0$ for $|x_1| \to \infty$. For Cooper pairs ($q_{\rm ext} = 1$) we obtain:

$$\chi_{\rm cl} = \text{sign}(x_1) \, 4 \arctan \mathrm{e}^{-\mu |x_1|} \ , \tag{45}$$

which gives eq. (37) with σ:

$$\sigma = \frac{\hbar \omega_P}{\ell} \sqrt{\frac{16}{\pi g \ell_0 \omega_P}} \sqrt{z} = \frac{\hbar \omega_P}{\ell} \sqrt{\frac{16}{\pi g \ell_0 \omega_P}} \, \mathrm{e}^{-\frac{\pi g}{2 \ell_0 \omega_P} G(0)} \ . \tag{46}$$

The string binds together charges, prevents charge transport on arrays of a sufficient size and is the origin of the infinite resistance characterizing superinsulation. If we consider, instead, a single electron probes, $q_{\rm ext} = 1/2$, with the same boundaries conditions we obtain, for the string tensions

$$\sigma_{\rm electrons} = \frac{1}{2} \sigma \ , \tag{47}$$

which implies that single electrons are also confined. This explains why charge transport mediated by thermally excited normal quasiparticles is not present in superinsulators.

From eq. (46) we can estimate the typical string size $\ell_{\rm string} = \sqrt{c\hbar/\sigma}$. Taking the following typical values for experimental JJA, $\ell = 100\,\text{nm}$ and $\omega_P = 10\,\text{GHz}$, and taking for $\ell_0 \omega_P = \mathcal{O}(1000)$ we arrive at $\ell_{\rm string}/\ell \approx 150$ lattice spacing, which represents the distance between the superconducting islands. This set a minimum dimension for the arrays to be able to accommodate an electric pion and to show superinsulations. To be able to see superinsulation in JJA, moreover, it is necessary to lower the value of g, that, in terms of the parameter of the arrays, implies increasing the vortex tunneling parameter E_C. While making $g > 1$ has been experimentally achieved in [41], the opposite limit seems to be more difficult. It will require a "more insulating" substrate to be able to increase vortex mobility and govern E_C. Our predictions is that in this case it will be possible to observe superinsulation in JJA.

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