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Article

Fuzzy Normed Linear Spaces Generated by Linear Functionals

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Abstract: For a nonzero normed linear space X, we will define some different classes of fuzzy norms on X generated by linear and bounded linear functionals. Also separate continuity of the elements of each class are investigated. The aim of this research is to introduce a source of examples and counterexamples in the field of fuzzy normed spaces.

Keywords: Fuzzy normed linear space; Linear functional; Separate continuity

1. Introduction

The idea of fuzzy norm on a linear space introduced by Katsaras [11] in 1984. Also a type of fuzzy metric introduced by O. Kaleva and S. Seikkala [10] in 1984. Felbin [7] in 1992 introduced an idea of fuzzy norm on a linear space, such that its corresponding fuzzy metric is of type of introduced by O. Kaleva and S. Seikkala. Another idea of a fuzzy norm on a linear space was introduced by Cheng and Mordeson [6] in 1994. Following Cheng and Mordeson, a definition of a fuzzy norm whose associated fuzzy metric is similar to Kramosil and Michalek type [12], was introduced by T. Bag and S. K. Samanta [1] in 2003. A large number of papers concerning fuzzy norms have been published by different authors such as papers [2–5,8,9].

In this paper we will consider X as a normed linear space over $\mathbb{C}$ or $\mathbb{R}$. Also let X' be the set of all linear functionals on X and let X^* be the set of all bounded linear functionals on X.

Given a normed linear space X, we will introduce some different classes of fuzzy normed linear spaces. Also separate continuity of the elements of each class are investigated. The results of this paper can be applied as a source of examples and counterexamples concerning fuzzy normed linear spaces.

2. FUZZY NORMS GENERATED BY LINEAR FUNCTIONALS

Definition 2.1. *[1] Let X be a linear space and let $N : X \times \mathbb{R} \longrightarrow [0,1]$ be a function with the following properties*

(1) $N(x,t) = 0$ *for all* $t \leq 0$,
(2) $N(x,t) = 1$ *for all* $t > 0$ *if and only if* $x = 0$,
(3) $N(cx,t) = N(x, \frac{t}{|c|})$ *for all* $c \neq 0$,
(4) $N(x+y, s+t) \geq \min(N(x,s), N(y,t))$,
(5) For each fixed $x \in X$, $N(x,.) : \mathbb{R} \longrightarrow [0,1]$ *is an increasing function, and* $\lim_{t \longrightarrow \infty} N(x,t) = 1$.

Then (X,N) *is called a fuzzy normed linear space.*

Example 2.2. *If X is a normed linear space, then $N : X \times \mathbb{R} \longrightarrow [0,1]$ defined by*

$$N(x,t) = \begin{cases} 0 & t \leq \|x\| \\ 1 & t > \|x\| \end{cases}$$

is a fuzzy norm on X.

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In the sequel we present some different classes of fuzzy normed linear spaces generated by linear functionals.

Proposition 2.3. *Let X be a normed linear space and $f \in X'$. Define $N_f : X \times \mathbb{R} \longrightarrow [0,1]$ by*

$$N_f(x,t) = \begin{cases} 0 & t \leq \|x\| + |f(x)| \\ \frac{t-\|x\|-|f(x)|}{t+\|x\|+|f(x)|} & t > \|x\| + |f(x)|. \end{cases}$$

Then N_f is a fuzzy norm on X.

Proof. We only prove parts (2), (4) and (5) of Definition 2.1.
(2) : If $x = 0$, then clearly $N_f(x,t) = 1$ for all $t > 0$. For the converse let $N_f(x,t) = 1$ for all $t > 0$ and suppose by contradiction that $x \neq 0$.
If $t = \|x\| + |f(x)|$, then $N_f(x,t) = 0$ that is a contradiction.
(4) : Let $x, y \in X$ and $s, t \in \mathbb{R}$. If $s + t \leq 0$, then $s \leq 0$ or $t \leq 0$. So $N_f(x,s) = 0$ or $N_f(y,t) = 0$.
Hence $0 = N_f(x+y, s+t) \geq \min(N_f(x,s), N_f(y,t)) = 0$.
If $s + t > 0$ and $s + t \leq \|x+y\| + |f(x) + f(y)|$, then $s \leq \|x\| + |f(x)|$ or $t \leq \|y\| + |f(y)|$. Therefore

$$0 = N_f(x+y, s+t) \geq \min(N_f(x,s), N_f(y,t)) = 0.$$

If $s + t > \|x+y\| + |f(x) + f(y)|$, then $N_f(x+y, s+t) = \frac{s+t-\|x+y\|-|f(x)+f(y)|}{s+t+\|x+y\|+|f(x)+f(y)|}$.
In this case if $s \leq \|x\| + |f(x)|$ or $t \leq \|y\| + |f(y)|$, then clearly

$$\frac{s+t-\|x+y\|-|f(x)+f(y)|}{s+t+\|x+y\|+|f(x)+f(y)|} \geq \min(N_f(x,s), N_f(y,t)) = 0.$$

If $s > \|x\| + |f(x)|$ and $t > \|y\| + |f(y)|$, then $N_f(x,s) = \frac{s-\|x\|-|f(x)|}{s+\|x\|+|f(x)|}$ and $N_f(y,t) = \frac{t-\|y\|-|f(y)|}{t+\|y\|+|f(y)|}$.
One can easily verify that if $N_f(x,s) \leq N_f(y,t)$, then

$$s(\|y\| + |f(y)|) \leq t(\|x\| + |f(x)|).$$

Also if $N_f(y,t) \leq N_f(x,s)$, then $t(\|x\| + |f(x)|) \leq s(\|y\| + |f(y)|)$.
A straightforward calculation reveals that
if $s(\|y\| + |f(y)|) \leq t(\|x\| + |f(x)|)$, then

$$\begin{aligned} N_f(x+y, s+t) &= \frac{s+t-\|x+y\|-|f(x)+f(y)|}{s+t+\|x+y\|+|f(x)+f(y)|} \\ &\geq \frac{s-\|x\|-|f(x)|}{s+\|x\|+|f(x)|} \\ &= \min(N_f(x,s), N_f(y,t)). \end{aligned}$$

Also if $t(\|x\| + |f(x)|) \leq s(\|y\| + |f(y)|)$, then

$$\begin{aligned} N_f(x+y, s+t) &= \frac{s+t-\|x+y\|-|f(x)+f(y)|}{s+t+\|x+y\|+|f(x)+f(y)|} \\ &\geq \frac{t-\|y\|-|f(y)|}{t+\|y\|+|f(y)|} \\ &= \min(N_f(x,s), N_f(y,t)). \end{aligned}$$

Therefore $N_f(x+y, s+t) \geq \min(N_f(x,s), N_f(y,t))$ for all $x, y \in X$ and $s, t \in \mathbb{R}$.
(5) : Let $x \in X$ and $s \leq t$. If $N_f(x,s) = 0$, then clearly $N_f(x,s) \leq N_f(x,t)$. If $N_f(x,s) \neq 0$, then

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$N_f(x,s) = \frac{s-\|x\|-|f(x)|}{s+\|x\|+|f(x)|}$ and $s >\|x\| + |f(x)|$. Hence $t >\|x\| + |f(x)|$ and $N_f(x,t) = \frac{t-\|x\|-|f(x)|}{t+\|x\|+|f(x)|}$. Since $s \leq t$,

$$s(\|x\| + |f(x)|) \leq t(\|x\| + |f(x)|).$$

Therefore a straightforward calculation reveals that

$$N_f(x,s) = \frac{s-\|x\| - |f(x)|}{s+\|x\| + |f(x)|} \leq \frac{t-\|x\| - |f(x)|}{t+\|x\| + |f(x)|} = N_f(x,t).$$

This shows that $N_f(x,.)$ is an increasing function for all $x \in X$.
Also $\lim\limits_{t\to\infty} N_f(x,t) = \lim\limits_{t\to\infty} \frac{t-\|x\|-|f(x)|}{t+\|x\|+|f(x)|} = 1$. □

Proposition 2.4. *Let X be a normed linear space, $f \in X^*$ and $\epsilon > 0$.*
Define $N_{f,\epsilon} : X \times \mathbb{R} \longrightarrow [0,1]$ by

$$N_{f,\epsilon}(x,t) = \begin{cases} 0 & t \leq\|x\|(\epsilon+\|f\|) \\ \frac{t-|f(x)|}{t+|f(x)|} & t >\|x\|(\epsilon+\|f\|). \end{cases}$$

Then $N_{f,\epsilon}$ is a fuzzy norm on X.

Proof. We only prove parts (2), (4) and (5) of Definition 2.1.
(2) : If $x = 0$, then clearly $N_{f,\epsilon}(x,t) = 1$ for all $t > 0$. For the converse let $N_{f,\epsilon}(x,t) = 1$ for all $t > 0$ and suppose by contradiction that $x \neq 0$.
If $t =\|x\|(\epsilon+\|f\|)$, then $N_{f,\epsilon}(x,t) = 0$ that is a contradiction.
(4) : Let $x, y \in X$ and $s, t \in \mathbb{R}$. If $s + t \leq 0$, then $s \leq 0$ or $t \leq 0$. So $N_{f,\epsilon}(x,s) = 0$ or $N_{f,\epsilon}(y,t) = 0$.
Hence $0 = N_{f,\epsilon}(x + y, s + t) \geq \min(N_{f,\epsilon}(x,s), N_{f,\epsilon}(y,t)) = 0$.
If $s + t > 0$ and $s + t \leq\|x + y\|(\epsilon+\|f\|)$, then $s \leq\|x\|(\epsilon+\|f\|)$ or $t \leq\|y\|(\epsilon+\|f\|)$. Therefore

$$0 = N_{f,\epsilon}(x + y, s + t) \geq \min(N_{f,\epsilon}(x,s), N_{f,\epsilon}(y,t)) = 0.$$

If $s + t >\|x + y\|(\epsilon+\|f\|)$, then $N_{f,\epsilon}(x + y, s + t) = \frac{s+t-|f(x)+f(y)|}{s+t+|f(x)+f(y)|}$.
In this case if $s \leq\|x\|(\epsilon+\|f\|)$ or $t \leq\|y\|(\epsilon+\|f\|)$, then clearly

$$\frac{s + t - |f(x) + f(y)|}{s + t + |f(x) + f(y)|} \geq \min(N_{f,\epsilon}(x,s), N_{f,\epsilon}(y,t)) = 0.$$

If $s >\|x\|(\epsilon+\|f\|)$ and $t >\|y\|(\epsilon+\|f\|)$, then $N_{f,\epsilon}(x,s) = \frac{s-|f(x)|}{s+|f(x)|}$ and $N_{f,\epsilon}(y,t) = \frac{t-|f(y)|}{t+|f(y)|}$.
One can easily verify that if $N_{f,\epsilon}(x,s) \leq N_{f,\epsilon}(y,t)$, then $2s|f(y)| \leq 2t|f(x)|$. Also if $N_{f,\epsilon}(y,t) \leq N_{f,\epsilon}(x,s)$, then $2t|f(x)| \leq 2s|f(y)|$.
A straightforward calculation reveals that if $2s|f(y)| \leq 2t|f(x)|$, then

$$\begin{aligned} N_{f,\epsilon}(x + y, s + t) &= \frac{s + t - |f(x) + f(y)|}{s + t + |f(x) + f(y)|} \\ &\geq \frac{s - |f(x)|}{s + |f(x)|} \\ &= \min(N_{f,\epsilon}(x,s), N_{f,\epsilon}(y,t)). \end{aligned}$$

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Also if $2t|f(x)| \leq 2s|f(y)|$, then

$$\begin{aligned} N_{f,\epsilon}(x+y,s+t) &= \frac{s+t-|f(x)+f(y)|}{s+t+|f(x)+f(y)|} \\ &\geq \frac{t-|f(y)|}{t+|f(y)|} \\ &= \min(N_{f,\epsilon}(x,s), N_{f,\epsilon}(y,t)). \end{aligned}$$

Therefore $N_{f,\epsilon}(x+y,s+t) \geq \min(N_{f,\epsilon}(x,s), N_{f,\epsilon}(y,t))$ for all $x,y \in X$ and $s,t \in \mathbb{R}$.
(5) : Let $x \in X$ and $s \leq t$. If $N_{f,\epsilon}(x,s) = 0$, then clearly $N_{f,\epsilon}(x,s) \leq N_{f,\epsilon}(x,t)$. If $N_{f,\epsilon}(x,s) \neq 0$, then $N_{f,\epsilon}(x,s) = \frac{s-|f(x)|}{s+|f(x)|}$ and $s > \|x\|(\epsilon + \|f\|)$. Hence $t > \|x\|(\epsilon + \|f\|)$ and $N_{f,\epsilon}(x,t) = \frac{t-|f(x)|}{t+|f(x)|}$. Since $s \leq t$, $2s|f(x)| \leq 2t|f(x)|$. Therefore a straightforward calculation reveals that $N_{f,\epsilon}(x,s) = \frac{s-|f(x)|}{s+|f(x)|} \leq \frac{t-|f(x)|}{t+|f(x)|} = N_{f,\epsilon}(x,t)$. This shows that $N_{f,\epsilon}(x,.)$ is an increasing function for all $x \in X$. Also

$$\lim_{t\to\infty} N_{f,\epsilon}(x,t) = \lim_{t\to\infty} \frac{t-|f(x)|}{t+|f(x)|} = 1.$$

□

Proposition 2.5. *Let X be a normed linear space and $g \in X'$. Then the maps*

$$N_g^{(1)} : X \times \mathbb{R} \longrightarrow [0,1]$$
$$N_g^{(1)}(x,t) = \begin{cases} 0 & t \leq \|x\| + |g(x)| \\ \frac{t}{t+\|x\|+|g(x)|} & t > \|x\| + |g(x)|, \end{cases}$$

and

$$N_g^{(2)} : X \times \mathbb{R} \longrightarrow [0,1]$$
$$N_g^{(2)}(x,t) = \begin{cases} 0 & t \leq |g(x)| \\ \frac{t}{t+\|x\|+|g(x)|} & t > |g(x)|, \end{cases}$$

are fuzzy norms on X.
Also if $\ker g = \{0\}$, then the map

$$N_g^{(3)} : X \times \mathbb{R} \longrightarrow [0,1]$$
$$N_g^{(3)}(x,t) = \begin{cases} 0 & t \leq |g(x)| \\ \frac{t}{t+|g(x)|} & t > |g(x)|, \end{cases}$$

is a fuzzy norm on X.

Proof. At the first we will prove that $N_g^{(2)}$ is a fuzzy norm. The proof of the other parts are similar. Note that for investigation of the fuzzy norm properties in the case $N_g^{(3)}$, $\ker g = \{0\}$ will be used in part (2) of Definition 2.1. Since $|g(x)| = 0$ implies that $x = 0$.

(1) If $t \leq 0$, then clearly $N_g^{(2)}(x,t) = 0$ for all $x \in X$.
(2) If $x = 0$, then $g(x) = 0$ and consequently for all $t > 0 = |g(x)|$, $N_g^{(2)}(x,t) = \frac{t}{t+0+0} = 1$. For the converse let $N_g^{(2)}(x,t) = 1$ for all $t > 0$. At the first we will show that $g(x) = 0$. Suppose by contradiction that $g(x) \neq 0$.
If $t = |g(x)|$, then $N_g^{(2)}(x,t) = 0$ that is a contradiction. This shows that $g(x) = 0$. So by

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hypothesis, for all $t > 0 = |g(x)|$, $\frac{t}{t+\|x\|+0} = N_g^{(2)}(x,t) = 1$. It follows that $\|x\| = 0$ that implies $x = 0$.

(3) If $c \neq 0$, then

$$N_g^{(2)}(cx,t) = \begin{cases} 0 & t \leq |g(cx)| \\ \frac{t}{t+\|cx\|+|g(cx)|} & t > |g(cx)| \end{cases}$$
$$= \begin{cases} 0 & \frac{t}{|c|} \leq |g(x)| \\ \frac{\frac{t}{|c|}}{\frac{t}{|c|}+\|x\|+|g(x)|} & \frac{t}{|c|} > |g(x)| \end{cases}$$
$$= N_g^{(2)}(x, \frac{t}{|c|}), x \in X, t \in \mathbb{R}.$$

(4) Let $x, y \in X$ and $s, t \in \mathbb{R}$. If $s + t \leq |g(x+y)| = |g(x) + g(y)|$, then $s \leq |g(x)|$ or $t \leq |g(y)|$. So

$$0 = N_g^{(2)}(x+y, s+t) \geq \min(N_g^{(2)}(x,s), N_g^{(2)}(y,t)) = 0.$$

Let $s + t > |g(x) + g(y)|$ and $s \leq |g(x)|$ or $t \leq |g(y)|$, then clearly,

$$N_g^{(2)}(x+y, s+t) = \frac{s+t}{s+t+\|x+y\| + |g(x)+g(y)|}$$
$$> 0$$
$$= \min(N_g^{(2)}(x,s), N_g^{(2)}(y,t)).$$

Let $s + t > |g(x) + g(y)|$ and $s > |g(x)|$ and $t > |g(y)|$. Then $N_g^{(2)}(x+y, s+t) = \frac{s+t}{s+t+\|x+y\|+|g(x)+g(y)|}$ and
$N_g^{(2)}(x,s) = \frac{s}{s+\|x\|+|g(x)|}$ and $N_g^{(2)}(y,t) = \frac{t}{t+\|y\|+|g(y)|}$.
If $\frac{s}{s+\|x\|+|g(x)|} \leq \frac{t}{t+\|y\|+|g(y)|}$, then

$$s(\|y\| + |g(y)|) \leq t(\|x\| + |g(x)|). \tag{1}$$

If $\frac{t}{t+\|y\|+|g(y)|} \leq \frac{s}{s+\|x\|+|g(x)|}$, then

$$t(\|x\| + |g(x)|) \leq s(\|y\| + |g(y)|). \tag{2}$$

In the case where $N_g^{(2)}(x,s) \leq N_g^{(2)}(y,t)$, inequality 1 implies that

$$\frac{s+t}{s+t+\|x+y\| + |g(x)+g(y)|} = N_g^{(2)}(x+y, s+t)$$
$$\geq \frac{s}{s+\|x\| + |g(x)|}$$
$$= \min(N_g^{(2)}(x,s), N_g^{(2)}(y,t)).$$

Also in the case where $N_g^{(2)}(y,t) \leq N_g^{(2)}(x,s)$, inequality 2 implies that $N_g^{(2)}(x+y, s+t) \geq \min(N_g^{(2)}(x,s), N_g^{(2)}(y,t))$.

(5) Let $s \leq t$. If $N_g^{(2)}(x,s) = 0$, then clearly $N_g^{(2)}(x,s) \leq N_g^{(2)}(x,t)$. Let $N_g^{(2)}(x,s) \neq 0$. Then $s > |g(x)|$. It follows that $t > |g(x)|$.

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Since $s(\|x\| + |g(x)|) \leq t(\|x\| + |g(x)|)$, $N_g^{(2)}(x,s) \leq N_g^{(2)}(x,t)$. So $N_g^{(2)}(x,.)$ is an increasing function and also

$$\lim_{t\to\infty} N_g^{(2)}(x,t) = \lim_{t\to\infty} \frac{t}{t+\|x\| + |g(x)|} = 1.$$

□

Proposition 2.6. *Let X be a normed linear space and $h \in X'$. Then the map*

$$N_h^{(1)} : X \times \mathbb{R} \longrightarrow [0,1]$$
$$N_h^{(1)}(x,t) = \begin{cases} 0 & t \leq \|x\| + |h(x)| \\ \frac{t-\|x\|-|h(x)|}{t} & t > \|x\| + |h(x)|, \end{cases}$$

is a fuzzy norm on X.
Also if $\ker h = \{0\}$, *then the map*

$$N_h^{(2)} : X \times \mathbb{R} \longrightarrow [0,1]$$
$$N_h^{(2)}(x,t) = \begin{cases} 0 & t \leq |h(x)| \\ \frac{t-|h(x)|}{t} & t > |h(x)|, \end{cases}$$

is a fuzzy norm on X.

Proof. A similar argument used to Proposition 2.5 can be applied. □

3. The separate continuity of $N_f, N_{f,\epsilon}, N_g^{(i)}, N_h^{(j)}, 1 \leq i \leq 3, 1 \leq j \leq 2$

In this section we characterize separate continuity of the fuzzy norms $N_f, N_{f,\epsilon}, N_g^{(i)}, N_h^{(j)}, 1 \leq i \leq 3, 1 \leq j \leq 2$, where $\epsilon > 0$ and $f, g, h \in X^*$.

Theorem 3.1. *Let X be a normed linear space and $f \in X^*$. If*
$N_f : X \times \mathbb{R} \longrightarrow [0,1]$ *is defined by*

$$N_f(x,t) = \begin{cases} 0 & t \leq \|x\| + |f(x)| \\ \frac{t-\|x\|-|f(x)|}{t+\|x\|+|f(x)|} & t > \|x\| + |f(x)|, \end{cases}$$

then the map

$$N_f(.,t) : X \longrightarrow [0,1]$$
$$x \longrightarrow N_f(x,t),$$

is continuous for all $t \in \mathbb{R}$.
Also the map

$$N_f(x,.) : \mathbb{R} \longrightarrow [0,1]$$
$$t \longrightarrow N_f(x,t),$$

is continuous for all $x \in X$ except $x = 0$.

Proof. If $t \leq 0$, then $N_f(x,t) = 0$ for all $x \in X$. So $N_f(.,t) : X \longrightarrow [0,1]$ is a constant function and consequently is continuous on X for all $t \leq 0$.
For a fixed $t > 0$, let $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$. If $t =$

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$\|x\| + |f(x)|$, then for all subsequences $\{x_{n_k}\}_{k=1}^{\infty} \subseteq \{x_n\}_{n=1}^{\infty}$ satisfying $t \leq \|x_{n_k}\| + |f(x_{n_k})|, k \in \mathbb{N}$, we have

$$\lim_{k\to\infty} N_f(x_{n_k}, t) = \lim_{k\to\infty} 0 = 0 = N_f(x,t).$$

Also for all subsequences $\{x_{n_k}\}_{k=1}^{\infty} \subseteq \{x_n\}_{n=1}^{\infty}$ satisfying $t > \|x_{n_k}\| + |f(x_{n_k})|, k \in \mathbb{N}$, we have

$$\begin{aligned}\lim_{k\to\infty} N_f(x_{n_k}, t) &= \lim_{k\to\infty} \frac{t - \|x_{n_k}\| - |f(x_{n_k})|}{t + \|x_{n_k}\| + |f(x_{n_k})|} \\ &= \frac{t-t}{t+t} \\ &= 0 \\ &= N_f(x,t).\end{aligned}$$

If $t < \|x\| + |f(x)|$, then there exists an $N \in \mathbb{N}$ such that $t < \|x_n\| + |f(x_n)|$ for all $n \geq N$. So $\lim_{n\to\infty} N_f(x_n, t) = \lim_{n\to\infty} 0 = 0 = N_f(x,t)$.
If $t > \|x\| + |f(x)|$, then there exists an $N \in \mathbb{N}$ such that $t > \|x_n\| + |f(x_n)|$ for all $n \geq N$. So

$$\begin{aligned}\lim_{n\to\infty} N_f(x_n, t) &= \lim_{n\to\infty} \frac{t - \|x_n\| - |f(x_n)|}{t + \|x_n\| + |f(x_n)|} \\ &= \frac{t - \|x\| - |f(x)|}{t + \|x\| + |f(x)|} \\ &= N_f(x,t).\end{aligned}$$

Hence for each $t > 0$, $N_f(., t)$ is continuous at every $x \in X$. So $N_f(., t)$ is continuous on X for all $t \in \mathbb{R}$.
For any fixed $x \neq 0$, let $t \in \mathbb{R}$ and $t_n \longrightarrow t$ as $n \longrightarrow \infty$. If $t = \|x\| + |f(x)|$, then for all subsequences $\{t_{n_k}\}_{k=1}^{\infty} \subseteq \{t_n\}_{n=1}^{\infty}$ satisfying $t_{n_k} \leq \|x\| + |f(x)|$ we have

$$\lim_{k\to\infty} N_f(x, t_{n_k}) = \lim_{k\to\infty} 0 = 0 = N_f(x,t).$$

Also for all subsequences $\{t_{n_k}\}_{k=1}^{\infty} \subseteq \{t_n\}_{n=1}^{\infty}$ satisfying $t_{n_k} > \|x\| + |f(x)|$ we have

$$\begin{aligned}\lim_{k\to\infty} N_f(x, t_{n_k}) &= \lim_{k\to\infty} \frac{t_{n_k} - \|x\| - |f(x)|}{t_{n_k} + \|x\| + |f(x)|} \\ &= \frac{t - \|x\| - |f(x)|}{t + \|x\| + |f(x)|} \\ &= \frac{t-t}{t+t} \\ &= 0 \\ &= N_f(x,t).\end{aligned}$$

If $t < \|x\| + |f(x)|$, then there exists an $N \in \mathbb{N}$ such that $t_n < \|x\| + |f(x)|$ for all $n \geq N$. So $\lim_{n\to\infty} N_f(x, t_n) = \lim_{n\to\infty} 0 = 0 = N_f(x,t)$.
If $t > \|x\| + |f(x)|$, then there exists an $N \in \mathbb{N}$ such that $t_n > \|x\| + |f(x)|$ for all $n \geq N$. So

$$\begin{aligned}\lim_{n\to\infty} N_f(x, t_n) &= \lim_{n\to\infty} \frac{t_n - \|x\| - |f(x)|}{t_n + \|x\| + |f(x)|} \\ &= \frac{t - \|x\| - |f(x)|}{t + \|x\| + |f(x)|} \\ &= N_f(x,t).\end{aligned}$$

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It follows that for any fixed $x \neq 0$, $t_n \longrightarrow t$ as $n \longrightarrow \infty$, implies that $\lim_{n\longrightarrow\infty} N_f(x, t_n) = N_f(x, t)$. Hence $N_f(x, .) : \mathbb{R} \longrightarrow [0, 1]$ is continuous for all $x \neq 0$.
We shall show that $N_f(0, .) : \mathbb{R} \longrightarrow [0, 1]$ is not continuous at $t = 0$.
Since

$$N_f(0, t) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0, \end{cases}$$

$\lim_{t\longrightarrow 0^+} N_f(0, t) = 1 \neq N_f(0, 0) = 0$. This shows that $N_f(0, .) : \mathbb{R} \longrightarrow [0, 1]$ is not continuous on $\mathbb{R}$. □

Theorem 3.2. *Let X be a normed linear space, $f \in X^*$ and $\epsilon > 0$. If $N_{f,\epsilon} : X \times \mathbb{R} \longrightarrow [0, 1]$ is defined by*

$$N_{f,\epsilon}(x, t) = \begin{cases} 0 & t \leq \|x\|(\epsilon + \|f\|) \\ \frac{t-|f(x)|}{t+|f(x)|} & t > \|x\|(\epsilon + \|f\|), \end{cases}$$

then

(1) the map

$$N_{f,\epsilon}(., t) : X \longrightarrow [0, 1]$$
$$x \longrightarrow N_{f,\epsilon}(x, t),$$

is continuous for all $t \leq 0$.

(2) if $t > 0$, then $N_{f,\epsilon}(., t)$ is continuous at every $x \in X \setminus S$, where $S = \{x \in X \mid \|x\| = \frac{t}{\epsilon+\|f\|}\}$.

(3) the map $N_{f,\epsilon}(x, .)$ is continuous at every $t \in \mathbb{R} \setminus T$, where $T = \{t \in \mathbb{R} \mid t = \|x\|(\epsilon+\|f\|)\}$.

Proof. (1) If $t \leq 0$, then $N_{f,\epsilon}(x, t) = 0$ for all $x \in X$. So $N_{f,\epsilon}(., t)$ is a constant function on X that is continuous.

(2) If $t > 0$ and $x \in S$, then $\|x\| = \frac{t}{\epsilon+\|f\|}$ and $N_{f,\epsilon}(x, t) = 0$. Set $x_n = \frac{t-\frac{t}{2n}}{\|x\|(\epsilon+\|f\|)} x$ for all $n \in \mathbb{N}$. So $x_n \longrightarrow x$ as $n \longrightarrow \infty$, and
$\|x_n\| = \frac{t-\frac{t}{2n}}{\|x\|(\epsilon+\|f\|)}\|x\| < \frac{t}{(\epsilon+\|f\|)}$ for all $n \in \mathbb{N}$. This shows that
$t > \|x_n\|(\epsilon+\|f\|)$ for all $n \in \mathbb{N}$.
Hence

$$\begin{aligned} \lim_{n\to\infty} N_{f,\epsilon}(x_n, t) &= \lim_{n\to\infty} \frac{t - |f(x_n)|}{t + |f(x_n)|} \\ &= \frac{t - |f(x)|}{t + |f(x)|} \\ &\neq N_{f,\epsilon}(x, t) = 0, \end{aligned}$$

since, $t = \|x\|(\epsilon+\|f\|) > \|x\|\|f\| \geq |f(x)|$. Therefore $N_{f,\epsilon}(., t)$ is discontinuous at every $x \in S$.
Let $x \notin S$ and let $x_n \longrightarrow x$ as $n \longrightarrow \infty$. So $t > \|x\|(\epsilon+\|f\|)$ or $t < \|x\|(\epsilon+\|f\|)$. If $t > \|x\|(\epsilon+\|f\|)$,

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then there exists an $N \in \mathbb{N}$ such that $t > \|x_n\|(\epsilon + \|f\|)$ for all $n \geq N$. Hence

$$\begin{aligned}\lim_{n\to\infty} N_{f,\epsilon}(x_n, t) &= \lim_{n\to\infty} \frac{t - |f(x_n)|}{t + |f(x_n)|} \\ &= \frac{t - |f(x)|}{t + |f(x)|} \\ &= N_{f,\epsilon}(x, t).\end{aligned}$$

If $t < \|x\|(\epsilon + \|f\|)$, then there exists an $N \in \mathbb{N}$ such that $t < \|x_n\|(\epsilon + \|f\|)$ for all $n \geq N$. So

$$\lim_{n\to\infty} N_{f,\epsilon}(x_n, t) = \lim_{n\to\infty} 0 = 0 = N_{f,\epsilon}(x, t).$$

Consequently in the case where $t > 0$, $N_{f,\epsilon}(., t)$ is continuous at every point $x \in X \setminus S$.

(3) Let $t \in T$. So $t = \|x\|(\epsilon + \|f\|)$ and $N_{f,\epsilon}(x, t) = 0$.
Set $t_n = (\|x\| + \frac{1}{n})(\epsilon + \|f\|)$ for all $n \in \mathbb{N}$. Clearly $t_n \longrightarrow t$ as $n \longrightarrow \infty$, and $t_n > \|x\|(\epsilon + \|f\|)$ for all $n \in \mathbb{N}$. Hence

$$\begin{aligned}\lim_{n\to\infty} N_{f,\epsilon}(x, t_n) &= \lim_{n\to\infty} \frac{t_n - |f(x)|}{t_n + |f(x)|} \\ &= \begin{cases} 1 & x = 0 \\ \frac{t - |f(x)|}{t + |f(x)|} & x \neq 0. \end{cases}\end{aligned}$$

It follows that $\lim_{n\to\infty} N_{f,\epsilon}(x, t_n) \neq N_{f,\epsilon}(x, t) = 0$. Note that if $t = \|x\|(\epsilon + \|f\|)$ and $x \neq 0$, then $t - |f(x)| \neq 0$. Since

$$t = \|x\|(\epsilon + \|f\|) > \|x\|\|f\| \geq |f(x)|.$$

We shall show that $N_{f,\epsilon}(x, .)$ is continuous at every $t \in \mathbb{R} \setminus T$.
Let $t \in \mathbb{R} \setminus T$ and let $t_n \longrightarrow t$ as $n \longrightarrow \infty$. Then $t < \|x\|(\epsilon + \|f\|)$ or $t > \|x\|(\epsilon + \|f\|)$. If $t < \|x\|(\epsilon + \|f\|)$, then there exists an $N \in \mathbb{N}$ such that $t_n < \|x\|(\epsilon + \|f\|)$ for all $n \geq N$. Hence

$$\lim_{n\to\infty} N_{f,\epsilon}(x, t_n) = \lim_{n\to\infty} 0 = 0 = N_{f,\epsilon}(x, t).$$

If $t > \|x\|(\epsilon + \|f\|)$, then there exists an $N \in \mathbb{N}$ such that $t_n > \|x\|(\epsilon + \|f\|)$ for all $n \geq N$. So

$$\begin{aligned}\lim_{n\to\infty} N_{f,\epsilon}(x, t_n) &= \lim_{n\to\infty} \frac{t_n - |f(x)|}{t_n + |f(x)|} \\ &= \frac{t - |f(x)|}{t + |f(x)|} \\ &= N_{f,\epsilon}(x, t).\end{aligned}$$

This shows that $N_{f,\epsilon}(x, .)$ is continuous at every $t \in \mathbb{R} \setminus T$.
□

Theorem 3.3. *Let X be a normed linear space and $g \in X^*$. If $N_g^{(1)} : X \times \mathbb{R} \longrightarrow [0,1]$ and $N_g^{(2)} : X \times \mathbb{R} \longrightarrow [0,1]$ are defined by*

$$N_g^{(1)}(x,t) = \begin{cases} 0 & t \leq \|x\| + |g(x)| \\ \frac{t}{t+\|x\|+|g(x)|} & t > \|x\| + |g(x)|, \end{cases}$$

$$N_g^{(2)}(x,t) = \begin{cases} 0 & t \leq |g(x)| \\ \frac{t}{t+\|x\|+|g(x)|} & t > |g(x)|, \end{cases}$$

and in the case where $\ker g = \{0\}$, $N_g^{(3)} : X \times \mathbb{R} \longrightarrow [0,1]$ *is defined by*

$$N_g^{(3)}(x,t) = \begin{cases} 0 & t \leq |g(x)| \\ \frac{t}{t+|g(x)|} & t > |g(x)|, \end{cases}$$

then

(1) *the maps $N_g^{(1)}(.,t)$, $N_g^{(2)}(.,t)$ and $N_g^{(3)}(.,t)$ are continuous on X for all $t \leq 0$.*
(2) *if $t > 0$, then the map $N_g^{(1)}(.,t)$ is continuous at every $x \in X \setminus S_1$, where $S_1 = \{x \in X \mid t = \|x\| + |g(x)|\}$.*
(3) *if $t > 0$, then the maps $N_g^{(2)}(.,t)$ and $N_g^{(3)}(.,t)$ are continuous at every $x \in X \setminus S_2$, where $S_2 = \{x \in X \mid t = |g(x)|\}$.*
(4) *for $x \in X$, the map $N_g^{(1)}(x,.)$ is continuous at every $t \in \mathbb{R} \setminus T_1$, where $T_1 = \{t \in \mathbb{R} \mid t = \|x\| + |g(x)|\}$.*
(5) *for $x \neq 0$, the map $N_g^{(2)}(x,.)$ is continuous at every $t \in \mathbb{R} \setminus T_2$, where $T_2 = \{t > 0 \mid t = |g(x)|\}$. Also the map $N_g^{(2)}(0,.)$ is continuous at every $t \in \mathbb{R}$ except $t = 0$.*
(6) *for $x \in X$, the map $N_g^{(3)}(x,.)$ is continuous at every $t \in \mathbb{R} \setminus T_3$, where $T_3 = \{t \in \mathbb{R} \mid t = |g(x)|\}$.*

Proof. (1) It is obvious.

(2) Let $t > 0$ and $x \in S_1$. So $t = \|x\| + |g(x)|$. Set $x_n = \frac{t-\frac{t}{2n}}{\|x\|+|g(x)|}x$ for all $n \in \mathbb{N}$. Obviously $x_n \longrightarrow x$ and so $g(x_n) \longrightarrow g(x)$ as $n \longrightarrow \infty$. Also

$$\begin{aligned} \|x_n\| &= \frac{t-\frac{t}{2n}}{\|x\|+|g(x)|}\|x\| \\ &< \frac{t}{\|x\|+|g(x)|}\|x\| \\ &= \|x\|, \quad n \in \mathbb{N}, \end{aligned}$$

and

$$\begin{aligned} |g(x_n)| &= \frac{t-\frac{t}{2n}}{\|x\|+|g(x)|}|g(x)| \\ &\leq \frac{t}{\|x\|+|g(x)|}|g(x)| \\ &= |g(x)|, \quad n \in \mathbb{N}. \end{aligned}$$

So $\|x_n\| + |g(x_n)| <\|x\| + |g(x)| = t$ for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned}\lim_{n\to\infty} N_g^{(1)}(x_n,t) &= \lim_{n\to\infty} \frac{t}{t+\|x_n\| + |g(x_n)|}\\ &= \frac{t}{t+t}\\ &= \frac{1}{2}\\ &\neq N_g^{(1)}(x,t)\\ &= 0.\end{aligned}$$

This shows that $N_g^{(1)}(.,t)$ is discontinuous at every $x \in S_1$. Now let $x \in X \setminus S_1$ and $\{z_n\}_{n=1}^{\infty}$ be a sequence such that $z_n \longrightarrow x$ as $n \longrightarrow \infty$. So $t >\|x\| + |g(x)|$ or $t <\|x\| + |g(x)|$. If $t >\|x\| + |g(x)|$, then there exists an $N \in \mathbb{N}$ such that $t >\|z_n\| + |g(z_n)|$ for all $n \geq N$. Hence

$$\begin{aligned}\lim_{n\to\infty} N_g^{(1)}(z_n,t) &= \lim_{n\to\infty} \frac{t}{t+\|z_n\| + |g(z_n)|}\\ &= \frac{t}{t+\|x\| + |g(x)|}\\ &= N_g^{(1)}(x,t).\end{aligned}$$

If $t <\|x\| + |g(x)|$, then there exists an $N \in \mathbb{N}$ such that $t <\|z_n\| + |g(z_n)|$ for all $n \geq N$. So

$$\lim_{n\to\infty} N_g^{(1)}(z_n,t) = \lim_{n\to\infty} 0 = 0 = N_g^{(1)}(x,t).$$

This shows that $N_g^{(1)}(.,t)$ is continuous at every $x \in X \setminus S_1$.

(3) Let $x \in S_2$. So $t = |g(x)|$, $x \neq 0$ and $N_g^{(2)}(x,t) = N_g^{(3)}(x,t) = 0$. Set $x_n = (1-\frac{1}{2n})x$ for all $n \in \mathbb{N}$. Clearly $x_n \longrightarrow x$, $g(x_n) \longrightarrow g(x)$ as $n \longrightarrow \infty$. Also $|g(x_n)| = (1-\frac{1}{2n})|g(x)| < |g(x)| = t$ for all $n \in \mathbb{N}$. Hence

$$\begin{aligned}\lim_{n\to\infty} N_g^{(2)}(x_n,t) &= \lim_{n\to\infty} \frac{t}{t+\|x_n\| + |g(x_n)|}\\ &= \frac{t}{t+\|x\| + t}\\ &= \frac{t}{2t+\|x\|}\\ &\neq 0\\ &= N_g^{(2)}(x,t).\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} N_g^{(3)}(x_n,t) &= \lim_{n\to\infty} \frac{t}{t+|g(x_n)|}\\ &= \frac{t}{t+t}\\ &= \frac{1}{2}\\ &\neq 0\\ &= N_g^{(3)}(x,t).\end{aligned}$$

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Hence $N_g^{(2)}(.,t)$ and $N_g^{(3)}(.,t)$ are discontinuous at every $x \in S_2$.
Now let $x \notin S_2$ and $z_n \longrightarrow x$ as $n \longrightarrow \infty$. Then $t < |g(x)|$ or $t > |g(x)|$. If $t < |g(x)|$, then there exists an $N \in \mathbb{N}$ such that $t < |g(z_n)|$ for all $n \geq N$. Hence

$$\lim_{n\to\infty} N_g^{(2)}(z_n,t) = 0 = N_g^{(2)}(x,t),$$

and

$$\lim_{n\to\infty} N_g^{(3)}(z_n,t) = 0 = N_g^{(3)}(x,t).$$

Also if $t > |g(x)|$, then there exists an $N \in \mathbb{N}$ such that
$t > |g(z_n)|$ for all $n \geq N$. So

$$\begin{aligned}\lim_{n\to\infty} N_g^{(2)}(z_n,t) &= \lim_{n\to\infty} \frac{t}{t+\|z_n\| + |g(z_n)|}\\ &= \frac{t}{t+\|x\| + |g(x)|}\\ &= N_g^{(2)}(x,t),\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} N_g^{(3)}(z_n,t) &= \lim_{n\to\infty} \frac{t}{t + |g(z_n)|}\\ &= \frac{t}{t + |g(x)|}\\ &= N_g^{(3)}(x,t).\end{aligned}$$

Hence $N_g^{(2)}(.,t)$ and $N_g^{(3)}(.,t)$ are continuous at every $x \in X \setminus S_2$.

(4) Let $t \in T_1$. Then $t = \|x\| + |g(x)|$ and $N_g^{(1)}(x,t) = 0$. Set $t_n = \|x\| + |g(x)| + \frac{1}{n}$ for all $n \in \mathbb{N}$. Clearly $t_n > \|x\| + |g(x)|$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} t_n = t$. So

$$\begin{aligned}\lim_{n\to\infty} N_g^{(1)}(x,t_n) &= \lim_{n\to\infty} \frac{t_n}{t_n+\|x\| + |g(x)|}\\ &= \begin{cases} 1 & x = 0\\ \frac{t}{t+t} = \frac{1}{2} & x \neq 0.\end{cases}\end{aligned}$$

It follows that $\lim_{n\to\infty} N_g^{(1)}(x,t_n) \neq N_g^{(1)}(x,t) = 0$. This shows that $N_g^{(1)}(x,.)$ is discontinuous at every $t \in T_1$. One can easily verify that if $t \in \mathbb{R} \setminus T_1$, then $N_g^{(1)}(x,.)$ is continuous at t.

(5) Let $x \neq 0$. If $t \in T_2$, then $t = |g(x)| > 0$ and $N_g^{(2)}(x,t) = 0$. Set $t_n = (1 + \frac{1}{n})|g(x)|$ for all $n \in \mathbb{N}$. Clearly $t_n \longrightarrow t$ as $n \longrightarrow \infty$, and $t_n > |g(x)|$ for all $n \in \mathbb{N}$. So

$$\begin{aligned}\lim_{n\to\infty} N_g^{(2)}(x,t_n) &= \lim_{n\to\infty} \frac{t_n}{t_n+\|x\| + |g(x)|}\\ &= \frac{t}{2t+\|x\|}\\ &\neq 0\\ &= N_g^{(2)}(x,t).\end{aligned}$$

This shows that $N_g^{(2)}(x,.)$ is discontinuous at every $t \in T_2$.
Let $t \in \mathbb{R} \setminus T_2$. Then $t \leq 0$ or $t \neq |g(x)|$. In the case where $t < 0$ or $t \neq |g(x)|$, the continuity of $N_g^{(2)}(x,.)$ at t can be obviously verified. If $t = 0$ and $t_n \longrightarrow 0$ as $n \longrightarrow \infty$, then for all

subsequences $\{t_{n_k}\}_{k=1}^{\infty} \subseteq \{t_n\}_{n=1}^{\infty}$ satisfying $t_{n_k} \leq |g(x)|$ we have, $\lim\limits_{k\longrightarrow\infty} N_g^{(2)}(x, t_{n_k}) = 0 = N_g^{(2)}(x, 0)$. Also for all subsequences $\{t_{n_k}\}_{k=1}^{\infty} \subseteq \{t_n\}_{n=1}^{\infty}$ satisfying $t_{n_k} > |g(x)|$ we have,

$$\lim_{k\to\infty} N_g^{(2)}(x, t_{n_k}) = \lim_{k\to\infty} \frac{t_{n_k}}{t_{n_k+\|x\|+|g(x)|}} = \frac{0}{0+\|x\| + |g(x)|} = 0 = N_g^{(2)}(x, 0).$$

So $N_g^{(2)}(x, .)$ is continuous at $t = 0$.
Clearly the map $N_g^{(2)}(0, .)$ is continuous at every $t \in \mathbb{R}$ except $t = 0$.

(6) Inspired by part (5), the proof is obvious.
□

Theorem 3.4. *Let X be a normed linear space and $h \in X^*$. If*
$N_h^{(1)} : X \times \mathbb{R} \longrightarrow [0, 1]$ is defined by

$$N_h^{(1)}(x, t) = \begin{cases} 0 & t \leq \|x\| + |h(x)| \\ \frac{t-\|x\|-|h(x)|}{t} & t > \|x\| + |h(x)|, \end{cases}$$

and in the case where $\ker h = \{0\}$, *$N_h^{(2)} : X \times \mathbb{R} \longrightarrow [0, 1]$ is defined by*

$$N_h^{(2)}(x, t) = \begin{cases} 0 & t \leq |h(x)| \\ \frac{t-|h(x)|}{t} & t > |h(x)|, \end{cases}$$

then

(1) *the map $N_h^{(1)}(., t)$ is continuous for all $t \in \mathbb{R}$.*
(2) *the map $N_h^{(1)}(x, .)$ is continuous for all $x \in X$ except $x = 0$.*
(3) *the map $N_h^{(2)}(., t)$ is continuous for all $t \in \mathbb{R}$.*
(4) *the map $N_h^{(2)}(x, .)$ is continuous for all $x \in X$ except $x = 0$.*

Proof. An argument similar to the proofs of the previous theorems can be applied. □

4. Declarations

4.1. Ethical Approval

All of the authors consent to participate and consent to publish the manuscript.

4.2. Competing interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

4.3. Authors' contributions

Ali Reza Khoddami and Fatemeh Kouhsari prepared and reviewed the manuscript.

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