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An Iterative Scheme for Evaluating Simultaneous Roots of Nonlinear Problems with Applications

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Abstract: In this article, an iterative method of two step is proposed for finding all roots simultaneously of polynomial equations. The order of convergence of the proposed algorithm is $2m$, by using any iterative scheme of order m. Numerical tests are performed to confirm the theoretical results and to compare the proposed scheme with existing methods for finding all roots simultaneously.

Keywords: nonlinear equations; simultaneous roots; convergence order; iterative processes

1. Introduction

Let us consider the problem of solving any polynomial equation $f(x) = 0$, where $f : D \subseteq \mathbb{C} \to \mathbb{C}$ a is nonlinear, continuous function on any domain D. Nonlinear equations have wide range of applications in various fields of science and engineering like in the theory of control systems, nonlinear circuits, analysis of transfer functions, various mathematical models, differential and difference equations, eigenvalue problems etc. [1]. This arises the development of large number of numerical methods which take the form of iterative procedures. These algorithms are distributed into single roots, multiple roots and simultaneous root finding methods.

In recent years, methods of finding simultaneous roots are very popular as these methods are more stable and can be applied for parallel computing (for example [2–5]). One of the best known schemes, for approximating the n simple roots α_i, $i = 1, 2, \ldots, n$, of a polynomial f of degree n, is the Weierstrass method [6] defined by the expression

$$x_i^{(k+1)} = x_i^{(k)} - W(x_i^{(k)}) = x_i^{(k)} - \frac{f(x_i^{(k)})}{\prod_{j=1, j\neq i}^{n}(x_i^k - x_j^k)}, \quad k = 0, 1, \ldots$$

In the 1960s, the Weierstrass method was rediscovered by Durand [7], Dochev [8], Kerner [9] and Prešić [10].

The main objective of this paper is to construct iterative schemes, of high order of convergence, for finding all roots of a polynomial equation, simultaneously. For any fixed point iterative method of order m, we construct a new root finding method of order of convergence $2m$, in Section 2. The convergence of the proposed technique is proved in Section 3. We compare the results obtained with the proposed methods and with other known schemes like Newton's method, Ostrowki method [11], Jarratt's method [12], Cordero et al. methods [13,14], Mir et al. schemes [15,16], Shams et al. method [17] and Petkovic et al. scheme [2]. Several numerical experiments are performed in Section 4 by using the mentioned algorithms. With some conclusion in Section 5, we finish the manuscript.

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2. Simultaneous methods construction

Let us consider a polynomial equation $f(x) = 0$ having n simple roots, denoted by ξ_i, $i = 1, 2, \ldots, n$. Consider well-known Newton's scheme, with quadratic convergence:

$$x_i^{(k+1)} = x_i^{(k)} - \frac{f(x_i^{(k)})}{f'(x_i^{(k)})}, k = 0, 1, \ldots \tag{1}$$

We take Weierstrass's correction as follows

$$\frac{f(x_i^{(k)})}{f'(x_i^{(k)})} = W(x_i^{(k)}) = \frac{f(x_i^{(k)})}{\prod_{j=1,j\neq i}^{n}(x_i^k - x_j^k)}, \tag{2}$$

Using (2.2) in (2.1), we have

$$x_i^{(k+1)} = x_i^{(k)} - \frac{f(x_i^{(k)})}{\prod_{j=1,j\neq i}^{n}(x_i^k - x_j^k)}, \tag{3}$$

Replacing x_i^k by $\bar{x}_i{}^k$, kth-iteration obtained by any scheme, in (2.3), we obtain

$$x_i^{(k+1)} = \bar{x}_i^{(k)} - \frac{f(\bar{x}_i^{(k)})}{\prod_{j=1,j\neq i}^{n}(\bar{x}_i^k - \bar{x}_j^k)}, \quad k = 0, 1, \ldots \tag{4}$$

Thus, a new two-step iterative method is constructed for finding all roots simultaneously by taking first step of any existing fixed point iterative method with order of convergence m, having the form $\bar{x}^{k+1} = \psi(x^k)$, and by taking second step as equation (2.4), which is denoted by ϕ_t and having order of convergence $2m$ as follows:

$$\begin{aligned} \bar{x}_i^k &= \psi(x_i^k), \\ x_i^{(k+1)} &= \bar{x}_i^{(k)} - \frac{f(\bar{x}_i^{(k)})}{\prod_{j=1,j\neq i}^{n}(\bar{x}_i^k - \bar{x}_j^k)}, \quad i = 1, 2, ..., n. \end{aligned} \tag{5}$$

2.1. Some special cases

We are going to describe some special cases that we will use in the numerical section.

(1) Taking as first step Newton's method we obtain the following iterative scheme of fourth-order of convergence, denoted by N_{2t}, with the following expression

$$\begin{aligned} \bar{x}_i^{(k)} &= x_i^{(k)} - \frac{f(x_i^{(k)})}{f'(x_i^{(k)})}, \\ x_i^{(k+1)} &= \bar{x}_i^{(k)} - \frac{f(\bar{x}_i^{(k)})}{\prod_{j=1,j\neq i}^{n}(\bar{x}_i^k - \bar{x}_j^k)}, \quad k = 0, 1, \ldots \end{aligned} \tag{6}$$

(2) Now, we consider as first step Ostrowski's method [11], having order of convergence four, then by applying the proposed idea, we obtain an eighth-order of convergence, denoted by T_{4t}, as follows

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$$
\begin{aligned}
y_i^{(k)} &= x_i^{(k)} - \frac{f(x_i^{(k)})}{f'(x_i^{(k)})}, \\
\bar{x}_i^{(k)} &= x_i^{(k)} - \left(\frac{f(y_i^{(k)}) - f(x_i^{(k)})}{2f(y_i^{(k)}) - f(x_i^{(k)})}\right)\left(\frac{f(x_i^{(k)})}{f'(x_i^{(k)})}\right), \\
x_i^{(k+1)} &= \bar{x}_i^{(k)} - \frac{f(\bar{x}_i^{(k)})}{\prod_{j=1,j\neq i}^{n}(\bar{x}_i^k - \bar{x}_j^k)}, \quad k = 0, 1, \ldots
\end{aligned} \tag{7}
$$

(3) Consider as first step of Jarratt's method [12], which has order of convergence four, then applying the second step of (5) we obtain an iterative method of eighth-order convergence, denoted by J_{4t}, as follows

$$
\begin{aligned}
y_i^{(k)} &= x_i^{(k)} - \frac{2f(x_i^{(k)})}{3f'(x_i^{(k)})}, \\
\bar{x}_i^{(k)} &= x_i^{(k)} - \left(1 - \frac{3}{2}\frac{f'(y_i^{(k)}) - f'(x_i^{(k)})}{3f'(y_i^{(k)}) - f'(x_i^{(k)})}\right)\left(\frac{f(x_i^{(k)})}{f'(x_i^{(k)})}\right), \\
x_i^{(k+1)} &= \bar{x}_i^{(k)} - \frac{f(\bar{x}_i^{(k)})}{\prod_{j=1,j\neq i}^{n}(\bar{x}_i^k - \bar{x}_j^k)}, \quad k = 0, 1, \ldots
\end{aligned} \tag{8}
$$

(4) Considering as first step the Cordero et al. scheme [13], which have four order of convergence and applying the proposed scheme, result an iterative scheme on eighth-order convergence, denoted by M_{4t}, as follows

$$
\begin{aligned}
y_i^{(k)} &= x_i^{(k)} - \frac{f(x_i^{(k)})}{f[w_i^{(k)}, x_i^{(k)}]}, \\
\bar{x}_i^{(k)} &= y_i^{(k)} - H(\mu_i^{(k)})\frac{f(y_i^{(k)})}{f[y_i^{(k)}, x_i^{(k)}] - f'(x_i^{(k)})}, \\
x_i^{(k+1)} &= \bar{x}_i^{(k)} - \frac{f(\bar{x}_i^{(k)})}{\prod_{j=1,j\neq i}^{n}(\bar{x}_i^k - \bar{x}_j^k)}, \quad k = 0, 1, \ldots,
\end{aligned} \tag{9}
$$

where $w_i^{(k)} = x_i^{(k)} + \beta f(x_i^{(k)})$ and $\mu_i^{(k)} = \frac{f(y_i^{(k)})}{f'(w_i^{(k)})}$, being β an arbitrary parameter.

3. Convergence Analysis

The convergence analysis of the family of simultaneous method (5) is presented in the following result.

Theorem 1. *Let $\xi_1, \xi_2, \ldots, \xi_n$ be all simple roots of the polynomial equation $f(x) = 0$. If $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}$ are the initial approximations of the roots $\xi_1, \xi_2, \ldots, \xi_n$, respectively, and they are sufficiently close of them, then the order of convergence of method (5) is $2m$.*

Proof. Let

$$\epsilon_i = x_i^{(k)} - \xi_i, \tag{10}$$

$$\epsilon_i' = \bar{x}_i^{(k)} - \xi_i, \tag{11}$$

$$\epsilon_i'' = x_i^{(k+1)} - \xi_i, \tag{12}$$

be the errors of approximations $x_i^{(k)}$, $\bar{x}_i^{(k)}$ and $x_i^{(k+1)}$, respectively.

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Now, we consider the second step of proposed scheme

$$x_i^{(k+1)} = \bar{x}_i^{(k)} - \frac{f(\bar{x}_i^{(k)})}{\prod_{j=1,j\neq i}^{n}(\bar{x}_i^k - \bar{x}_j^k)}, \tag{13}$$

and subtract ξ_i from both sides of (13)

$$x_i^{(k+1)} - \xi_i = \bar{x}_i^{(k)} - \xi_i - \frac{f(\bar{x}_i^{(k)})}{\prod_{j=1,j\neq i}^{n}(\bar{x}_i^k - \bar{x}_j^k)}, \tag{14}$$

Then using the errors defined before, we have

$$\epsilon_i'' = \epsilon_i' - \frac{f(\bar{x}_i^{(k)})}{\prod_{j=1,j\neq i}^{n}(\bar{x}_i^k - \bar{x}_j^k)}, \tag{15}$$

$$\epsilon_i'' = \epsilon_i' \left(1 - \frac{1}{\epsilon_i'} \frac{f(\bar{x}_i^{(k)})}{\prod_{j=1,j\neq i}^{n}(\bar{x}_i^k - \bar{x}_j^k)}\right), \tag{16}$$

$$\epsilon_i'' = \epsilon_i'(1 - E_i), \tag{17}$$

where

$$E_i = \frac{1}{\epsilon_i'} \frac{f(\bar{x}_i^{(k)})}{\prod_{j=1,j\neq i}^{n}(\bar{x}_i^k - \bar{x}_j^k)}. \tag{18}$$

Now, as $f(x)$ can be described in the form

$$f(x) = \prod_{j=1}^{n}(x - \xi_j), \tag{19}$$

equation (18), becomes

$$E_i = \prod_{j=1,\ j\neq i}^{n} \frac{(\bar{x}_i^k - \xi_j)}{(\bar{x}_i^k - \bar{x}_j^k)}, \tag{20}$$

$$\frac{(\bar{x}_i^k - \xi_j)}{(\bar{x}_i^k - \bar{x}_j^k)} = 1 + \frac{(\bar{x}_j^k - \xi_j)}{(\bar{x}_i^k - \bar{x}_j^k)}, \tag{21}$$

and from (3.2) $(\bar{x}_j^k - \xi_j) = O(\epsilon')$. For a simple root ξ and small enough ϵ, $|(\bar{x}_i^k - \bar{x}_j^k)|$ is bounded and away from zero, so equation (3.12) becomes,

$$\frac{(\bar{x}_i^k - \xi_j)}{(\bar{x}_i^k - \bar{x}_j^k)} = 1 + \frac{(\bar{x}_j^k - \xi_j)}{(\bar{x}_i^k - \bar{x}_j^k)} = 1 + O(\epsilon'), \tag{22}$$

Using (3.13) in (3.11)

$$\prod_{j\neq ij=1}^{n} \frac{(\bar{x}_i^{(k)} - \xi_j)}{(\bar{x}_i^k - \bar{x}_j^k)} = (1 + O(\epsilon'))^{n-1} = 1 + (n-1)O(\epsilon') = 1 + O(\epsilon'),$$

$$E_i = 1 + O(\epsilon'),$$

$$E_i - 1 = O(\epsilon').$$

Thus, (3.8) gives

$$\epsilon_i'' = O(\epsilon')^2. \tag{23}$$

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Since the first step has order m, we have $\epsilon_i' = O(\epsilon^m)$. Thus

$$\epsilon_i'' = O(\epsilon^m)^2 = O(\epsilon)^{2m}. \tag{24}$$

Therefore, the order of convergence $2m$ is proved. □

4. Numerical experiments

In this section, different numerical experiments are performed in order to analyze the behavior of the proposed scheme. As discussed in the previous section, we consider Newton's method (denoted by N_{2t}), Ostrowski's scheme (denoted by T_{4t}), Jarratt's method (denoted by J_{4t}), Cordero et al. method (denoted by M_{4t}) modified by applying the second step of our proposal. We compare the results obtained from these modified methods with the known schemes for finding simultaneous roots given by Mir et al. (denoted by MMN_8, MR_6, MR_4), Shams (denoted by S_5) and Petkovic et al. (denoted by P_{10}). The iterative expressions of these schemes for finding all roots simultaneously are:

1. Mir et al. scheme [15], (MMN_8):

$$\begin{aligned}
y_i^{(k)} &= x_i^{(k)} - \frac{f(x_i^{(k)})}{f'(x_i^{(k)})},\\
z_i^{(k)} &= x_i^{(k)} - \frac{1}{\frac{f'(x_i^{(k)})}{f(x_i^{(k)})} - \sum_{j=1,j\neq i}^{n} \frac{1}{(x_i^{(k)}-y_j^{(k)})}},\\
x_i^{(k+1)} &= z_i^{(k)} - \frac{1}{\frac{f'(z_i^{(k)})}{f(z_i^{(k)})} - \sum_{j=1,j\neq i}^{n} \frac{1}{(z_i^{(k)}-z_j^{(k)})} - \alpha}.
\end{aligned}$$

2. Mir et al. scheme [16], (MR_6):

$$\begin{aligned}
z_i^{(k)} &= x_i^{(k)} - \frac{1}{\frac{f'(x_i^{(k)})}{f(x_i^{(k)})} - \sum_{j=1,j\neq i}^{n} \frac{1}{(x_i^{(k)}-x_j^{(k)})}},\\
x_i^{(k+1)} &= z_i^{(k)} - \frac{1}{\frac{f'(z_i^{(k)})}{f(z_i^{(k)})} - \sum_{j=1,j\neq i}^{n} \frac{1}{(z_i^{(k)}-z_j^{(k)})} - \alpha}.
\end{aligned}$$

3. Mir et al. scheme [16], (MR_4):

$$\begin{aligned}
w_i(x_i) &= \frac{f(x_i)}{\prod_{j=1,j\neq i}^{n}(x_i - y_j)},\\
y_i &= x_i - w_i(x_i),\\
z_i &= y_i - \frac{W_i(y_i)}{1 - \alpha W_i(y_i)}.
\end{aligned}$$

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4. Shams et al. scheme [17], (S_5):

$$
\begin{aligned}
y_i^{(k)} &= x_i^{(k)} - \frac{f(x_i^{(k)})}{f'(x_i^{(k)})}, \\
z_i^{(k)} &= y_i^{(k)} - \frac{f'(x_i^{(k)}) - f'(y_i^{(k)})}{\alpha f'(y_i^{(k)}) + (2-\alpha) f'(x_i^{(k)})} \frac{f(x_i^{(k)})}{f'(x_i^{(k)})}, \\
x_i^{(k+1)} &= x_i^{(k)} - \frac{1}{\frac{f'(x_i^{(k)})}{f(x_i^{(k)})} - \sum_{j=1, j\neq i}^{n} \frac{1}{(x_i^{(k)} - z_j^{(k)})}}.
\end{aligned}
$$

5. Petkovic et al. scheme [2], (P_{10}):

$$
\begin{aligned}
y_i^{(k)} &= x_i^{(k)} - \frac{f(x_i^{(k)})}{f'(x_i^{(k)})}, \\
z_i^{(k)} &= y_i^{(k)} - \frac{f(y_i^{(k)}) f(x_i^{(k)})}{(f(x_i^{(k)}) - f(y_i^{(k)}))^2} \frac{f(x_i^{(k)})}{f'(x_i^{(k)})}, \\
t_i^{(k)} &= z_i^{(k)} - \left(\frac{f(y_i^{(k)})(f(x_i^{(k)}))^2}{(f(y_i^{(k)}) - f(z_i^{(k)}))^2} \right) \left(\frac{x_i^{(k)} - z_i^{(k)}}{(f(x_i^{(k)}) - f(z_i^{(k)}))^2} - \right. \\
&\quad \left. \frac{1}{(f(x_i^{(k)}) - f(z_i^{(k)})) f'(x_i^{(k)})} - \frac{f(y_i^{(k)})}{f'(x_i^{(k)})(f(x_i^{(k)}) - f(y_i^{(k)}))^2} \right), \\
x_i^{(k+1)} &= x_i^{(k)} - \frac{1}{\frac{f'(x_i^{(k)})}{f(x_i^{(k)})} - \sum_{j=1, j\neq i}^{n} \frac{1}{(x_i^{(k)} - t_j^{(k)})}}.
\end{aligned}
$$

In the Mir et al. scheme and Shames et al. scheme, we consider the value of $\alpha = 30$. The outcomes of the experiments have been completed in Mathematica version 11.1 software. We choose

$$\|x^{(k+1)} - x^{(k)}\|_2 + \|F(x^{(k+1)})\|_2 < Tol$$

as stopping criterion, with a tolerance, $Tol = 10^{-300}$, where

$$F(x^{(k+1)}) = \left(f(x_1^{(k+1)}), ..., f(x_n^{(k+1)}) \right).$$

The norm of the distance between two approximations is given by $\|x^{(k+1)} - x^{(k)}\|_2$ and the approximated computational order of convergence (ACOC), [18], which is an estimation of the theoretical order of convergence ρ whose expression is

$$\rho \sim ACOC = \frac{\ln \left(\frac{\|x^{(k+2)} - x^{(k+1)}\|_2}{\|x^{(k+1)} - x^{(k)}\|_2} \right)}{\ln \left(\frac{\|x^{(k+1)} - x^{(k)}\|_2}{\|x^{(k)} - x^{(k-1)}\|_2} \right)}.$$

Example 1. *Consider the polynomial*

$$f(x) = (x+1)(x+3)(x^2 - 2x + 2)(x-1)(x^2 - 4x + 5)(x^2 + 4x + 5),$$

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whose exact roots are

$$\xi_1 = -1,\ \xi_2 = -3,\ \xi_3 = 1+i,\ \xi_4 = 1-i,\ \xi_5 = 1,\ \xi_6 = -2+i,\ \xi_7 = -2-i,\ \xi_8 = 2+i,\ \xi_9 = 2-i.$$

The initial approximation has been taken as:
$x_1^{(0)} = -1.3+0.2i,\ x_2^{(0)} = -2.8-0.2i,\ x_3^{(0)} = 1.2+1.3i,\ x_4^{(0)} = 0.8-1.2i,\ x_5^{(0)} = 0.8-0.3i,\ x_6^{(0)} = -1.8+1.2i,$
$x_7^{(0)} = -1.8-1.2i,\ x_8^{(0)} = 1.8+0.8i,\ x_9^{(0)} = 1.8-0.8i.$

Table 1 represents the results obtained from the different iterative methods. We can observed that the proposed methods T_{4t}, J_{4t}, N_{2t} and M_{4t} have less absolute error than other methods and are faster than existing schemes.

Table 1. Numerical results for Example 1 based on various methods.

Method	$\|\|x^{(3)}-x^{(2)}\|\|$	$\|\|x^{(4)}-x^{(3)}\|\|$	$\|\|x^{(5)}-x^{(4)}\|\|$	$\|\|f(x^{(5)})\|\|$	ρ	Time (s)
T_{4t}	2.2(−7)	5.7(−53)	1.3(−417)	1.9(−3331)	8.0232	0.671
J_{4t}	1.9(−7)	2.2(−53)	7.2(−421)	1.7(−3357)	8.0230	1.017
N_{2t}	1.0(−2)	2.3(−8)	6.8(−31)	7.0(−118)	4.1001	0.532
M_{4t}	1.6(−2)	3.2(−5)	1.4(−24)	4.4(−175)	9.8547	0.657
MMN_8	2.3(−5)	1.2(−37)	2.5(−297)	4.9(−2375)	8.1617	0.734
MR_6	9.1(−5)	2.5(−26)	6.4(−158)	1.0(−948)	6.1057	0.812
MR_4	3.5(−1)	8.1(−2)	2.3(−4)	4.0(−11)	4.3266	0.531
S_5	5.4(−4)	9.8(−16)	8.0(−77)	2.5(−379)	5.2340	1.390
P_{10}	2.5(−7)	2.2(−68)	8.6(−681)	4.9(−6800)	10.020	3.313

Example 2. *In this example, let us consider the polynomial*

$$f(x) = (x-4)(x^2-1)(x^4-16)(x^2+9)(x^2+16)(x^2+2x+5)(x^2+2x+2)(x^2-2x+2)$$
$$(x^2-4x+5)(x^2-2x+10),$$

with the exact roots
$\xi_1 = 4,\ \xi_2 = -1,\ \xi_3 = 2,\ xi_4 = -2,\ \xi_5 = 2i,\ \xi_6 = -2i,\ \xi_7 = 3i,\ \xi_8 = -3i,\ \xi_9 = -1+2i,\ \xi_10 = -1-2i,$
$\xi_{11} = -1+i,\ \xi_{12} = -1-i,\ \xi_{13} = 1+i,\ \xi_{14} = 1-i,\ \xi_{15} = 2+i,\ \xi_{16} = 2-i,\ \xi_{17} = 1+3i,\ \xi_{18} = 1-3i,\ \xi_{19} = 4i,$
$\xi_{20} = -4i,\ \xi_{21} = 1.$

The initial approximations we have taken are:
$x_1^{(0)} = 4.2+0.1i,\ x_2^{(0)} = -1.2+0.1i,\ x_3^{(0)} = 2.2+0.1i,\ x_4^{(0)} = -2.2-0.1i,\ x_5^{(0)} = 0.2+2.1i,\ x_6^{(0)} = 0.2-2.1i,$
$x_7^{(0)} = 0.2+3.1i,\ x_8^{(0)} = 0.2-3.1i,\ x_9^{(0)} = -1.2+2.1i,\ x_{10}^{(0)} = -1.2-2.1i,\ x_{11}^{(0)} = -1.2+1.1i,\ x_{12}^{(0)} = -1.2-1.1i,$
$x_{13}^{(0)} = 1.2+1.1i,\ x_{14}^{(0)} = 1.2-1.1i,\ x_{15}^{(0)} = 2.2+1.1i,\ x_{16}^{(0)} = 2.2-1.1i,\ x_{17}^{(0)} = 1.2+3.1i,\ x_{18}^{(0)} = 1.2-3.1i,$
$x_{19}^{(0)} = 0.2+4.1i,\ x_{20}^{(0)} = 0.2-4.1i,\ x_{21}^{(0)} = 1.1+0.2i.$

Table 2 represents the results obtained from different iterative methods, observing that the number of iterations in most of the proposed method is less than the number of iterations of the known schemes. We can observe that T_{4t} and M_{4t} obtain the best results as compared to existing schemes.

Table 2. Numerical results for Example 2 based on various methods.

Method	$\|\|x^{(3)}-x^{(2)}\|\|$	$\|\|x^{(4)}-x^{(3)}\|\|$	$\|\|x^{(5)}-x^{(4)}\|\|$	$\|\|f(x^{(5)})\|\|$	$ACOC$	Time (s)
T_{4t}	5.6(−15)	4.8(−113)	8.4(−897)	5.8(−7155)	8.0159	3.203
J_{4t}	9.5(−15)	4.1(−111)	3.0(−881)	1.5(−7030)	8.0145	6.812
N_{2t}	2.7(−4)	1.3(−14)	2.0(−55)	6.7(−207)	4.0848	2.359
M_{4t}	2.7(−4)	1.3(−14)	2.4(−64)	1.9(−448)	8.0343	1.703
MMN_8	3.0(−14)	1.9(−108)	3.1(−862)	5.5(−6883)	8.0215	4.345
MR_6	9.7(−7)	7.2(−38)	6.3(−227)	4.9(−1358)	6.0741	3.734
MR_4	2.6(−1)	1.3(−2)	1.0(−7)	9.7(−19)	5.7548	1.328
S_5	8.8(−7)	4.2(−30)	6.0(−146)	2.9(−716)	5.0487	9.390
P_{10}	1.7(−17)	7.1(−166)	3.7(−1652)	7.8(−16500)	10.018	31.219

Example 3. *Consider the polynomial of degree seven*

$$f(x) = x^7 + x^5 - 10x^4 - x^3 - x + 10,$$

with exact roots

$$\xi_1 = 2,\ \xi_2 = 1,\ \xi_3 = -1,\ \xi_4 = i,\ \xi_5 = -i,\ \xi_6 = -1+2i,\ \xi_7 = -1-2i.$$

The initial approximations taken are:
$x_1^{(0)} = 1.66 + 0.23i,\ x_2^{(0)} = 1.36 - 0.31i,\ x_3^{(0)} = -0.76 + 0.18i,\ x_4^{(0)} = -0.35 + 1.17i,\ x_5^{(0)} = 0.29 - 1.37i,$
$x_6^{(0)} = -0.75 + 2.36i,\ x_7^{(0)} = -1.27 - 1.62i.$

Table 3 represents the results obtained from the different iterative methods used until now.

Table 3. Numerical results for Example 3 based on various methods.

Method	$\|\|x^{(3)}-x^{(2)}\|\|$	$\|\|x^{(4)}-x^{(3)}\|\|$	$\|\|x^{(5)}-x^{(4)}\|\|$	$\|\|f(x^{(5)})\|\|$	$ACOC$	Time (s)
T_{4t}	6.8(−5)	7.7(−34)	2.2(−265)	2.3(−2115)	8.0366	0.985
J_{4t}	1.2(−4)	8.5(−32)	4.9(−249)	1.7(−1989)	8.0390	0.927
N_{2t}	5.5(−2)	6.9(−6)	2.0(−21)	2.1(−81)	4.1600	0.311
M_{4t}	5.7(−2)	4.6(−6)	2.1(−35)	2.6(−266)	8.0321	1.109
MMN_8	3.6(−3)	1.9(−23)	1.5(−183)	1.3(−1462)	8.1090	0.735
MR_6	8.5(−3)	4.8(−13)	7.5(−76)	4.7(−453)	6.1383	0.422
MR_4	1.9(−1)	5.8(−3)	9.0(−9)	9.2(−30)	4.4467	0.376
S_5	6.5(−3)	2.5(−11)	3.1(−52)	4.6(−256)	5.1550	0.530
P_{10}	7.7(−5)	6.8(−42)	3.7(−411)	3.9(−4103)	10.121	2.219

Example 4. *Now, let us consider the polynomial*

$$f(x) = (x+1)(x+2)(x^2 - 2x + 2)(x^2 + 1)(x-2)(x+2-i),$$

whose exact roots are

$$\xi_1 = -1,\ \xi_2 = -2,\ \xi_3 = 1+i,\ \xi_4 = 1-i,\ \xi_5 = +i,\ \xi_6 = -i,\ \xi_7 = 2,\ \xi_8 = -2+i.$$

In this example, we take as initial approximations:
$x_1^{(0)} = -1.3 + 0.2i,\ x_2^{(0)} = -2.2 - 0.3i,\ x_3^{(0)} = 1.3 + 1.2i,\ x_4^{(0)} = 0.7 - 1.2i,\ x_5^{(0)} = -0.2 + 0.8i,\ x_6^{(0)} = 0.2 - 1.3i,$
$x_7^{(0)} = 2.2 - 0.3i,\ x_8^{(0)} = -2.2 + 0.7i.$

In Table 4 we can observe the results for this example. Some proposed methods show much better results than the other ones.

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Table 4. Numerical results for Example 4 based on various methods.

Method	$\|\|x^{(3)} - x^{(2)}\|\|$	$\|\|x^{(4)} - x^{(3)}\|\|$	$\|\|x^{(5)} - x^{(4)}\|\|$	$\|\|f(x^{(5)})\|\|$	$ACOC$	Time (s)
T_{4t}	6.0(−11)	3.2(−82)	2.7(−652)	1.9(−5210)	8.0151	1.047
J_{4t}	1.2(−10)	1.3(−79)	1.7(−631)	4.4(−5044)	8.0154	1.738
N_{2t}	1.3(−3)	3.2(−12)	1.7(−46)	2.1(−181)	4.0672	0.511
M_{4t}	1.2(−3)	1.1(−15)	3.9(−111)	4.3(−872)	8.1326	0.500
MMN_8	3.0(−9)	7.4(−69)	5.2(−545)	3.8(−4357)	8.0358	1.113
MR_6	2.0(−4)	3.4(−22)	8.4(−129)	2.8(−766)	6.0439	1.578
MR_4	8.9(−2)	2.5(−4)	2.2(−14)	1.9(−52)	4.4202	0.390
S_5	3.2(−5)	9.6(−22)	2.4(−104)	3.5(−515)	5.0449	1.828
P_{10}	9.1(−5)	1.1(−42)	6.4(−420)	1.2(−4192)	10.139	2.593

Example 5. *In this example, we consider*

$$f(x) = (x+3)(x-2i)(x^2+4x+5)(x^2-4x+5),$$

whose roots are

$$\xi_1 = -3,\ \xi_2 = 2i,\ \xi_3 = -2+i,\ \xi_4 = -2-i,\ \xi_5 = 2+i,\ \xi_6 = 2-i.$$

In this case, we use the initial approximations:
$x_1^{(0)} = -0.33+0.2i,\ x_2^{(0)} = 0.3+2.3i,\ x_3^{(0)} = -2.3+1.2i,\ x_4^{(0)} = -2.3-1.2i,\ x_5^{(0)} = 2.3+1.2i,\ x_6^{(0)} = 2.3-1.2i.$

Based on the results obtained, shown in Table 5, it is clear that proposed methods are, in general, faster than the existing ones.

Table 5. Numerical results for Example 5 based on various methods.

Method	$\|\|x^{(3)} - x^{(2)}\|\|$	$\|\|x^{(4)} - x^{(3)}\|\|$	$\|\|x^{(5)} - x^{(4)}\|\|$	$\|\|f(x^{(5)})\|\|$	$ACOC$	Time (s)
T_{4t}	4.4(−30)	2.4(−236)	1.7(−1886)	2.9(−15085)	8.0051	0.735
J_{4t}	6.3(−30)	4.4(−235)	2.9(−1876)	1.6(−15003)	8.0051	0.875
N_{2t}	1.7(−7)	1.2(−28)	3.7(−113)	6.7(−449)	4.0305	0.36
M_{4t}	1.6(−5)	4.3(−32)	1.2(−244)	1.3(−1942)	8.0400	0.360
MMN_8	1.0(−18)	8.8(−146)	3.7(−1162)	5.9(−9291)	8.0086	0.422
MR_6	1.6(−7)	9.2(−43)	2.6(−256)	1.4(−1538)	6.0637	0.468
MR_4	2.2(−2)	1.5(−7)	1.5(−27)	1.2(−105)	4.0698	0.344
S_5	8.9(−9)	6.3(−41)	8.6(−202)	9.9(−1004)	5.0354	0.937
P_{10}	9.4(−37)	2.1(−362)	1.9(−3617)	1.9(−36170)	10.014	0.891

Example 6. *To show the real life problems of nonlinear equations, we apply the proposed methods, for Classical Dynamics of Blocks on an inclined planes [19]. The role of eigenvalue/eigenvector pairs is presented here. It is shown that the lower frequencies of eigenvalue/eigenvector pairs dominant the solution of this system. The dynamical problem is to determine block acceleration a_B and cable tensions T_A, T_B and T_C just after the system is released from rest. The fourth order matrix from Newton's laws are given by:*

$$\begin{bmatrix} 0.5m_A & -1 & 0 & 0 \\ m_B & 0 & 1 & 0 \\ 0.5\frac{J}{R} & 0 & -R & R \\ 0.5m_p & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} a_B \\ T_A \\ T_B \\ T_C \end{bmatrix} = \begin{bmatrix} -\mu m_A g \\ m_B g(sin(45) - \mu cos(45)) \\ 0 \\ 0 \end{bmatrix}.$$

The following numerical values are used:

1. *Block & Pulley Weights:*
   $W_A = 200\ lb,\ W_B = 300\ lb,\ W_P = 50\ lb,$

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2. *Surface coefficient of friction* $\mu = 0.2$,
3. *Pulley Mass Moment of Inertia*
$J = \frac{1}{2}m_P R^2,\ A_B = \frac{a_B}{g},\ F_A = \frac{T_A}{m_A g},\ F_A = \frac{T_B}{m_A g},\ F_C = \frac{T_C}{m_A g},\ K_B = \frac{m_B}{m_A},\ K_B = \frac{T_B}{m_A}$
with $K_B = 0.67, K_P = 0.25$.

The dimensionless numerical matrix A is as follows:

$$A = \begin{bmatrix} 0.5 & -1 & 0 & 0 \\ 0.67 & 0 & 1 & 0 \\ 0.063 & 0 & -1 & 1 \\ 0.13 & 1 & -1 & -1 \end{bmatrix}.$$

The characteristic equation of matrix A is given by:

$$f(x) = (x^2 - 0.807213x + 0.072386)(x^2 + 0.230721x + 0.280855),$$

whose roots are
$\xi_1 \approx -1.153605 + 1.215626i,\ \xi_2 \approx -1.153605 - 1.215626i,\ \xi_3 \approx 0.4036065 + 0.7489738i,\ \xi_4 \approx 0.4036065 - 0.7489738i.$

The initial approximation used are :
$x_1^{(0)} = -3 + 3i,\ x_2^{(0)} = -3 - 3i,\ x_3^{(0)} = 2 + 2i,\ x_4^{(0)} = 2 - 2i.$

The results obtained by the different methods appear in Table 6. The characteristics of these numerical results are similar to those in the other examples.

Table 6. Numerical results for Example 6 based on various methods.

Method	$\|\|x^{(3)} - x^{(2)}\|\|$	$\|\|x^{(4)} - x^{(3)}\|\|$	$\|\|x^{(5)} - x^{(4)}\|\|$	$\|\|f(x^{(5)})\|\|$	*ACOC*	Time (s)
T_{4t}	2.7(−5)	1.9(−39)	1.9(−312)	2.7(−2495)	8.0544	0.359
J_{4t}	2.8(−5)	3.3(−39)	1.8(−3101)	2.0(−2479)	8.0541	0.359
N_{2t}	1.9(−1)	2.5(−4)	8.7(−16)	1.1(−60)	4.3190	0.375
M_{4t}	1.9(−1)	1.6(−5)	7.9(−38)	4.2(−295)	8.4883	0.515
MMN_8	1.2(−1)	2.4(−8)	6.9(−62)	2.3(−489)	8.1543	0.406
MR_6	3.9(−1)	2.6(−3)	3.0(−16)	4.6(−93)	6.4975	0.406
MR_4	8.7(−1)	5.0(−1)	2.1(−1)	7.2(−2)	-1.4440	0.327
S_5	1.2(−1)	4.0(−5)	2.8(−22)	4.4(−107)	5.2972	0.407
P_{10}	5.4(−1)	6.6(−6)	5.5(−54)	1.1(−533)	10.047	0.641

Example 7. *In this example, we apply the proposed methods for a mechanism composed of a block and disc connected by a pulley system [19]. The role of eigenvalue/eigenvector pairs is shown that the lower frequencies of eigenvalue/eigenvector pairs dominant the solution of this system. The dynamical problem is to determine block acceleration a_B and disc acceleration a_A disc friction force F_A, cable tensions T_A, T_B, T_C and T_D just after the system is released from rest. The seventh order matrix from Newton's laws are given by:*

$$\begin{bmatrix} m_A & 0 & -1 & 1 & 0 & 0 & 0 \\ 5J_A & 0 & 0.2 & 0 & 0 & 0 & 0 \\ 10J_C & 0 & 0 & -0.1 & 0 & 0.1 & 0 \\ 0 & m_D & 0 & 0 & 1 & -1 & -1 \\ 0 & 10J_D & 0 & 0 & 0 & -0.1 & 0.1 \\ 0 & m_B & 0 & 0 & -1 & 0 & 0 \\ -0.5 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_A \\ a_B \\ F_A \\ T_A \\ T_B \\ T_C \\ T_D \end{bmatrix} = \begin{bmatrix} 0 \\ \tau_A \\ 0 \\ -m_D g \\ 0 \\ -m_B g \\ 0 \end{bmatrix}.$$

The following numerical values are used:

1. *Disk & Block Masses:* $m_A = 20kg,\ m_B = 10kg$,

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2. *Pulley Masses:* $m_C = m_D = 15kg$,
3. *Disk Mass Moment of Inertia:* $J_A = 0.4kg - m^2$,
4. *Pulleys Mass Moment of Inertia:* $J_C = J_D = 0.075kg - m^2$,
5. *External Torque:* $\tau_A = 10N - m$.

The dimensionless numerical matrix A is as follows:

$$A = \begin{bmatrix} 20 & 0 & -1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0.75 & 0 & 0 & -0.1 & 0 & 0.1 & 0 \\ 0 & 15 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0.75 & 0 & 0 & 0 & -0.1 & 0.1 \\ 0 & 10 & 0 & 0 & -1 & 0 & 0 \\ -0.5 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The characteristic equation of matrix A is given by

$$\begin{aligned} f(x) &= (x - 20.0364)(x - 0.765264)(x - 0.0923177)(x^2 + 0.388177x + 0.180708) \\ &\quad (x^2 + 0.505777x + 0.071346), \end{aligned}$$

whose roots are
$\xi_1 \approx 20.0364$, $\xi_2 \approx 0.765264$, $\xi_3 \approx 0.0923177$, $\xi_4 \approx -0.194088 + 1.33019i$, $\xi_5 \approx -0.194088 - 1.33019i$, $\xi_6 \approx -0.252888 + 0.085985i$, $\xi_7 \approx -0.252888 - 0.085985i$.

The initial approximation has been taken as:
$x_1^{(0)} = 21.5 + 0.01i$, $x_2^{(0)} = 1.7 + 0.02i$, $x_3^{(0)} = 0.5 + 0.03i$, $x_4^{(0)} = -0.55 + 2.3i$, $x_5^{(0)} = -0.55 - 2.3i$, $x_6^{(0)} = -0.55 + 0.02i$, $x_7^{(0)} = -0.55 - 0.02i$.

The obtained numerical results for this example appear in Table 7.

Table 7. Numerical results for Example 7 based on various methods.

Method	$\|\|x^{(3)} - x^{(2)}\|\|$	$\|\|x^{(4)} - x^{(3)}\|\|$	$\|\|x^{(5)} - x^{(4)}\|\|$	$\|\|f(x^{(5)})\|\|$	*ACOC*	Time (s)
T_{4t}	2.3(−2)	1.9(−9)	6.3(−66)	2.3(−517)	8.0187	0.953
J_{4t}	2.4(−2)	3.3(−9)	6.1(−64)	1.8(−501)	8.1425	2.125
N_{2t}	1.6(−1)	1.5(−2)	5.4(−6)	2.3(−19)	3.9359	0.766
M_{4t}	9.0(−2)	3.4(−4)	3.3(−22)	6.5(−166)	8.0586	0.812
MMN_8	5.5(−1)	8.7(−2)	1.2(−5)	1.2(−35)	6.8550	5.359
MR_6	4.7(−1)	1.7(−1)	2.5(−3)	1.9(−14)	8.0862	5.435
MR_4	3.0(−1)	4.3(−1)	1.6(−1)	3.6(−2)	0.16920	2.735
S_5	5.5(−1)	1.3(−1)	2.9(−3)	1.1(−9)	4.8264	1.172
P_{10}	1.4(−1)	3.0(−4)	1.1(−28)	9.0(−276)	10.067	5.688

In the last table, we show the number of iterations needed in each example for satisfying the stopping criterion.

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Table 8. Comparison of the number of iterations with different examples and methods.

Method	NI_{ex1}	NI_{ex2}	NI_{ex3}	NI_{ex4}	NI_{ex5}	NI_{ex6}	NI_{ex7}	Total NI	Average NI
T_{4t}	5	5	6	5	5	5	6	37	5.29
J_{4t}	5	5	6	5	5	5	6	37	5.29
N_{2t}	7	7	7	7	6	8	9	51	7.29
M_{4t}	7	6	7	6	6	7	7	46	6.57
MMN_8	6	5	6	5	5	6	8	41	5.86
MR_6	6	6	6	6	6	7	8	45	6.43
MR_4	9	8	8	8	7	10	11	61	8.71
S_5	6	6	7	6	6	7	9	47	6.71
P_{10}	5	5	5	5	4	5	7	36	5.14

NI = Number of iterations.

5. Conclusions

In this manuscript, we combine the Weierstrass method with any iterative scheme for designing new iterative algorithms for approximating simultaneously all the roots of a real or complex polynomial. The order of convergence of the proposed schemes is $2m$, being m the order of the iterative method that combine with Weierstrass's scheme.

Several numerical tests confirm the theoretical results and allow us to compare the proposed schemes with other known in the literature.

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