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The Fourier Continuous Derivative: A New Approach to Fractional Differentiation

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The Fourier Continuous Derivative: A New Approach to Fractional Differentiation

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Abstract: The Fourier Continuous Derivative (D_C) offers a unique perspective on fractional differentiation grounded in the theory of Fourier series. This approach has the potential to address problems across various disciplines, including physics, engineering, and mathematics. The primary insight underpinning this approach is that a convex function defined on $\mathbb{Z}$ retains its convexity on $\mathbb{R}$. This paper delves into the Fourier Continuous Derivative, compares it with traditional fractional derivatives, and outlines its possible real-world applications, such as modeling viscoelastic materials, solving wave equations, and financial data analysis.

Keywords: fractional calculus; fourier series; fourier transform; differentiation operators; continuous derivatives; riemann-liouville derivative; mathematical analysis

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1. Introduction

Fractional calculus is a branch of mathematical analysis that generalizes differentiation and integration to non-integer orders. In recent years, fractional differentiation has attracted significant attention due to its demonstrated applications in various scientific fields such as physics, engineering, and bioengineering. Among the existing fractional differentiation approaches, the Fourier Continuous Derivative (D_C) offers a novel perspective firmly grounded in Fourier analysis.

The key motivation behind the D_C operator is the need for a well-behaved fractional derivative that retains the convexity properties of ordinary integer derivatives. Additionally, the D_C aims to overcome limitations in previous fractional derivatives regarding non-smooth functions and dependency preservation.

Therefore, this paper introduces the D_C, systematically explores its fundamental mathematical properties, and exemplifies its application on archetypal functions. The central hypotheses tested are:

1. The D_C satisfies key invariance properties including linearity, exponential function preservation, and chain rule extension.
2. The D_C retains convexity and dependency compared to classical fractional derivatives.
3. The D_C provides an efficient approach for fractional differentiation across various periodic functions.

The Continuous Fourier Derivative (D_C) represents a new perspective for fractional differentiation, based on solid foundations of Fourier transform theory and series.

The D_C operator possesses distinctive characteristics that make it promising for solving complex problems in mathematics, physics, and engineering:

- It is uniquely and coherently defined for every real order $\mu \in \mathbb{R}$, maximizing its flexibility of use.
- It preserves fundamental properties such as linearity, preservation of the exponential function, and the chain rule, ensuring its formal correctness.
- It preserves the convexity of functions, unlike other commonly used fractional derivatives.
- It can represent non-differentiable functions locally, thus generalizing the classical notion of derivative.
- It naturally connects with Fourier series, making it suitable for periodic problems.
- It has solid spectral foundations in Fourier transforms, enhancing its numerical applicability and computational stability compared to other methods.

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The D_C operator shows promising potential for obtaining more realistic results in certain applications due to its maintained properties, with the preservation of convexity and its connection with function analysis. However, thorough validation and careful consideration of computational complexity and physical interpretation are required to conclusively determine its superiority over other fractional operators in a general context. The choice of the most suitable and realistic fractional operator will ultimately depend on the specific problem at hand and will require a case-by-case analysis.

In summary, the D_C operator provides a promising tool for addressing problems traditionally elusive to conventional formalism. Its future impact appears disruptive.

Advances of the Continuous Fourier Derivative Approach

The D_C operator addresses long-standing issues in innovative ways:

- **Fractional Derivative Properties**: Previous work sought fractional derivatives preserving key traits like convexity, transformation affinity, and natural function dependence. D_C elegantly achieves this by anchoring in Fourier theory.
- **Non-Locally Differential Functions**: The feasibility of deriving locally non-differentiable functions was debated. D_C's spectral view lays groundwork to study this intriguing new math class.
- **Periodic Context Generalization**: Generalizing derivatives to periodic contexts posed challenges. D_C seamlessly connects to Fourier series, fulfilling this hurdle.
- **Non-Local Differential Models**: Problems like viscoelastic material modeling demanded non-local operators beyond classic schemes. D_C shows promise tackling such systems.
- **Broad Impacts**: In summary, D_C nourishes fractional calculus with renewed vision, solving issues predecessors missed while opening entirely new questions. Its impact will surely be tremendous.

2. Implications of the Fourier Continuous Derivative Operator

The Fourier Continuous Derivative (DC) introduces a novel approach to fractional differentiation, grounded in Fourier series theory. This methodology is distinguished by its capability to retain the convexity of functions and is coherently defined for every real order $\mu \in \mathbb{R}$, significantly expanding its flexibility of use compared to traditional fractional derivatives. The key contributions and debates that the DC operator aims to resolve or elucidate include:

- **Convexity Retention:** Unlike common fractional derivatives that do not preserve function convexity, DC maintains this crucial property, essential in various applications where convexity is a desirable or necessary feature for mathematical analysis or in modeling physical and engineering phenomena.
- **Coherent Definition for All Real Orders:** The DC operator is unique in its capacity to be coherently and uniquely defined for each real order μ, thus maximizing its utility across a wide range of applications. This contrasts with other fractional differentiation approaches that may have definition restrictions or applicability limitations.
- **Preservation of Fundamental Properties:** The DC operator preserves fundamental properties such as linearity, the preservation of the exponential function, and the chain rule. This consistency with classical differentiation ensures its formal correctness and facilitates its interpretation and application in mathematical and engineering problems.
- **Applicability to Non-differentiable Functions and Periodicity:** DC can locally represent non-differentiable functions, thus generalizing the classical notion of derivative. Furthermore, its natural connection with Fourier series makes it particularly suitable for periodic problems, offering a solid framework for the fractional differentiation of functions representable as Fourier series.
- **Numerical Challenges and Noise Sensitivity:** Although the DC operator has many advantages, it also faces challenges such as numerical complexity in certain applications and sensitivity to

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noise, which can affect the accuracy of the results obtained with this operator. These challenges underscore the importance of ongoing research to develop robust and efficient numerical methods for its implementation.

In summary, the Fourier Continuous Derivative (DC) offers a new and promising perspective for fractional differentiation, addressing previous limitations and opening new avenues for research and application across various disciplines. However, as with any mathematical tool, it is crucial to understand both its strengths and limitations to maximize its potential.

3. Concepts and Definitions

The Fourier Continuous Derivative (D_C) is an extension of differentiation to fractional orders. This section elucidates the foundational properties and definitions anchoring the D_C operator.

Definition 1. *The operator D_C is defined as a Fourier Continuous Derivative (D_C) if, for all $\mu \in \mathbb{R}$:*

$$\begin{aligned} D^\mu(af(x)+bg(x)) &= aD^\mu(f(x))+bD^\mu(g(x)), \\ D^\mu(e^x) &= e^x, \\ D^\mu(f(g(x))) &= D^\mu(f(u))(D^1 g(x))^\mu, \end{aligned}$$

where $u = g(x) = ax + b$ and $a \in \mathbb{R}$ or $a \in \mathbb{C}$.

Property 1. *The differentiation rule for the linear combination of functions is given by:*

$$\frac{d^\mu}{dx^\mu}(af(x)+bg(x)) = a\frac{d^\mu f(x)}{dx^\mu} + b\frac{d^\mu g(x)}{dx^\mu}, \quad \text{for } \mu \in \mathbb{N}_0.$$

Property 2. *For the exponential function, the differentiation rule of order μ is:*

$$\frac{d^\mu e^x}{dx^\mu} = e^x, \quad \text{for } \mu \in \mathbb{N}_0.$$

Property 3. *Let $f : \mathbb{R} \to \mathbb{R}$ and $u : \mathbb{C} \to \mathbb{C}$, where $u = g(x) = ax + b$, with $a \in \mathbb{R}$ or $a \in \mathbb{C}$, $x \in \mathbb{R}$, and $b \in \mathbb{R}$. The differentiation rule of order μ for composite functions, when $g(x)$ is linear, is:*

$$\frac{d^\mu f(g(x))}{dx^\mu} = \frac{d^\mu f(u)}{du^\mu}\left(\frac{d^1 g(x)}{dx^1}\right)^\mu, \quad \text{for } \mu \in \mathbb{N}_0.$$

The highlighted properties set the stage for a derivative operator that is congruent with both classical differentiation and Fourier series derivatives. Significantly, the Fourier Continuous Derivative is commutative with linear functions, retains the exponential function, and preserves the order of composite functions if the inner function is linear.

4. Limitations of D_C

Though the D_C operator boasts several benefits, it is not without constraints. Primary challenges encompass:

- **Numerical Complexity:** The intricacy of D_C can pose numerical challenges in certain applications.
- **Sensitivity to Noise:** Noise can detrimentally impact the precision of results garnered via the D_C operator.
- **Frequency Representation:** To harness the full potential of D_C, functions under examination should be suitably represented in the frequency domain.

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4.1. Significance of the Fourier Continuous Derivative's Properties

The properties of the Fourier Continuous Derivative are pivotal, as they certify the operator's well-defined character and its capacity to yield accurate outcomes.

- **Linearity:** The inaugural property, ensuring linearity, validates the operator's alignment with classical differentiation. Classical differentiation's linearity mandates that a linear combination of functions' derivative is the derivatives' linear combination. This trait is mirrored by the Fourier Continuous Derivative, enabling differentiation of functions expressed as linear combinations.
- **Preservation of Exponential Function:** By upholding the exponential function, the second property assures the operator's compatibility with the Fourier series' derivative. The Fourier series derivative of an exponential function remains an exponential function with identical arguments. This is conserved by the Fourier Continuous Derivative, allowing for differentiation of Fourier series-represented functions.
- **Preservation of Order of Composite Functions:** The third property ensures the operator's coherence with fractional derivatives of composed functions. The Fourier Continuous Derivative conserves the order of composite functions having linear inner components, facilitating the differentiation of functions integrating a linear function with another.

Such properties make the Fourier Continuous Derivative versatile and influential for diverse applications, encompassing fractional differential equation solutions, non-smooth wave and fluid analyses, non-linear system stability assessments, innovative image and signal processing techniques, and mathematical theory evaluations. As a result, the Fourier Continuous Derivative is a potent tool in the realms of mathematics, physics, and engineering.

5. Invariants in Mathematics

Invariants represent mathematical object properties that remain unchanged under certain transformations. For instance, a square's area is invariant; it remains unaltered irrespective of the square's rotation or translation. Invariants have broad applications in numerous mathematical fields, including geometry, topology, algebra, and number theory.

Regarding the Fourier Continuous Derivative, invariance properties are essential to affirm the operator's well-defined nature. For example, the Fourier Continuous Derivative should remain invariant under linear transformations, such as translations and rotations, given that the Fourier transform shares this invariance. Furthermore, the addition of constants should not alter the Fourier Continuous Derivative, since a constant function's derivative is zero.

6. Motivation for the Fourier Continuous Derivative

The motivation behind the Fourier Continuous Derivative (D_C) lies in the need for a well-defined fractional derivative operator that aligns with classical differentiation. Ensuring its validity, D_C meets all invariant property criteria. Moreover, D_C facilitates the differentiation of non-smooth functions, which remains a limitation of numerous other fractional derivative operators. With relatively straightforward implementation, D_C promises practicality across a range of applications.

7. Advantages over Other Methods

The D_C boasts several advantages over alternative fractional differentiation methods:

- It is well-defined for all real values of differentiation order.
- Consistency with classical differentiation offers easier result interpretation.
- Enables differentiation of non-smooth functions.

8. Example of D_C

Consider $f(x) = \cos(x)$ and let D^μ symbolize a D_C operator where $\mu \in \mathbb{R}$.

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$$f(x) = \cos(x) = \frac{e^{ix} - e^{-ix}}{2} \tag{1}$$
$$D^{\mu}(\cos(x)) = \frac{1}{2}D^{\mu}(e^{ix}) - \frac{1}{2}D^{\mu}(e^{-ix}) \tag{2}$$
$$D^{\mu}(\cos(x)) = \frac{1}{2}i^{\mu}(e^{ix}) - \frac{1}{2}(-i)^{\mu}(e^{-ix}) \tag{3}$$
$$D^{\mu}(\cos(x)) = \frac{1}{2}e^{i(x+\frac{\pi\mu}{2})} - \frac{1}{2}e^{-i(x+\frac{\pi\mu}{2})} \tag{4}$$
$$D^{\mu}(\cos(x)) = \cos(x + \frac{\pi\mu}{2}) \tag{5}$$

This exemplifies the utility of the Fourier Continuous Derivative to differentiate any function representable as a Fourier series.

9. Derivative over a Fourier Series

9.1. Fourier Series

A function f that satisfies the following condition, known as the weak Fourier condition:

$$\int_{-\infty}^{\infty} |f(t)|dt < \infty \tag{6}$$

can be expressed as a Fourier series, which relies on sine and cosine functions and periodicity.

$$f(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty}(a_j \cos(wjx) + b_j \sin(wjx)) \tag{7}$$

Here, $w = \frac{2\pi}{T}$ is the fundamental frequency, and T is the integration interval (periodicity).

$$a_j = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos\left(\frac{2j\pi}{T}\right)tdt \tag{8}$$
$$b_j = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin\left(\frac{2j\pi}{T}\right)tdt \tag{9}$$

Since the qth-order derivative of a function is equal in value to the qth-order derivative of its Fourier series representation, its derivative formula becomes:

$$\frac{d^q f(x)}{dx^q} = \sum_{j=1}^{\infty}(wj)^{\mu}(a_j \cos(wjx + \frac{\pi}{2}\mu) + b_j \sin(wjx + \frac{\pi}{2}\mu)) \tag{10}$$

$$D_C^{\mu} f(x) = \sum_{j=1}^{\infty}(w_j)^{\mu}(a_j \cos(w_j x + \frac{\pi}{2}\mu) + b_j \sin(w_j x + \frac{\pi}{2}\mu)) \tag{11}$$

- D^{μ}: The Fourier Continuous Derivative operator.
- $f(x)$: The function to be differentiated.
- j: The index of the Fourier coefficient.
- w_j: The frequency of the jth Fourier coefficient.
- a_j: The real part of the jth Fourier coefficient.
- b_j: The imaginary part of the jth Fourier coefficient.
- μ: The order of the derivative.

The Fourier series also has a complex form in its representation:

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$$f(x) = \sum_{j=-\infty}^{\infty} (c_j e^{(wj)ix}) \tag{12}$$

$$c_j = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-2\pi i \frac{j}{T} t} dt \tag{13}$$

$$\frac{d^\mu f(x)}{dx^\mu} = \sum_{j=-\infty}^{\infty} (c_j (wji)^\mu e^{wjix}) \tag{14}$$

10. D_C over a Fourier Series

The expression of a function through a Fourier series allows us to generalize the derivative of such a series by extending the coefficient μ to $\mathbb{R}$. It suffices to demonstrate that the application of the D_C operator to such a series complies with its conditions.

Theorem 1. *Let f be a function defined on the interval $[a, b]$ that satisfies the weak Fourier condition, and let D^μ be an operator denoted as D_C for all $\mu \in \mathbb{R}$. Then, it holds that:*

$$D^\mu(f(x)) = \sum_{j=-\infty}^{\infty} c_j (w_j i)^\mu e^{w_j ix},$$

where $c_j = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-2\pi i \frac{j}{T} t} dt$.

Proof. We proceed by the following steps:

Lemma 1. *The operator D^μ is linear.*

Proof. For any functions f and g and scalars a and b:

$$D^\mu(af(x) + bg(x)) = aD^\mu f(x) + bD^\mu g(x)$$

This follows from the definition of D_C. □

Lemma 2. *For any constant k, $D^\mu(e^{kx}) = k^\mu e^{kx}$.*

Proof. We prove this by induction on the integer part of μ, then extend to all real μ.

Base case: For $\mu = 0$, $D^0(e^{kx}) = e^{kx} = k^0 e^{kx}$.

Inductive step: Assume $D^n(e^{kx}) = k^n e^{kx}$ for some integer n. Then:

$$\begin{aligned} D^{n+1}(e^{kx}) &= D^1(D^n(e^{kx})) \\ &= D^1(k^n e^{kx}) \\ &= k^n D^1(e^{kx}) \\ &= k^n \cdot k e^{kx} \\ &= k^{n+1} e^{kx} \end{aligned}$$

For non-integer μ, we can use the property that D^μ is a continuous function of μ to extend this result to all real μ. □

Now, we can express $f(x)$ as a Fourier series:

$$f(x) = \sum_{j=-\infty}^{\infty} c_j e^{w_j ix}$$

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Applying D^μ to both sides and using linearity:

$$D^\mu(f(x)) = D^\mu\left(\sum_{j=-\infty}^{\infty} c_j e^{w_j ix}\right) = \sum_{j=-\infty}^{\infty} c_j D^\mu(e^{w_j ix})$$

Using Lemma 2 with $k = w_j i$:

$$D^\mu(f(x)) = \sum_{j=-\infty}^{\infty} c_j (w_j i)^\mu e^{w_j ix}$$

This completes the proof. □

11. Symmetry of D_C

Theorem 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function expressible as a combination of sine and cosine functions, and let D^q be the continuous Fourier derivative operator of order q. Then, it holds that:*

$$D^q f(x) = D^{2q/\pi} f\left(\frac{\pi}{2}x\right)$$

Proof. First, let's recall the properties of sine and cosine functions under the continuous Fourier derivative:

$$D^q \sin(x) = \sin\left(x + \frac{\pi}{2}q\right)$$
$$D^q \cos(x) = \cos\left(x + \frac{\pi}{2}q\right)$$

Now, let's assume that $f(x)$ can be expressed as:

$$f(x) = F(\sin(x), \cos(x))$$

where F is a function combining sine and cosine functions.

Then, applying the continuous Fourier derivative of order q to $f(x)$, we get:

$$\begin{aligned} D^q f(x) &= F(D^q \sin(x), D^q \cos(x)) \\ &= F\left(\sin\left(x + \frac{\pi}{2}q\right), \cos\left(x + \frac{\pi}{2}q\right)\right) \end{aligned}$$

Now, we can write:

$$\sin\left(x + \frac{\pi}{2}q\right) = \sin\left(\frac{\pi}{2}q + \frac{\pi}{2}\left(\frac{2x}{\pi}\right)\right)$$
$$\cos\left(x + \frac{\pi}{2}q\right) = \cos\left(\frac{\pi}{2}q + \frac{\pi}{2}\left(\frac{2x}{\pi}\right)\right)$$

So:

$$D^q f(x) = F\left(\sin\left(\frac{\pi}{2}q + \frac{\pi}{2}\left(\frac{2x}{\pi}\right)\right), \cos\left(\frac{\pi}{2}q + \frac{\pi}{2}\left(\frac{2x}{\pi}\right)\right)\right)$$

Let's define a new function G such that:

$$G(q, x) = F\left(\frac{\pi}{2}q, \frac{2x}{\pi}\right)$$

Then, we can write:

$$D^q f(x) = G(q, x)$$

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Exchanging the variables q and x due to their independence:

$$D^x f(q) = G(x,q)$$

But in the continuous Fourier derivative, the order of differentiation is related to the angular frequency. So, to maintain this relationship, we adjust the argument of the function f:

$$D^{2x/\pi} f\left(\frac{\pi}{2}q\right) = G(x,q)$$

Therefore, we have shown that:

$$D^q f(x) = D^{2q/\pi} f\left(\frac{\pi}{2}x\right)$$

□

Property 4. *Let $f(x)$ be a function expressible as a Fourier series, i.e., $f(x) = \sum_{n=0}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$. Then, for the continuous fractional Fourier derivative D^q, it holds that:*

$$D^q f(x) = D^{2q/\pi} f\left(\frac{\pi}{2}x\right)$$

To derive the periodicity formula for the continuous Fourier derivative, let's start by applying the continuous Fourier derivative to the Fourier series of $f(x)$:

Given:

$$f(x) = \sum_{n=0}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$$

Applying D^q to $f(x)$:

$$D^q f(x) = \sum_{n=0}^{\infty}(a_n D^q \cos(nx) + b_n D^q \sin(nx))$$

Using the properties of D^q on sine and cosine functions:

$$D^q \cos(x) = \cos\left(x + \frac{\pi}{2}q\right)$$
$$D^q \sin(x) = \sin\left(x + \frac{\pi}{2}q\right)$$

Substituting these into the equation:

$$D^q f(x) = \sum_{n=0}^{\infty}\left(a_n \cos\left(nx + \frac{\pi}{2}q\right) + b_n \sin\left(nx + \frac{\pi}{2}q\right)\right)$$

Using the angle sum formulas for sine and cosine:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$D^q f(x) = \sum_{n=0}^{\infty} \left(a_n \left(\cos(nx) \cos\left(\frac{\pi}{2}q\right) - \sin(nx) \sin\left(\frac{\pi}{2}q\right) \right) + b_n \left(\sin(nx) \cos\left(\frac{\pi}{2}q\right) + \cos(nx) \sin\left(\frac{\pi}{2}q\right) \right) \right)$$

Rearranging the terms:

$$D^q f(x) = \sum_{n=0}^{\infty} \left(\left(a_n \cos\left(\frac{\pi}{2}q\right) + b_n \sin\left(\frac{\pi}{2}q\right) \right) \cos(nx) + \left(b_n \cos\left(\frac{\pi}{2}q\right) - a_n \sin\left(\frac{\pi}{2}q\right) \right) \sin(nx) \right)$$

Let $A_n = a_n \cos\left(\frac{\pi}{2}q\right) + b_n \sin\left(\frac{\pi}{2}q\right)$ and $B_n = b_n \cos\left(\frac{\pi}{2}q\right) - a_n \sin\left(\frac{\pi}{2}q\right)$, then:

$$D^q f(x) = \sum_{n=0}^{\infty} (A_n \cos(nx) + B_n \sin(nx))$$

This is the Fourier series of $D^q f(x)$. Now, let's apply $D^{2q/\pi}$ to $f\left(\frac{\pi}{2}x\right)$:

$$f\left(\frac{\pi}{2}x\right) = \sum_{n=0}^{\infty} \left(a_n \cos\left(\frac{\pi}{2}nx\right) + b_n \sin\left(\frac{\pi}{2}nx\right) \right)$$

Applying $D^{2q/\pi}$:

$$D^{2q/\pi} f\left(\frac{\pi}{2}x\right) = \sum_{n=0}^{\infty} \left(a_n D^{2q/\pi} \cos\left(\frac{\pi}{2}nx\right) + b_n D^{2q/\pi} \sin\left(\frac{\pi}{2}nx\right) \right)$$

Using the properties of $D^{2q/\pi}$ on sine and cosine functions:

$$D^{2q/\pi} \cos\left(\frac{\pi}{2}x\right) = \cos\left(\frac{\pi}{2}x + \frac{\pi}{2}\frac{2q}{\pi}\right) = \cos\left(\frac{\pi}{2}x + q\right)$$
$$D^{2q/\pi} \sin\left(\frac{\pi}{2}x\right) = \sin\left(\frac{\pi}{2}x + \frac{\pi}{2}\frac{2q}{\pi}\right) = \sin\left(\frac{\pi}{2}x + q\right)$$

Substituting these into the equation:

$$D^{2q/\pi} f\left(\frac{\pi}{2}x\right) = \sum_{n=0}^{\infty} \left(a_n \cos\left(\frac{\pi}{2}nx + q\right) + b_n \sin\left(\frac{\pi}{2}nx + q\right) \right)$$

Using the angle sum formulas for sine and cosine:

$$D^{2q/\pi} f\left(\frac{\pi}{2}x\right) = \sum_{n=0}^{\infty} \Bigg(a_n \left(\cos\left(\frac{\pi}{2}nx\right) \cos(q) - \sin\left(\frac{\pi}{2}nx\right) \sin(q) \right) + b_n \left(\sin\left(\frac{\pi}{2}nx\right) \cos(q) + \cos\left(\frac{\pi}{2}nx\right) \sin(q) \right) \Bigg)$$

Rearranging the terms:

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$$D^{2q/\pi} f\left(\frac{\pi}{2}x\right) = \sum_{n=0}^{\infty}\left(\Big(a_n \cos(q) + b_n \sin(q)\Big)\cos\left(\frac{\pi}{2}nx\right) + \Big(b_n \cos(q) - a_n \sin(q)\Big)\sin\left(\frac{\pi}{2}nx\right)\right)$$

Substituting $q = \frac{\pi}{2}q$:

$$D^{2q/\pi} f\left(\frac{\pi}{2}x\right) = \sum_{n=0}^{\infty}\left(\left(a_n \cos\left(\frac{\pi}{2}q\right) + b_n \sin\left(\frac{\pi}{2}q\right)\right)\cos\left(\frac{\pi}{2}nx\right) + \left(b_n \cos\left(\frac{\pi}{2}q\right) - a_n \sin\left(\frac{\pi}{2}q\right)\right)\sin\left(\frac{\pi}{2}nx\right)\right)$$

Substituting A_n and B_n:

$$D^{2q/\pi} f\left(\frac{\pi}{2}x\right) = \sum_{n=0}^{\infty}\left(A_n \cos\left(\frac{\pi}{2}nx\right) + B_n \sin\left(\frac{\pi}{2}nx\right)\right)$$

Comparing this with the result for $D^q f(x)$, we see that:

$$D^q f(x) = D^{2q/\pi} f\left(\frac{\pi}{2}x\right)$$

Therefore, we have derived the periodicity formula for the continuous Fourier derivative. To generalize the formula, let's start with the base case we derived:

$$D^q f(x) = D^{2q/\pi} f\left(\frac{\pi}{2}x\right)$$

Now, let's apply the same operation to the right-hand side of this equation:

$$D^{2q/\pi} f\left(\frac{\pi}{2}x\right) = D^{2(2q/\pi)/\pi} f\left(\frac{\pi}{2}\left(\frac{\pi}{2}x\right)\right)$$

Simplifying the exponents and fractions:

$$D^{2q/\pi} f\left(\frac{\pi}{2}x\right) = D^{2^2 q/\pi^2} f\left(\frac{\pi^2}{2^2}x\right)$$

Combining this with the base case:

$$D^q f(x) = D^{2q/\pi} f\left(\frac{\pi}{2}x\right) = D^{2^2 q/\pi^2} f\left(\frac{\pi^2}{2^2}x\right)$$

We can continue this process iteratively. For the next step:

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$$D^{2^2q/\pi^2} f\left(\frac{\pi^2}{2^2}x\right) = D^{2(2^2q/\pi^2)/\pi} f\left(\frac{\pi}{2}\left(\frac{\pi^2}{2^2}x\right)\right)$$

Simplifying:

$$D^{2^2q/\pi^2} f\left(\frac{\pi^2}{2^2}x\right) = D^{2^3q/\pi^3} f\left(\frac{\pi^3}{2^3}x\right)$$

Combining with the previous results:

$$D^q f(x) = D^{2q/\pi} f\left(\frac{\pi}{2}x\right) = D^{2^2q/\pi^2} f\left(\frac{\pi^2}{2^2}x\right) = D^{2^3q/\pi^3} f\left(\frac{\pi^3}{2^3}x\right)$$

We can generalize this pattern. For any non-negative integer n:

$$D^q f(x) = D^{2^nq/\pi^n} f\left(\frac{\pi^n}{2^n}x\right)$$

This formula shows that the continuous Fourier derivative exhibits a self-similar, fractal-like behavior under repeated application of the periodicity formula. Each application of the formula results in a new continuous Fourier derivative with an order that is a power of 2 divided by a power of π, applied to the function with its argument scaled by a power of π divided by a power of 2.

This generalized formula provides deeper insight into the periodicity and self-similarity properties of the continuous Fourier derivative.

This property opens up several possibilities in different areas:

1. **Symmetry in fractional derivative**: The property shows an interesting symmetry between the differentiation order and the scale of the function in the context of Fourier series.
2. **Simplification of calculations**: For functions expressible as Fourier series, this property could simplify the calculation of fractional derivatives.
3. **Scale invariance**: The property suggests a form of scale invariance in the continuous Fourier derivative, where a change in the argument scale can be compensated by a change in the differentiation order.
4. **Connection with Fourier transform**: The property could have implications in the frequency domain and suggest a relationship between the differentiation order and the frequency components of the function.
5. **Generalization to other periodic functions**: The property could be generalized to other periodic functions representable by Fourier series.
6. **Applications in signal processing**: In signal processing, this property could have applications in filtering, analysis, and transformation of periodic signals using fractional derivatives.
7. **Connection with physics**: The property could have implications in understanding and analyzing physical phenomena modeled by periodic functions, using fractional derivatives.
8. **Solving fractional differential equations**: The property could be useful in transforming, solving, and analyzing fractional differential equations, particularly those involving periodic functions or periodic boundary conditions.

The generalized formula we have derived, $D^q f(x) = D^{2^nq/\pi^n} f\left(\frac{\pi^n}{2^n}x\right)$, has several interesting and potentially significant implications:

1. **Self-similarity and fractality**: The formula suggests that the continuous Fourier derivative exhibits self-similar or fractal behavior under repeated application of the periodicity formula.

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This means that the structure of the derivative repeats at different scales, a characteristic property of fractals.

2. **Connection between derivative order and scale**: The formula establishes an intrinsic relationship between the derivative order and the scale of the function's argument. As the derivative order increases by powers of 2 divided by powers of π, the scale of the function's argument decreases by powers of π divided by powers of 2.
3. **Scale invariance**: The formula implies a form of scale invariance in the continuous Fourier derivative. If we scale the function's argument by a factor of $\frac{\pi}{2}$ and multiply the derivative order by $\frac{2}{\pi}$, we obtain the same derivative. This property could be useful in analyzing functions and systems exhibiting scale invariance.
4. **Generalized periodicity**: The generalized periodicity formula suggests that the continuous Fourier derivative has a more complex periodicity than simple periodicity in the function's argument. The periodicity is intertwined with the derivative order and the scale of the argument.
5. **Connection with fractional calculus**: The appearance of fractional powers in the formula suggests a connection with fractional calculus. This formula could provide a new perspective on the interpretation and properties of fractional derivatives.
6. **Potential applications**: The formula could have applications in various fields such as signal processing, image analysis, quantum mechanics, and the study of complex systems. It could be particularly relevant for problems involving self-similarity, scale invariance, or periodicity.
7. **Direction for future research**: The formula opens new avenues for future research on the properties and applications of the continuous Fourier derivative. It could inspire new generalizations, connections with other mathematical concepts, and the development of new tools and techniques based on this formula.

If we set $q = 0$ in the generalized formula $D^q f(x) = D^{2^n q/\pi^n} f\left(\frac{\pi^n}{2^n} x\right)$, we obtain an interesting and potentially significant result.

Substituting $q = 0$:

$$D^0 f(x) = D^{2^n 0/\pi^n} f\left(\frac{\pi^n}{2^n} x\right)$$

Simplifying the expression:

$$D^0 f(x) = D^0 f\left(\frac{\pi^n}{2^n} x\right)$$

We know that D^0 is the identity operator, which means $D^0 f(x) = f(x)$ for any function f. Therefore, the equation reduces to:

$$f(x) = f\left(\frac{\pi^n}{2^n} x\right)$$

This equation implies that the function f is invariant under the scale $\frac{\pi^n}{2^n}$ for any non-negative integer n. In other words, the function has a generalized periodicity or scale symmetry.

Some observations about this result:

1. Periodic functions: If f is a periodic function with period T, then $f(x) = f(x + T)$ for all x. The equation we have derived suggests that f also satisfies $f(x) = f(\alpha x)$ for $\alpha = \frac{\pi^n}{2^n}$. This implies a generalized form of periodicity or self-similarity.
2. Self-similar functions: Self-similarity is a characteristic property of fractals, where a part of the object resembles the whole. The equation $f(x) = f(\alpha x)$ is a form of self-similarity, suggesting that functions satisfying this equation may have fractal properties.

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3. Connection with scale theory: Scale invariance is a fundamental concept in scale theory, which studies systems and phenomena that are invariant under scale transformations. The equation we have derived is a form of scale invariance and could have applications in this field.
4. Fractional bases: The appearance of $\frac{\pi^n}{2^n}$ suggests a connection with fractional bases. Fractional bases are number systems where powers of the base are fractions instead of integers. This equation could provide a new perspective on fractional bases and their properties.

Although this equation seems to impose additional constraints on periodic functions, it could be seen as a generalization of periodicity. It defines a special class of functions with periodic and self-similar properties, which could be of interest in certain contexts. However, further research is needed to fully understand the implications of this equation and its relationship with Fourier series theory.

It is important to note that this special class of functions is a result derived from the periodicity formula under the specific condition $q = 0$. It is not an inherent limitation of the derivative itself.

It is important to note that these are theoretical observations and further research would be needed to fully explore the implications and applications of this result. However, this equation opens up new and intriguing possibilities for the study of functions with generalized periodicity, self-similarity, and scale invariance properties.

Substituting $q = 1$ in the generalized formula:

$$D^1 f(x) = D^{2^n/\pi^n} f\left(\frac{\pi^n}{2^n} x\right)$$

We know that D^1 is the classical first-order derivative, denoted as $f'(x)$. Therefore:

$$f'(x) = D^{2^n/\pi^n} f\left(\frac{\pi^n}{2^n} x\right)$$

Taking the limit as n tends to infinity:

$$\lim_{n\to\infty} D^{2^n/\pi^n} f\left(\frac{\pi^n}{2^n} x\right) = f'(x)$$

We observe that:

$$\lim_{n\to\infty} \frac{2^n}{\pi^n} = 0$$
$$\lim_{n\to\infty} \frac{\pi^n}{2^n} = \infty$$

Therefore, in the limit, the equation reduces to:

$$D^0 f(\infty \cdot x) = f'(x)$$

Or equivalently:

$$f(\infty \cdot x) = f'(x)$$

This result suggests that, in the limit, the classical first-order derivative of a function is equivalent to the function evaluated at a infinitely scaled argument.

However, this result should be interpreted with caution. The notation $f(\infty \cdot x)$ is not well-defined in a strict mathematical sense. It would be more accurate to say that the classical first-order derivative is the limit of the continuous Fourier derivative as the order of derivation tends to zero and the function argument is scaled towards infinity in a specific manner.

This result imposes no additional restrictions on the function f, as it applies to any function that is differentiable in the classical sense.

In summary, when $q = 1$, the generalized formula leads us back to the classical first-order derivative in the limit as n tends to infinity. This result provides an interesting connection between the continuous Fourier derivative and the classical derivative, but imposes no additional restrictions on the function.

12. Minimum Interval for Determining q

It is possible to determine a minimum interval for q taking advantage of the symmetry and self-similarity properties of the D_C operator.

Let's recall the generalized periodicity formula:

$$D^q f(x) = D^{(2^n q/\pi^n)} f(\pi^n/2^n x)$$

for any non-negative integer n.

If we set $n = 1$ in this formula, we obtain:

$$D^q f(x) = D^{(2q/\pi)} f(\pi/2x)$$

This suggests that the D_C operator of order q is equivalent to the D_C operator of order $2q/\pi$ applied to the function with the argument scaled by $\pi/2$.

Therefore, if we know the behavior of D_C for values of q in the interval $[0, \pi/2]$, we can extend it to any real value of q using this self-similarity property.

Specifically, for any real q, we can find an integer n such that:

$$2^n q/\pi^n \in [0, \pi/2]$$

Then, we can calculate $D^q f(x)$ using:

$$D^q f(x) = D^{(2^n q/\pi^n)} f(\pi^n/2^n x)$$

where $2^n q/\pi^n$ is in the interval $[0, \pi/2]$.

In conclusion, the minimum interval to define q that guarantees the continuous extension of D_C to all real numbers taking advantage of its symmetry and self-similarity properties is $[0, \pi/2]$. By knowing the behavior of D_C in this interval, we can determine its behavior for any real order of differentiation.

13. A Local Approximation for Continuous Fourier Derivative

Let's begin with the definition of the continuous Fourier derivative of order q of a function $f(x)$:

$$D^q f(x) = \sum_{n=-\infty}^{\infty} (i2\pi n)^q \hat{f}(n) e^{i2\pi nx}$$

where $\hat{f}(n)$ are the Fourier coefficients of $f(x)$.

Now, let's consider a local approximation of $f(x)$ around a point x_0:

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$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

This is the Taylor series of $f(x)$ around x_0.
Let's take the Fourier coefficients of this approximation:

$$\hat{f}(n) \approx \int_{-\infty}^{\infty} \left(f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots \right) e^{-i2\pi nx} dx$$

Integrating term by term:

$$\hat{f}(n) \approx f(x_0)\delta(n) + f'(x_0)\frac{1}{i2\pi n} + \frac{f''(x_0)}{2!}\frac{1}{(i2\pi n)^2} + \cdots$$

where $\delta(n)$ is the Dirac delta function.
Substituting this approximation into the definition of the continuous Fourier derivative:

$$D^q f(x_0) \approx \sum_{n=-\infty}^{\infty} (i2\pi n)^q \left(f(x_0)\delta(n) + f'(x_0)\frac{1}{i2\pi n} + \frac{f''(x_0)}{2!}\frac{1}{(i2\pi n)^2} + \cdots \right) e^{i2\pi n x_0}$$

Simplifying:

$$\begin{aligned} D^q f(x_0) \approx\ & f(x_0) + f'(x_0) \sum_{n=-\infty}^{\infty} \frac{(i2\pi n)^{q-1}}{i2\pi} e^{i2\pi n x_0} \\ & + \frac{f''(x_0)}{2!} \sum_{n=-\infty}^{\infty} \frac{(i2\pi n)^{q-2}}{(i2\pi)^2} e^{i2\pi n x_0} + \cdots \end{aligned}$$

This expression suggests that the continuous Fourier derivative of order q at a point x_0 can be expressed in terms of the classical derivatives of f at x_0 and certain infinite series depending on q and x_0.

To obtain a formula for an infinitesimal limit, we could consider the difference $D^q f(x_0 + h) - D^q f(x_0)$ and take the limit as $h \to 0$. However, evaluating this limit would require careful analysis of the infinite series appearing in the expression.

14. Limit of Difference of Continuous Fourier Derivatives

Theorem 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function with a continuous Fourier transform $\hat{f}(\omega)$. For $\mu > 0$, the Fourier Continuous Derivative $D_C^{\mu} f(x)$ satisfies:*

$$\lim_{h \to 0} \frac{D_C^{\mu} f(x+h) - D_C^{\mu} f(x)}{h} = D_C^{\mu+1} f(x)$$

Proof. We proceed in several steps:

Lemma 3. *The Fourier transform of $D_C^{\mu} f(x)$ is given by $(i\omega)^{\mu} \hat{f}(\omega)$.*

Proof. This follows directly from the definition of the Fourier Continuous Derivative in the frequency domain. □

Now, let's consider the difference quotient:

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$$\frac{D_C^\mu f(x+h) - D_C^\mu f(x)}{h}$$

Taking the Fourier transform of this expression:

$$\mathcal{F}\left\{\frac{D_C^\mu f(x+h) - D_C^\mu f(x)}{h}\right\}(\omega) = \frac{1}{h}\left(e^{i\omega h}(i\omega)^\mu \hat{f}(\omega) - (i\omega)^\mu \hat{f}(\omega)\right)$$
$$= (i\omega)^\mu \hat{f}(\omega)\frac{e^{i\omega h} - 1}{h}$$

Now, we use the following lemma:

Lemma 4. *For any complex number z,*

$$\lim_{h\to 0} \frac{e^{zh} - 1}{h} = z$$

Proof. This follows from the definition of the derivative of the exponential function at zero. □

Applying this lemma with $z = i\omega$, we get:

$$\lim_{h\to 0} \frac{e^{i\omega h} - 1}{h} = i\omega$$

Therefore,

$$\lim_{h\to 0} \mathcal{F}\left\{\frac{D_C^\mu f(x+h) - D_C^\mu f(x)}{h}\right\}(\omega) = (i\omega)^{\mu+1}\hat{f}(\omega)$$

By Lemma 1, the right-hand side is the Fourier transform of $D_C^{\mu+1} f(x)$.
To complete the proof, we need the following lemma:

Lemma 5. *If* $\lim_{h\to 0} \mathcal{F}\{g_h\}(\omega) = \mathcal{F}\{g\}(\omega)$ *pointwise and the limit is uniform on compact sets, then* $\lim_{h\to 0} g_h(x) = g(x)$ *pointwise.*

Proof. This follows from the continuity of the inverse Fourier transform under uniform convergence on compact sets. □

To apply Lemma 3, we need to show that the convergence is uniform on compact sets. Let K be a compact subset of $\mathbb{R}$. Then for $\omega \in K$:

$$\begin{aligned}
&\left|\frac{e^{i\omega h} - 1}{h} - i\omega\right| \\
&= \left|\frac{1}{h}\int_0^h (i\omega e^{i\omega t} - i\omega)dt\right| \\
&\le \frac{1}{h}\int_0^h |\omega||e^{i\omega t} - 1|dt \\
&\le \frac{1}{h}\int_0^h |\omega|^2 t dt = \frac{|\omega|^2 h}{2} \le \frac{M^2 h}{2}
\end{aligned}$$

where $M = \max_{\omega\in K} |\omega|$. This shows that the convergence is uniform on K.
Therefore, by Lemma 3:

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$$\lim_{h \to 0} \frac{D_C^{\mu} f(x+h) - D_C^{\mu} f(x)}{h} = D_C^{\mu+1} f(x)$$

This completes the proof. □

15. Practical Applications

The flexibility of the D_C^{μ} operator can be demonstrated across various functions suitable for study:

- **Rectangular Pulse Function**: This is an essential function in signal processing.
- **Sawtooth Wave**: Gives insights into periodic functions.
- **Gaussian Function**: It is critical for probability and statistical studies.
- **Logarithmic Function**: Explored in both mathematics and engineering.
- **Piecewise Continuous Functions**: Useful in control systems and physics.
- And many more.

16. Detailed Implementation of D_C

To implement the D_C operator in a practical scenario, it is essential to consider the following steps:

1. **Selection of Numerical Libraries**: Choose environments like Python or MATLAB.
2. **Discretization of the Domain**: Define your function's domain.
3. **Calculation of Coefficients c_j**
4. **Frequency Range Selection**
5. **Calculation of $D_C^{\mu}(f(x))$**
6. **Parameter Tuning**
7. **Error Analysis**
8. **Optimization and Parallelization**
9. **Documentation and Testing**

These steps guide enthusiasts in effectively using the D_C operator for different applications.

17. Example Implementation for $f(x) = x^2$

Consider the function $f(x) = x^2$.
First, let's calculate the Fourier coefficients:
Coefficient a_0:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx \tag{15}$$

$$a_0 = \frac{1}{\pi} \left[\frac{x^3}{3} \right]_{-\pi}^{\pi} \tag{16}$$

$$a_0 = \frac{1}{\pi} \left[\frac{\pi^3}{3} - \frac{(-\pi)^3}{3} \right] \tag{17}$$

$$a_0 = \frac{2\pi^2}{3} \tag{18}$$

Coefficient a_n (for $n \geq 1$):

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos\left(\frac{2\pi n x}{\pi} \right) dx \tag{19}$$

$$a_n = 0 \tag{20}$$

Coefficient b_n (for $n \geq 1$): Since $f(x) = x^2$ is an even function, all b_n coefficients will be zero.

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$$b_n = 0 \tag{21}$$

The continuous Fourier derivative $D_C^{\mu_0}$ is given by:

$$D_C^{\mu_0} f(x) = \sum_{n=1}^{\infty} \left[-n^{\mu_0} a_n \sin\left(\frac{2\pi nx}{\pi}\right) + n^{\mu_0} b_n \cos\left(\frac{2\pi nx}{\pi}\right) \right] \tag{22}$$

Substituting in the coefficients, we get:

$$D_C^{\mu_0} f(x) = \sum_{n=1}^{\infty} n^{\mu_0} \frac{1}{4} (-1)^n \cos(2\pi nx) \tag{23}$$

Conclusions:

This development illustrates that the continuous Fourier derivative can be used to compute the derivative of power functions using the Fourier series expansion. This is a function that cannot be straightforwardly addressed using traditional differentiation methods.

18. Proofs of the properties of the D_C Operator

Proof of linearity. Let $f(x)$ and $g(x)$ be two functions, and let a and b be two constants. Then,

$$\begin{aligned} D_C(af(x) + bg(x)) &= \sum_{j=1}^{\infty} j^{\mu} (a_j \cos(wjx) + b_j \sin(wjx)) \\ &= a \sum_{j=1}^{\infty} j^{\mu} a_j \cos(wjx) + b \sum_{j=1}^{\infty} j^{\mu} b_j \cos(wjx) \\ &= aD_C(f(x)) + bD_C(g(x)). \end{aligned}$$

□

19. Other Examples of D_C Applications

19.1. Modeling Nonlinear Wave Behavior, Korteweg-de Vries (KdV) Equation and the D_C Operator

The KdV equation for the evolution of nonlinear waves:

$$\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

Can be written using the D_C operator:

$$D_C^{\alpha} \frac{\partial u}{\partial t} + 6u D_C^{\alpha} \frac{\partial u}{\partial x} + D_C^{\alpha} \frac{\partial^3 u}{\partial x^3} = 0$$

Where the fractional derivation order α allows for adjusting the relative influence of nonlinear and dispersive terms.

The D_C operator facilitates stability analysis and numerical simulations by converging faster than integer derivatives. Where the fractional derivation order α allows tuning the relative influence of the nonlinear and dispersive terms.

The nonlinear term in the KdV equation is: $6u\frac{\partial u}{\partial x}$. This term models nonlinear effects. The dispersive term is: $\frac{\partial^3 u}{\partial x^3}$. It represents wave dispersion in the medium. By replacing the integer derivative with the fractional Continuous Fourier Derivative of order α, we can "tune" the relative

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importance of each term. For instance, with $\alpha = 1$ the original KdV equation is recovered. But with $\alpha < 1$, more weight is given to the nonlinear term.

This provides a useful degree of freedom when studying nonlinear systems with competing terms like the KdV equation. It allows for exploring different regimes. The D_C operator facilitates stability analysis and numerical simulations by converging faster than integer derivatives.

The Continuous Fourier Derivative converges faster than numerical methods based on integer derivatives (finite differences), requiring fewer sampling points.

This is because the Continuous Fourier Derivative has an optimal bandwidth that maximizes the spectral decay rate, allowing for a smoother representation of functions with fewer samples. This translates into faster simulation speeds and better estimation of solution stability in nonlinear problems.

20. How Invariance Ensures That the Operator is Well-Defined?

The invariance properties of the D_C ensure that it is a well-defined operator. This is because the invariance properties guarantee that the operator does not change the essential properties of the function being differentiated. For example, the D_C preserves the convexity of functions, which is a property that is important in many applications.

In addition to ensuring that the D_C is well-defined, the invariance properties also make it easier to calculate the derivative of functions. This is because the invariance properties allow us to reduce the problem of differentiating a function to the problem of differentiating a simpler function.

For example, the invariance property of the D_C for linear functions allows us to calculate the derivative of a composite function where the inner function is linear by simply differentiating the outer function. This can be a significant simplification, as it can often be difficult to differentiate composite functions directly.

Overall, the invariance properties of the D_C make it a powerful and versatile tool for fractional differentiation.

21. Properties of Invariance of the Fourier Continuous Derivative (D_C)

The properties of invariance of the D_C are crucial to ensure that it is a well-defined operator. These properties not only guarantee the D_C produces consistent results with classical differentiation definitions but also ensure its compatibility with the inherent properties of functions.

21.1. Invariance with Linearity

The first foundational property of invariance for the D_C is its commutativity with linear operations. In mathematical terms, this property signifies that the D_C applied to a linear combination of functions results in the same linear combination of the D_C applied to each individual function.

$$D_C^\mu(a \cdot f(x) + b \cdot g(x)) = a \cdot D_C^\mu(f(x)) + b \cdot D_C^\mu(g(x)) \tag{24}$$

Where:

- $D_C^\mu(f(x))$ represents the D_C of the function $f(x)$.
- a and b are constants.

Proof. Let $\mathbb{F}$ be the set of functions, $\mathbb{R}$ the set of real numbers, and D_C^μ the Fourier Continuous Derivative operator of order μ.

Axioms: $\forall f, g \in \mathbb{F}, \forall \alpha, \beta \in \mathbb{R}$:

$D_C^\mu(f + g) = D_C^\mu(f) + D_C^\mu(g)$ (Linearity of the sum) $D_C^\mu(\alpha f) = \alpha D_C^\mu(f)$ (Linearity of the constant α)

Theorem:

$D_C^\mu(\alpha f + \beta g) = \alpha D_C^\mu(f) + \beta D_C^\mu(g)$

Proof:

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$D_C^\mu((\alpha f) + (\beta g)) = D_C^\mu(\alpha f) + D_C^\mu(\beta g)$ (Axiom 1 - Linearity of the sum) $D_C^\mu(\alpha f) = \alpha D_C^\mu(f)$ (Axiom 2 - Linearity of the constant α) $D_C^\mu(\beta g) = \beta D_C^\mu(g)$ (Similarly from Axiom 2 for β)

Substituting 2 and 3 into 1: $D_C^\mu(\alpha f + \beta g) = \alpha D_C^\mu(f) + \beta D_C^\mu(g)$ □

21.2. Preservation of Exponential Functions

Theorem 4 (Exponential Function Preservation)**.** *For the Fourier Continuous Derivative operator D_C^μ, where $\mu \in \mathbb{R}$, and the exponential function $f(x) = e^{ax}$ with $a \in \mathbb{C}$, the following property holds:*

$$D_C^\mu(e^{ax}) = a^\mu e^{ax}$$

Proof. We proceed with the following steps:

1. Let $f(x) = e^{ax}$ where $a \in \mathbb{C}$.
2. Define the Fourier transform of $f(x)$:

$$\mathcal{F}\{f(x)\}(\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} e^{ax} e^{-i\omega x} dx$$

3. Evaluate the Fourier transform:

$$\hat{f}(\omega) = 2\pi\delta(\omega - ai)$$

where δ is the Dirac delta function.

4. By definition of the Fourier Continuous Derivative:

$$D_C^\mu f(x) = \mathcal{F}^{-1}\{(i\omega)^\mu \hat{f}(\omega)\}(x)$$

5. Substitute the Fourier transform:

$$D_C^\mu f(x) = \mathcal{F}^{-1}\{(i\omega)^\mu 2\pi\delta(\omega - ai)\}(x)$$

6. Apply the sifting property of the delta function:

$$D_C^\mu f(x) = \mathcal{F}^{-1}\{(ia)^\mu 2\pi\delta(\omega - ai)\}(x)$$

7. Simplify:

$$D_C^\mu f(x) = a^\mu \mathcal{F}^{-1}\{2\pi\delta(\omega - ai)\}(x)$$

8. Recognize that $\mathcal{F}^{-1}\{2\pi\delta(\omega - ai)\}(x) = e^{ax}$:

$$D_C^\mu f(x) = a^\mu e^{ax}$$

Therefore, we have proved that $D_C^\mu(e^{ax}) = a^\mu e^{ax}$ for all $\mu \in \mathbb{R}$ and $a \in \mathbb{C}$. □

Lemma 6. *The Fourier transform of e^{ax} is $2\pi\delta(\omega - ai)$.*

Proof.

$$\begin{aligned}\mathcal{F}\{e^{ax}\}(\omega) &= \int_{-\infty}^{\infty} e^{ax} e^{-i\omega x} dx \\ &= \int_{-\infty}^{\infty} e^{(a-i\omega)x} dx \\ &= 2\pi\delta(\omega - ai)\end{aligned}$$

This result follows from the properties of the Fourier transform of exponential functions. □

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21.3. *Invariance in Composed Functions*

The third critical property of invariance for the D_C pertains to its behavior with composed functions.

$$D_C^{\mu}(f(g(x))) = D_C^{\mu}(f(u)) \cdot D_C^{\mu}(g(x)) \tag{25}$$

Where:

- $D_C^{\mu}(f(g(x)))$ represents the D_C of the composed function $f(g(x))$.
- $D_C^{\mu}(f(u))$ denotes the D_C of the outer function $f(u)$.
- $D_C^{\mu}(g(x))$ is the D_C of the inner function $g(x)$.
- u is an intermediate variable.

22. Invariance of Convexity in Leibniz's Rule with $g(x) = ax + b$

Theorem 5. *For any linear function $g(x) = ax + b$, where $a, b \in \mathbb{R}$, and any function f for which the Fourier Continuous Derivative is defined, the following property holds:*

$$D_C^{\mu}(f(g(x))) = D_C^{\mu}(f(u)) \cdot (a)^{\mu}$$

where $u = g(x) = ax + b$ and $\mu \in \mathbb{R}$.

Proof. We proceed with the following steps:

1. Let f be a function for which the Fourier Continuous Derivative is defined, and let $g(x) = ax + b$ be a linear function.
2. Define the composition $h(x) = f(g(x)) = f(ax + b)$.
3. Let $\mathcal{F}$ denote the Fourier transform operator. By definition of the Fourier Continuous Derivative:
$$D_C^{\mu}h(x) = \mathcal{F}^{-1}\{(i\omega)^{\mu}\mathcal{F}\{h(x)\}\}$$
4. Apply the Fourier transform to $h(x)$:
$$\mathcal{F}\{h(x)\} = \mathcal{F}\{f(ax + b)\} = \frac{1}{|a|}e^{-i\omega b/a}\hat{f}(\omega/a)$$
where $\hat{f}$ is the Fourier transform of f.
5. Substitute this into the Fourier Continuous Derivative:
$$D_C^{\mu}h(x) = \mathcal{F}^{-1}\{(i\omega)^{\mu}\frac{1}{|a|}e^{-i\omega b/a}\hat{f}(\omega/a)\}$$
6. Change variables: Let $\eta = \omega/a$. Then $\omega = a\eta$ and $d\omega = ad\eta$. Substituting:
$$D_C^{\mu}h(x) = \mathcal{F}^{-1}\{(ia\eta)^{\mu}\frac{1}{|a|}e^{-i\eta b}\hat{f}(\eta)\} \cdot |a|$$
7. Simplify:
$$D_C^{\mu}h(x) = a^{\mu}\mathcal{F}^{-1}\{(i\eta)^{\mu}e^{-i\eta b}\hat{f}(\eta)\}$$
8. Recognize that $\mathcal{F}^{-1}\{(i\eta)^{\mu}\hat{f}(\eta)\} = D_C^{\mu}f(u)$, where $u = ax + b$:
$$D_C^{\mu}h(x) = a^{\mu}D_C^{\mu}f(ax + b)$$

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9. Therefore:

$$D^{\mu}_{C}(f(g(x))) = D^{\mu}_{C}(f(u)) \cdot (a)^{\mu}$$

where $u = g(x) = ax + b$.

This completes the proof. □

Definition 2 (Convexity). *A function $f : I \to \mathbb{R}$ is convex on an interval I if for all $x_1, x_2 \in I$ and $\lambda \in [0, 1]$:*

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

Theorem 6. *For non-integer $\mu > 0$, the Fourier Continuous Derivative $D^{\mu}_{C} f(u)$ can be expressed as:*

$$D^{\mu}_{C} f(u) = \frac{1}{\Gamma(-\mu)} \int_0^{\infty} \frac{f(u+t) - f(u)}{t^{\mu+1}}\, dt$$

where Γ is the Gamma function.

Proof. We proceed in several steps:

Lemma 7. *The Fourier transform of $D^{\mu}_{C} f(u)$ is given by $(i\omega)^{\mu} \hat{f}(\omega)$, where $\hat{f}(\omega)$ is the Fourier transform of $f(u)$.*

Proof. This follows from the definition of the Fourier Continuous Derivative in the frequency domain. □

Lemma 8. *For $\mu > 0$, the Fourier transform of $t_+^{-\mu-1}$ is given by:*

$$\mathcal{F}\{t_+^{-\mu-1}\}(\omega) = \frac{\Gamma(-\mu)}{(-i\omega)^{\mu}}$$

where $t_+^{-\mu-1} = t^{-\mu-1}$ for $t > 0$ and 0 otherwise.

Proof. This is a well-known result in Fourier analysis. It can be derived using contour integration and the properties of the Gamma function. □

Now, let's consider the expression:

$$g(u) = \frac{1}{\Gamma(-\mu)} \int_0^{\infty} \frac{f(u+t) - f(u)}{t^{\mu+1}}\, dt$$

Taking the Fourier transform of both sides:

$$\begin{aligned}
\hat{g}(\omega) &= \frac{1}{\Gamma(-\mu)} \mathcal{F}\left\{ \int_0^{\infty} \frac{f(u+t) - f(u)}{t^{\mu+1}}\, dt \right\}(\omega) \\
&= \frac{1}{\Gamma(-\mu)} \int_0^{\infty} \frac{\mathcal{F}\{f(u+t)\}(\omega) - \mathcal{F}\{f(u)\}(\omega)}{t^{\mu+1}}\, dt \\
&= \frac{1}{\Gamma(-\mu)} \int_0^{\infty} \frac{e^{i\omega t} \hat{f}(\omega) - \hat{f}(\omega)}{t^{\mu+1}}\, dt \\
&= \frac{\hat{f}(\omega)}{\Gamma(-\mu)} \int_0^{\infty} \frac{e^{i\omega t} - 1}{t^{\mu+1}}\, dt
\end{aligned}$$

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Now, let's consider the integral:

$$I = \int_0^\infty \frac{e^{i\omega t} - 1}{t^{\mu+1}} \, dt$$

We can rewrite this as:

$$I = \int_0^\infty \frac{e^{i\omega t} - 1}{(i\omega t)^\mu} \cdot \frac{(i\omega)^\mu}{t} \, dt$$

Substituting $s = \omega t$:

$$I = (i\omega)^\mu \int_0^\infty \frac{e^{is} - 1}{s^\mu} \cdot \frac{1}{s} \, ds = (i\omega)^\mu \Gamma(-\mu)$$

Therefore:

$$\hat{g}(\omega) = (i\omega)^\mu \hat{f}(\omega)$$

By Lemma 1, this is exactly the Fourier transform of $D_C^\mu f(u)$. Since the Fourier transform is invertible, we can conclude that:

$$D_C^\mu f(u) = \frac{1}{\Gamma(-\mu)} \int_0^\infty \frac{f(u+t) - f(u)}{t^{\mu+1}} \, dt$$

This completes the proof. □

Theorem 7 (Invariance of Convexity)**.** *Let D_C^μ be the Fourier Continuous Derivative operator of order μ. If $f(\theta)$ is convex on $\mathbb{N}_0$, then $D_C^\mu(f(g(x)))$ with $g(x) = ax + b$ is convex on $\mathbb{R}$ for $a > 0$.*

Proof. We proceed in several steps:

Lemma 9. *For $g(x) = ax + b$ with $a > 0$, $D_C^\mu(f(g(x))) = a^\mu D_C^\mu(f(u))$, where $u = g(x)$.*

Proof. This follows from the chain rule property of D_C^μ and its linearity. □

Lemma 10. *If $f(\theta)$ is convex on $\mathbb{N}_0$, then its extension $f(u)$ is convex on $\mathbb{R}$.*

Proof. Let $u_1, u_2 \in \mathbb{R}$ and $\lambda \in [0,1]$. Define $\theta_1 = \lfloor u_1 \rfloor$ and $\theta_2 = \lfloor u_2 \rfloor$. Let $\alpha_1 = u_1 - \theta_1$ and $\alpha_2 = u_2 - \theta_2$.

Then:

$$\begin{aligned} f(u_1) &= (1-\alpha_1) f(\theta_1) + \alpha_1 f(\theta_1 + 1) \\ f(u_2) &= (1-\alpha_2) f(\theta_2) + \alpha_2 f(\theta_2 + 1) \end{aligned}$$

Now, $\lambda u_1 + (1-\lambda) u_2 = \theta_3 + \alpha_3$ for some $\theta_3 \in \mathbb{N}_0$ and $\alpha_3 \in [0,1)$.

Using the convexity of f on $\mathbb{N}_0$:

$$\begin{aligned} f(\lambda u_1 + (1-\lambda) u_2) &= (1-\alpha_3) f(\theta_3) + \alpha_3 f(\theta_3 + 1) \\ &\leq (1-\alpha_3)[\lambda f(\theta_1) + (1-\lambda) f(\theta_2)] \\ &\quad + \alpha_3[\lambda f(\theta_1 + 1) + (1-\lambda) f(\theta_2 + 1)] \\ &= \lambda[(1-\alpha_1) f(\theta_1) + \alpha_1 f(\theta_1 + 1)] \\ &\quad + (1-\lambda)[(1-\alpha_2) f(\theta_2) + \alpha_2 f(\theta_2 + 1)] \\ &= \lambda f(u_1) + (1-\lambda) f(u_2) \end{aligned}$$

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This proves that $f(u)$ is convex on $\mathbb{R}$. □

Lemma 11. *The operator D_C^μ preserves convexity for $\mu > 0$.*

Proof. For $\mu \in \mathbb{N}$, this is a well-known property of integer-order derivatives.

For non-integer $\mu > 0$, we can express $D_C^\mu f(u)$ as:

$$D_C^\mu f(u) = \frac{1}{\Gamma(-\mu)} \int_0^\infty \frac{f(u+t) - f(u)}{t^{\mu+1}} \, dt$$

Let $u_1, u_2 \in \mathbb{R}$ and $\lambda \in [0,1]$. Then:

$$\begin{aligned}
&D_C^\mu f(\lambda u_1 + (1-\lambda)u_2) \\
&= \frac{1}{\Gamma(-\mu)} \int_0^\infty \frac{f(\lambda u_1 + (1-\lambda)u_2 + t) - f(\lambda u_1 + (1-\lambda)u_2)}{t^{\mu+1}} \, dt \\
&\leq \frac{1}{\Gamma(-\mu)} \int_0^\infty \frac{\lambda f(u_1+t) + (1-\lambda)f(u_2+t) - \lambda f(u_1) - (1-\lambda)f(u_2)}{t^{\mu+1}} \, dt \\
&= \frac{\lambda}{\Gamma(-\mu)} \int_0^\infty \frac{f(u_1+t) - f(u_1)}{t^{\mu+1}} \, dt + \frac{1-\lambda}{\Gamma(-\mu)} \int_0^\infty \frac{f(u_2+t) - f(u_2)}{t^{\mu+1}} \, dt \\
&= \lambda D_C^\mu f(u_1) + (1-\lambda) D_C^\mu f(u_2)
\end{aligned}$$

The inequality in the third line follows from the convexity of f. Specifically, for any $t \geq 0$:

$$f(\lambda u_1 + (1-\lambda)u_2 + t) \leq \lambda f(u_1+t) + (1-\lambda)f(u_2+t)$$

and

$$f(\lambda u_1 + (1-\lambda)u_2) \leq \lambda f(u_1) + (1-\lambda)f(u_2)$$

Subtracting these inequalities and dividing by $t^{\mu+1}$ (which is positive for $t > 0$) preserves the inequality.

The final equality demonstrates that $D_C^\mu f(u)$ satisfies the definition of convexity:

$$D_C^\mu f(\lambda u_1 + (1-\lambda)u_2) \leq \lambda D_C^\mu f(u_1) + (1-\lambda) D_C^\mu f(u_2)$$

This proves that $D_C^\mu f(u)$ is convex for $\mu > 0$. □

Combining these lemmas, we can conclude that $D_C^\mu(f(g(x))) = a^\mu D_C^\mu(f(u))$ is convex on $\mathbb{R}$ for $a > 0$, as it is a positive scalar multiple of a convex function. □

23. Extension of Convexity from $\mathbb{N}$ to $\mathbb{R}$ for the Fourier Continuous Derivative

We aim to rigorously prove that if a function $f(\mu)$ is convex on $\mathbb{N}$, then its extension to $\mathbb{R}$ via the Fourier Continuous Derivative D_C^μ is also convex.

Definition 3 (Convexity on $\mathbb{N}$)**.** *A function $f : \mathbb{N} \to \mathbb{R}$ is convex if for all $n_1, n_2 \in \mathbb{N}$ and $\lambda \in [0,1]$ such that $\lambda n_1 + (1-\lambda)n_2 \in \mathbb{N}$:*

$$f(\lambda n_1 + (1-\lambda)n_2) \leq \lambda f(n_1) + (1-\lambda)f(n_2)$$

Definition 4 (Convexity on $\mathbb{R}$)**.** *A function $F : \mathbb{R} \to \mathbb{R}$ is convex if for all $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0,1]$:*

$$F(\lambda x_1 + (1-\lambda)x_2) \leq \lambda F(x_1) + (1-\lambda)F(x_2)$$

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Theorem 8. *Let $f : \mathbb{N} \to \mathbb{R}$ be a convex function. Define $F : \mathbb{R} \to \mathbb{R}$ as:*

$$F(\mu) = D_C^{\mu} g(x)$$

where $g(x)$ is a function such that $D_C^n g(x) = f(n)$ for all $n \in \mathbb{N}$. Then $F(\mu)$ is convex on $\mathbb{R}$.

Proof. We proceed in several steps:

Lemma 12. *For any $\mu \in \mathbb{R}$, there exist $n \in \mathbb{N}$ and $\alpha \in [0,1)$ such that $\mu = n + \alpha$.*

Proof. Let $n = \lfloor \mu \rfloor$ be the floor function of μ. Then $\alpha = \mu - n \in [0,1)$. □

Lemma 13 (Interpolation Property). *For $\mu = n + \alpha$ with $n \in \mathbb{N}$ and $\alpha \in [0,1)$:*

$$F(\mu) = (1-\alpha)F(n) + \alpha F(n+1)$$

Proof. This follows from the definition of the Fourier Continuous Derivative and its linearity property:

$$\begin{aligned} F(\mu) &= D_C^{\mu} g(x) \\ &= D_C^{n+\alpha} g(x) \\ &= D_C^{n}(D_C^{\alpha} g(x)) \\ &= (1-\alpha)D_C^{n} g(x) + \alpha D_C^{n+1} g(x) \\ &= (1-\alpha)F(n) + \alpha F(n+1) \end{aligned}$$

□

Now, let $\mu_1, \mu_2 \in \mathbb{R}$ and $\lambda \in [0,1]$. We need to prove:

$$F(\lambda\mu_1 + (1-\lambda)\mu_2) \le \lambda F(\mu_1) + (1-\lambda)F(\mu_2)$$

Let $\mu_1 = n_1 + \alpha_1$ and $\mu_2 = n_2 + \alpha_2$ where $n_1, n_2 \in \mathbb{N}$ and $\alpha_1, \alpha_2 \in [0,1)$.
Then $\lambda\mu_1 + (1-\lambda)\mu_2 = (\lambda n_1 + (1-\lambda)n_2) + (\lambda\alpha_1 + (1-\lambda)\alpha_2)$.
Let $n_3 = \lfloor \lambda n_1 + (1-\lambda)n_2 \rfloor$ and $\alpha_3 = (\lambda n_1 + (1-\lambda)n_2) - n_3 + (\lambda\alpha_1 + (1-\lambda)\alpha_2)$.
By Lemma 2:

$$\begin{aligned} F(\lambda\mu_1 + (1-\lambda)\mu_2) &= (1-\alpha_3)F(n_3) + \alpha_3 F(n_3+1) \\ &\le (1-\alpha_3)[\lambda F(n_1) + (1-\lambda)F(n_2)] \\ &\quad + \alpha_3[\lambda F(n_1+1) + (1-\lambda)F(n_2+1)] \end{aligned}$$

The inequality follows from the convexity of f on $\mathbb{N}$.
Now, using Lemma 2 again:

$$\begin{aligned} \lambda F(\mu_1) + (1-\lambda)F(\mu_2) &= \lambda[(1-\alpha_1)F(n_1) + \alpha_1 F(n_1+1)] \\ &\quad + (1-\lambda)[(1-\alpha_2)F(n_2) + \alpha_2 F(n_2+1)] \end{aligned}$$

Comparing these expressions term by term, we can see that:

$$F(\lambda\mu_1 + (1-\lambda)\mu_2) \le \lambda F(\mu_1) + (1-\lambda)F(\mu_2)$$

This completes the proof that $F(\mu)$ is convex on $\mathbb{R}$. □

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Table 1. Comparison of Original and New Formulations of Fourier Continuous Derivative (D_C)

Characteristic	Original Fourier Continuous Derivative (D_C)	New Integral-based D_C
Definition	$D^{\mu}_{C}f(x) = \mathcal{F}^{-1}\{(i\omega)^{\mu}\hat{f}(\omega)\}$	$D^{\mu}_{C}f(u) = \frac{1}{\Gamma(-\mu)}\int_0^{\infty}\frac{f(u+t)-f(u)}{t^{\mu+1}}\,dt$
Domain	Frequency domain	Time/Space domain
Fourier transform dependency	Explicit	Implicit
Applicability to non-periodic functions	Limited	High
Computational approach	FFT-based	Direct integration
Intuitive interpretation	Frequency-based	Difference-based
Similarity to classical derivative	Moderate	High
Non-locality	Global (frequency domain)	Controlled global (time/space domain)
Flexibility for different orders	High	High
Convexity preservation	Yes	Yes
Ease of analytical manipulation	High for Fourier-friendly functions	Moderate for general functions
Numerical implementation complexity	Low (using FFT)	Moderate (requires numerical integration)
Memory effects modeling	Implicit	Explicit
Connection to fractional calculus theory	Through Fourier analysis	Through integral formulation

24. Applicability of the Fourier Continuous Derivative to Non-Periodic Functions

The original formulation of D_C based on the Fourier transform:

$$D^{\mu}_{C}f(x) = \mathcal{F}^{-1}\{(i\omega)^{\mu}\mathcal{F}[f](\omega)\}$$

was inherently limited to periodic functions or functions that could be adequately represented by Fourier series.

The equivalent version using the Gamma function and an integral:

$$D^{\mu}_{C}f(u) = \frac{1}{\Gamma(-\mu)}\int_0^{\infty}\frac{f(u+t)-f(u)}{t^{\mu+1}}dt$$

This formulation successfully addresses the applicability to non-periodic functions for the following reasons:

1. **Non-dependence on periodicity:** This definition does not assume or require the function to be periodic.
2. **Applicability to finite domain:** It can be applied to functions defined on a finite domain, as the integral can be truncated or adapted as necessary.
3. **Non-smooth functions:** It can handle functions that are not smooth or have discontinuities, as it does not rely on Fourier series expansion.
4. **Local interpretation:** It provides a more local interpretation of the fractional derivative, as it compares the function's value at a point with its values in a neighborhood.
5. **Connection to classical calculus:** This formulation has a clearer connection to classical definitions of derivatives and integrals.

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6. **Flexibility:** It can be applied to a wider range of functions, including those that do not have a well-defined Fourier transform.
7. **Boundary behavior:** It allows for a more direct study of the fractional derivative's behavior near the boundaries of the function's domain.

This formulation, therefore, significantly expands the scope of application of the Fourier Continuous Derivative, making it applicable to a much wider range of functions and problems, including those involving non-periodic or finite-domain functions.

25. Convolution Property

Consider the Fourier Continuous Derivative D^{α}_{C}. One of its remarkable properties is given by:

$$D^{\alpha}_{C}[(f * g)(x)] = (D^{\alpha}_{C} f * g)(x).$$

To elucidate this property, we will leverage both the definition of the Fourier Continuous Derivative operator D^{α}_{C} and the definition of function convolution $(f * g)(x)$.

25.1. Definition of Convolution

For two functions $f(x)$ and $g(x)$ defined on the interval $[a, b]$, their convolution is defined as:

$$(f * g)(x) = \int_a^b f(t) g(x - t) dt. \tag{26}$$

25.2. Fourier Series of Convolution

If $F(x)$ and $G(x)$ represent the Fourier series of $f(x)$ and $g(x)$ respectively, then the Fourier series of their convolution is given by:

$$\begin{aligned}
(F * G)(x) &= \int_a^b F(t) G(x - t) dt \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m \cos\left(\frac{2\pi n t}{T}\right) \cos\left(\frac{2\pi m x}{T} - \frac{2\pi m t}{T}\right) \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m \cos\left(\frac{2\pi (n+m) x}{T}\right) \\
&= \sum_{k=-\infty}^{\infty} d_k \cos\left(\frac{2\pi k x}{T}\right),
\end{aligned} \tag{27}$$

where:

$$d_k = \sum_{n=-\infty}^{\infty} c_n c_{k-n}. \tag{28}$$

Thus, the Fourier series of the convolution of f and g is a trigonometric series with coefficients represented by d_k.

25.3. Fourier Continuous Derivative of Convolution

In the context of the Fourier Continuous Derivative, the coefficients d_k are modified as:

$$d_k = \sum_{n=-\infty}^{\infty} c_n c_{k-n} (w_k i)^{\alpha}, \tag{29}$$

with:

$$w_k = \frac{T}{2\pi k},$$

being the kth Fourier coefficient of $g(x)$.

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This indicates that the Fourier series of the convolution of f and g under the influence of the Fourier Continuous Derivative is a trigonometric series. The coefficients here are a more generalized form than the classical coefficients.

26. Classical Fractional Derivatives

We introduce the Caputo fractional derivative as an exemplar.

Definition 5. *Let f be a function defined on the interval $[a,b]$ with $0 < \mu < 1$. The expression*

$$D_0^\mu f(t_0) = \frac{1}{\Gamma(1-\mu)} \int_0^{t_0} (t_0 - t)^{-\mu} f(t)\, dt \tag{30}$$

is termed the Caputo fractional derivative of order μ for the function f.

26.1. Classical Fractional Derivatives versus D_C

An examination is necessary to determine if D^μ_{GL} meets the same standards as the Continuous Fourier Derivatives. Prior to that, we'll delineate two criteria which will help in favoring one family of differential operators over the other.

$$D^\mu(f(g(x))) = D^\mu(f(u))I(\mu), \tag{31}$$

where

$$I(\mu) = \begin{cases} a^\mu, & \mu \in \mathbb{N}_0 \\ a^\mu + r(\mu), & \mu \in \mathbb{R} - \mathbb{N}_0 \end{cases} \tag{32}$$

A shift in the generalized smoothness of the curve can be observed. Between two subsequent generalized smoothness points, the smoothness is affected, resulting in a lack of convexity.

A scenario that further impacts the preservability is:

$$D^\mu(f(g(x))) = D^\mu(f(u))I(\mu, x), \tag{33}$$

where

$$I(\mu, x) = \begin{cases} a^\mu, & \mu \in \mathbb{N}_0 \\ a^\mu + r(\mu, x), & \mu \in \mathbb{R} - \mathbb{N}_0 \end{cases} \tag{34}$$

Here, the risk is twofold: apart from the previously mentioned factors, alterations in the value of x modify the curve to μ, further compromising preservance (dependency isn't upheld).

The final two principles are designated as: *convexity* and *preservation of dependency*.

It's worth noting that the roster of these rules remains open to additions upon the discovery of new properties associated with functions $f_i(\mu)$.

27. The New List of Criteria to Define D_C

In order for a differential operator to be a valid Fourier Continuous Derivative, it should satisfy certain conditions. Here, we propose five criteria that any Fourier Continuous Derivative should satisfy:

1. **Invariance of Convexity:** If $f(\theta)$ is a convex function involved in a property of the classical derivative (such as the chain rule for a linear function) in $\mathbb{N}_0$, then its generalization in $\mathbb{R}$ should be a convex function (it implies the generalization of ordinary calculus to fractional calculus).
2. **Invariance of Dependency:** If $D^\mu_C(f(x))$ depends on a parameter θ for $\mu \in \mathbb{N}$, then $D^\mu_C(f(x))$ should also depend only on θ for $\mu \in \mathbb{R}$.
3. **Consistency:** The Fourier Continuous Derivative should reduce to the classical derivative when the order of differentiation is an integer. This means that $D^n_C[f(x)] = \frac{d^n}{dx^n}[f(x)]$ for all $n \in \mathbb{N}_0$.

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4. **Linearity:** The Fourier Continuous Derivative should be a linear operator. This means that $D_C^\mu[\alpha f(x) + \beta g(x)] = \alpha D_C^\mu[f(x)] + \beta D_C^\mu[g(x)]$ for all $\mu \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$, and $f(x), g(x)$ defined on $\mathbb{R}$.
5. **Derivative of Constants:** The Fourier Continuous Derivative of a constant should be zero. This means that $D_C^\mu[c] = 0$ for all $\mu \in \mathbb{R}$ and $c \in \mathbb{R}$.

28. On the Locality of the Fourier Continuous Derivative Operator

The nature of the Fourier Continuous Derivative operator D_C with respect to locality requires careful examination. We will provide precise definitions and then analyze the operator under these definitions.

Definition 6 (Traditional Locality). *An operator T is traditionally local if, for any function f and any open set U, $supp(Tf) \cap U$ depends only on the values of f in U.*

Definition 7 (Generalized Locality). *An operator T is generalized local if, for any function f and any point x, $(Tf)(x)$ depends only on the values of f and a finite number of its derivatives at x.*

Now, let's examine the Fourier Continuous Derivative operator D_C^μ under these definitions.

Proposition 1. *The Fourier Continuous Derivative operator D_C^μ is not traditionally local for non-integer $\mu > 0$.*

Proof. Recall that for non-integer $\mu > 0$, D_C^μ can be expressed as:

$$D_C^\mu f(x) = \frac{1}{\Gamma(-\mu)} \int_0^\infty \frac{f(x+t) - f(x)}{t^{\mu+1}}\, dt$$

Consider a function f that is zero in a neighborhood of x. The integral in the definition of $D_C^\mu f(x)$ depends on values of f outside this neighborhood. Therefore, $D_C^\mu f(x)$ does not satisfy the traditional definition of locality. □

However, the situation is more nuanced when we consider the generalized definition of locality.

Theorem 9. *For integer values of μ, the Fourier Continuous Derivative operator D_C^μ is generalized local.*

Proof. For integer $\mu = n$, $D_C^n f(x)$ is equivalent to the classical n-th derivative $f^{(n)}(x)$, which depends only on the value of f and its first n derivatives at x. □

Proposition 2. *For non-integer $\mu > 0$, the Fourier Continuous Derivative operator D_C^μ can be approximated by generalized local operators.*

Sketch of Proof. Consider the Taylor expansion of $f(x+t)$ around x:

$$f(x+t) = f(x) + f'(x)t + \frac{f''(x)}{2!}t^2 + \cdots$$

Substituting this into the integral definition of $D_C^\mu f(x)$:

$$\begin{aligned} D_C^\mu f(x) &= \frac{1}{\Gamma(-\mu)} \int_0^\infty \frac{f'(x)t + \frac{f''(x)}{2!}t^2 + \cdots}{t^{\mu+1}}\, dt \\ &= \frac{f'(x)}{\Gamma(-\mu)} \int_0^\infty t^{-\mu}\, dt + \frac{f''(x)}{2!\Gamma(-\mu)} \int_0^\infty t^{1-\mu}\, dt + \cdots \end{aligned}$$

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This series involves only the derivatives of f at x. While the full series is infinite, it can be approximated to any desired accuracy using a finite number of terms, each of which depends only on a finite number of derivatives of f at x. □

Remark 1. *The apparent contradiction between the non-locality in the traditional sense and the potential locality in the generalized sense arises from the different perspectives on what constitutes "local" information. The traditional definition considers only function values in a neighborhood, while the generalized definition allows for derivative information at a point.*

In conclusion, the Fourier Continuous Derivative operator D_C^μ exhibits a dual nature with respect to locality:

1. It is non-local in the traditional sense for non-integer μ, as it requires information about the function over an extended domain.

2. It can be viewed as approximately local in a generalized sense, as it can be expressed (or approximated) in terms of derivatives at a single point.

This duality resolves the apparent contradiction by clarifying the different notions of locality involved.

29. Seeking the Local D_C

Let $f(x)$ be a real-valued function defined on the interval $[a, b]$.

Let $D_C^\alpha(f(x))$ denote the global Fourier Continuous Derivative of order α of $f(x)$, defined by:

$$D_C^\alpha(f(x)) = \sum_{n=-\infty}^{\infty} c_n (i2\pi n)^\alpha e^{i2\pi nx}$$

where c_n are the Fourier coefficients of $f(x)$.

Let $\phi_{h,k}^\alpha(f(x))$ denote the finite difference approximation of order α using spacing h and using k points centered around x:

$$\phi_{h,k}^\alpha(f(x)) = \left(\frac{1}{h^\alpha}\right) \sum (\text{coef})[f(x + jh) - f(x)]$$

where (coef) refers to the finite difference coefficients.

We wish to determine if:

$$\lim_{h \to 0} \phi_{h,k}^\alpha(f(x)) = D_C^\alpha(f(x))$$

That is, whether the localized finite difference formulations converge to the global Fourier Continuous Derivative under the limit of the stencil size h approaching 0.

To analyze this, we need to explore if suitable finite difference formulations can approximate the Fourier coefficients c_n and complex exponential terms when $h \to 0$. Appropriate smoothing of higher frequency terms may also be required.

We can explore numerical experiments with varying stencil configurations and parameters to minimize errors between the finite differences and the Fourier Continuous Derivative.

30. Taylor Expansion of FCD

We start with the given definition of the Fourier Continuous Derivative:

$$D_C^\mu f(u) = \frac{1}{\Gamma(-\mu)} \int_0^\infty \frac{f(u+t) - f(u)}{t^{\mu+1}}\, dt$$

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Let's apply Taylor's theorem to expand $f(u+t)$ around u:

$$f(u+t) = f(u) + f'(u)t + \frac{f''(u)}{2!}t^2 + \frac{f'''(u)}{3!}t^3 + \cdots$$

Substituting this into the FCD formula:

$$\begin{aligned}
D_C^{\mu} f(u) &= \frac{1}{\Gamma(-\mu)} \int_0^{\infty} \frac{(f(u) + f'(u)t + \frac{f''(u)}{2!}t^2 + \frac{f'''(u)}{3!}t^3 + \cdots) - f(u)}{t^{\mu+1}}\, dt \\
&= \frac{1}{\Gamma(-\mu)} \int_0^{\infty} \frac{f'(u)t + \frac{f''(u)}{2!}t^2 + \frac{f'''(u)}{3!}t^3 + \cdots}{t^{\mu+1}}\, dt \\
&= \frac{1}{\Gamma(-\mu)} \left[f'(u) \int_0^{\infty} t^{-\mu}\, dt + \frac{f''(u)}{2!} \int_0^{\infty} t^{1-\mu}\, dt + \frac{f'''(u)}{3!} \int_0^{\infty} t^{2-\mu}\, dt + \cdots \right]
\end{aligned}$$

Now, let's evaluate these integrals. For $k \geq 0$:

$$\int_0^{\infty} t^{k-\mu}\, dt = \frac{\Gamma(k+1-\mu)}{k+1-\mu}$$

Substituting back:

$$\begin{aligned}
D_C^{\mu} f(u) &= \frac{1}{\Gamma(-\mu)} \left[f'(u) \frac{\Gamma(1-\mu)}{1-\mu} + \frac{f''(u)}{2!} \frac{\Gamma(2-\mu)}{2-\mu} + \frac{f'''(u)}{3!} \frac{\Gamma(3-\mu)}{3-\mu} + \cdots \right] \\
&= f'(u) \frac{\Gamma(1-\mu)}{\Gamma(-\mu)(1-\mu)} + \frac{f''(u)}{2!} \frac{\Gamma(2-\mu)}{\Gamma(-\mu)(2-\mu)} + \frac{f'''(u)}{3!} \frac{\Gamma(3-\mu)}{\Gamma(-\mu)(3-\mu)} + \cdots
\end{aligned}$$

This series can be written more compactly as:

$$D_C^{\mu} f(u) = \sum_{k=1}^{\infty} \frac{f^{(k)}(u)}{k!} \frac{\Gamma(k-\mu)}{\Gamma(-\mu)(k-\mu)}$$

31. Analysis of the Local Approximation

1. This series representation expresses the FCD in terms of local derivatives of f at u, providing a connection to classical calculus.

2. The coefficients $\frac{\Gamma(k-\mu)}{\Gamma(-\mu)(k-\mu)}$ determine the weight of each derivative term.

3. For integer values of μ, many terms in this series vanish due to poles in the Gamma function, potentially recovering classical derivative results.

4. The series converges for functions that are analytic in a neighborhood of u, but the rate of convergence depends on the smoothness of f and the value of μ.

5. This representation allows for truncation to any desired order, providing a local approximation of the FCD operator.

32. Conclusions

This Taylor series expansion provides a local representation of the Fourier Continuous Derivative. It expresses the fractional derivative in terms of integer-order derivatives at a single point, offering a bridge between fractional and classical calculus. This formulation could be particularly useful for numerical approximations and for gaining intuition about the behavior of the FCD operator in terms of familiar calculus concepts.

However, it's important to note that while this representation is local in terms of where the function is evaluated, it still captures global behavior through the infinite series. The convergence and

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practical applicability of this series representation for various classes of functions and different values of μ warrant further investigation.

33. Fractional Derivative Vs. Fourier Continuous Derivative

Fractional derivatives have been a cornerstone in advanced calculus for some time, offering a means to differentiate functions to non-integer orders. However, they are not without challenges:

1. **Non-local Nature**: Fractional derivatives are intrinsically non-local, demanding knowledge of the function across its entire span. This non-locality can make certain applications cumbersome.
2. **Complexity**: The non-integer nature of the derivative makes it inherently challenging to apply in certain scenarios and to gain intuitive insights.

Conversely, the Fourier Continuous Derivative has notable benefits:

1. **Local Operation (Under Certain Definitions)**: As discussed, under some definitions, D_C can be perceived as local, potentially simplifying its application in specific contexts.
2. **Preservation of Functional Properties**: The D_C maintains certain properties of the original function, such as convexity, offering potential advantages in various applications.
3. **Computational Simplicity with Fourier Series**: A striking advantage of D_C is its straightforward computation using Fourier series. The relationship:

$$D_C^{\mu}(f(x)) = \sum_{n=-\infty}^{\infty} (2\pi i n)^{\mu} c_n e^{2\pi i n x} \tag{35}$$

makes this clear. Here, c_n represents the Fourier coefficients of the function $f(x)$, and this equation essentially offers a direct method to compute the Fourier Continuous Derivative.

34. Potential Shortcomings of the Fourier Continuous Derivative

While the Fourier Continuous Derivative offers several advantages, it's essential to acknowledge its potential drawbacks:

1. **Computational Overhead**: Utilizing the Fourier transform can be computationally taxing, particularly for large-scale functions or those with intricate frequency compositions.
2. **Noise Sensitivity**: Like many differentiation operators, D_C can be susceptible to noise. Small disturbances or perturbations in the input data might lead to pronounced errors in the derivative, especially for high-frequency components.
3. **Incomplete Understanding of Certain Properties**: Even though D_C's invariance properties are touted as strengths, a comprehensive understanding of these attributes is still a work in progress.
4. **Application Constraints**: D_C's efficiency is not universal. It may not always be the optimal choice, especially when dealing with functions that don't naturally align with its advantages.

In light of these points, while D_C promises to be an influential tool in fractional differentiation, researchers must approach its applications judiciously, keeping both its strengths and limitations in mind.

34.1. Limitations of FCD

1. **Numerical Complexity:** The D_C involves Fourier transforms and can be computationally intensive, especially for large datasets or functions with complex frequency content. This can lead to long computation times and resource requirements.
2. **Sensitivity to Noise:** Like other derivative operators, the D_C can be sensitive to noise in the data. Noise in the input function can lead to significant errors in the derivative estimation, especially for high-frequency components.

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3. **Limited Understanding of Invariance Properties:** While the invariance properties of D_C are a strength, there is still ongoing research to fully understand these properties and how they apply to different types of functions and datasets.
4. **Application Specificity:** The effectiveness of D_C depends on the characteristics of the problem at hand. It may not be the best choice for all applications, especially when dealing with functions that do not exhibit the desired invariance properties.

In conclusion, while the D_C is a promising tool for fractional differentiation, it is not without its limitations. Ongoing research and development efforts aim to address these limitations and enhance its applicability across different domains.

35. Application of Fourier Derivative

Consider the fractional differential equation for modeling one-dimensional anomalous diffusion:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = K_\alpha \frac{\partial^2 u(x,t)}{\partial x^2}$$

Where $0 < \alpha < 1$ and K_α is the fractional diffusion coefficient. Applying the Continuous Fourier Derivative D_C^α:

$$D_C^\alpha \frac{\partial u(x,t)}{\partial t} = K_\alpha \frac{\partial^2 u(x,t)}{\partial x^2}$$

Representing $u(x,t)$ as a Fourier series:

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n(t) e^{inx}$$

And applying D_C^α we obtain a set of ordinary differential equations for the coefficients $c_n(t)$:

$$D_C^\alpha \dot{c}_n(t) = -K_\alpha n^2 c_n(t)$$

Which can be easily solved due to the fractional nature of the temporal derivative.

36. Signal Noise Identification with Fourier Continuous Derivative

Consider a signal $x(t)$ composed of a periodic signal $s(t)$ and noise $n(t)$:

$$x(t) = s(t) + n(t)$$

We apply the Continuous Fourier Derivative of order α:

$$D_C^\alpha X(\omega) = D_C^\alpha S(\omega) + D_C^\alpha N(\omega)$$

Where:

$$X(\omega) = \text{Fourier Transform of } x(t)$$
$$S(\omega) = \text{Fourier Transform of } s(t)$$
$$N(\omega) = \text{Fourier Transform of } n(t)$$

By filtering the low-frequency components in $D_C^\alpha X(\omega)$, we highlight the periodic signal $s(t)$ and attenuate the noise due to the properties of D_C^α.

Then we apply the inverse transform to reconstruct $s(t)$. The Continuous Fourier Derivative thus allows for identifying and filtering the signal of interest.

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37. Example: Modeling Viscoelastic Relaxation Response Using the Continuous Fourier Derivative

Consider a simple fractional derivative model of a viscoelastic solid:

$$m\frac{d^2x(t)}{dt^2} + c\, D_C^{\alpha}\, \frac{dx(t)}{dt} + kx(t) = F(t)$$

Where:

$$m = \text{Mass} \quad (36)$$
$$c = \text{Viscous damping coefficient} \quad (37)$$
$$k = \text{Elastic constant} \quad (38)$$
$$\alpha = \text{Fractional derivation order} \quad (39)$$
$$D_C^{\alpha} = \text{Continuous Fourier Derivative operator} \quad (40)$$
$$F(t) = \text{Applied force} \quad (41)$$

The fractional derivative models the viscoelastic behavior. Applying the Fourier Transform on both sides:

$$-m\omega^2 X(\omega) - c(i\omega)^{\alpha} X(\omega) + kX(\omega) = F(\omega)$$

Solving for the frequency response $X(\omega)$:

$$X(\omega) = \frac{F(\omega)}{-m\omega^2 - c(i\omega)^{\alpha} + k}$$

The Continuous Fourier Derivative allows modeling of the viscoelastic response and obtain time-domain responses by inverse transform.

38. Application of the D_C Operator: Modeling Seismic Wave Propagation

The propagation of seismic waves through porous underground media involves anomalous diffusion phenomena better described by fractional differential equations. Traditionally, the fractional wave equation is used:

$$\rho\partial^2 u/\partial t^2 = (\partial^{\alpha}/\partial|x|^{\alpha})u + f \quad (42)$$

Where ρ is the density and α is a fractional order depending on medium properties.

However, by representing the displacement field u via its Fourier series, we can rewrite the equation in the spectral domain:

$$\rho\partial^2 \tilde{u}/\partial t^2 = (-i\omega)^{\alpha}\tilde{u} + \tilde{f} \quad (43)$$

Here, applying the D_C operator of order α yields:

$$\rho\partial^2 \tilde{u}/\partial t^2 = {D_C}^{\alpha}\tilde{u} + \tilde{f} \quad (44)$$

This formulation would enable numerical simulation of wave propagation adapted to the observed fractal behavior.

39. Application of the Fourier Continuous Derivative to Anomalous Diffusion in Heterogeneous Porous Media

Anomalous diffusion refers to diffusion processes that deviate from the classical Fickian diffusion behavior, where the mean squared displacement of particles is proportional to time. In heterogeneous porous media, such as rocks or biological materials, diffusion often exhibits anomalous behavior due to the complex structure of the medium and the interactions between the particles and the medium.

The fractional diffusion equation is commonly used to model anomalous diffusion:

$$D_C^{1-\alpha}\left[\frac{\partial}{\partial t}u(x,t)\right] = D_\alpha \nabla_x^\alpha u(x,t) \tag{45}$$

where $D_C^{1-\alpha}$ represents the Fourier Continuous Derivative of order $(1-\alpha)$ with respect to time, ∇_x^α represents the fractional spatial Laplacian operator of order α, and D_α is the fractional diffusion coefficient.

The application of the D_C could allow for a more efficient solution of the fractional diffusion equation by leveraging the properties of the D_C and its relationship with Fourier series. The solution could provide valuable insights into the anomalous diffusion behavior in heterogeneous porous media, such as the spatial and temporal distribution of the diffusing particles.

Furthermore, the convexity-preserving properties of the D_C could be beneficial in this context, as they could ensure physically realistic and stable solutions.

This example illustrates how the Fourier Continuous Derivative could be applied in the realm of physics and engineering, specifically in modeling complex diffusion processes in heterogeneous media. The application of the D_C could lead to new insights and more efficient solutions in this field.

It is important to note that this is a conceptual proposal, and its feasibility and effectiveness would require further research and validation. However, it demonstrates the potential of the D_C to address challenging problems in various disciplines beyond those mentioned in the article.

40. Practical Applications

The practical applications of the Fourier Continuous Derivative encompass a wide array of fields:

- **Signal Processing**: It finds use in signal analysis, noise reduction, and feature extraction from signals. The D_C could be used to design filters that are more effective at removing certain types of noise or isolating specific signal features.
- **Optics**: In wave optics, the Fourier Transform is used to model wave propagation through various media. The D_C can assist in studying the effects of diffraction and refraction.
- **Vibration Analysis**: When studying mechanical vibrations, the Fourier Transform helps in the frequency domain analysis of the system's response to different inputs. Using D_C, we can effectively model damping and other nonlinear effects.
- **Electrical Engineering**: In circuit analysis, the Fourier Transform provides insights into the behavior of circuits in the frequency domain. The Fourier Continuous Derivative can be instrumental in understanding the effects of parasitic capacitances, inductances, and other phenomena.
- **Fluid Dynamics**: The study of the propagation of waves in fluids can be analyzed using the Fourier Transform. The D_C can offer insights into phenomena like dispersion and nonlinearity in wave propagation.

While knowledge of realized applications is still limited, the D_C operator shows promise in several areas:

- **Telecommunications**: Modeling long-memory noises or anomalous propagation in communication channels using D_C could improve filter and coding designs.
- **Materials Simulation**: Researchers may apply D_C to simulate flows in porous media, crack propagation in rocks, or develop more realistic viscoelastic material models.

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- **Financial and Economic Modeling**: Given economic/financial data's fractal memory nature, D_C could illuminate long-term autocorrelations in asset price time series.
- **Digital Image Processing**: D_C is potentially being explored for edge detection in blurred images, facial recognition, texture compression, or deteriorated image restoration.
- **Climate Simulation**: D_C could impact geophysical fluid dynamics models, atmospheric wave propagation simulations, or self-similar pollutant dispersion at varying scales.

In conclusion, the Fourier Continuous Derivative is a powerful mathematical tool that, when combined with the Fourier Transform, offers deeper insights into the analysis and solutions of various problems across different fields of study.

41. Detailed Implementation of D_C in Practical Applications

This section provides in-depth explanations and examples of how the Fourier Continuous Derivative (D_C) operator can be implemented in various practical applications.

41.1. Signal Processing: Fractional Edge Detection

In image processing, edge detection is a crucial task. The D_C operator can be used to implement a fractional edge detector, which can be more flexible than traditional integer-order methods.

Example 1 (Fractional Edge Detection)**.** *Consider a 2D image $f(x,y)$. We can implement a fractional edge detector using D_C as follows:*

Algorithm 1 Fractional Edge Detection using D_C

```
procedure FractionalEdgeDetection(f, μ)
    F ← FFT2(f)                                ▷ 2D Fast Fourier Transform
    ω_x, ω_y ← FrequencyGrid(F)
    H ← (iω_x)^μ + (iω_y)^μ                    ▷ Fractional derivative filter
    G ← H · F                                  ▷ Element-wise multiplication
    g ← IFFT2(G)                               ▷ Inverse FFT
    return |g|                                 ▷ Magnitude of the result
end procedure
```

The fractional order μ allows for fine-tuning the edge detection sensitivity. Values of μ between 1 and 2 can detect edges at different scales.

41.2. Anomalous Diffusion in Porous Media

The D_C operator can be used to model anomalous diffusion processes in porous media, where the diffusion deviates from the classical Fickian model.

Example 2 (Anomalous Diffusion Equation)**.** *Consider the fractional diffusion equation:*

$$\frac{\partial u}{\partial t} = K_\alpha D_C^\alpha u$$

where $u(x,t)$ is the concentration, K_α is the generalized diffusion coefficient, and $0 < \alpha \leq 2$. We can solve this numerically using the following scheme:

Algorithm 2 Numerical Solution of Fractional Diffusion Equation

```
procedure FractionalDiffusion(u_0, α, K_α, Δt, T)
    u ← u_0
    t ← 0
    while t < T do
        U ← FFT(u)
        ω ← FrequencyGrid(U)
        U_new ← U exp(K_α (iω)^α Δt)
        u ← IFFT(U_new)
        t ← t + Δt
    end while
    return u
end procedure
```

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This scheme uses the spectral method, leveraging the efficiency of FFT for spatial derivatives and exact integration in the frequency domain for time-stepping.

41.3. Viscoelastic Material Modeling

The D_C operator can be used to model the behavior of viscoelastic materials, which exhibit both viscous and elastic characteristics.

Example 3 (Fractional Kelvin-Voigt Model)**.** *Consider a fractional Kelvin-Voigt model:*

$$\sigma(t) = E\epsilon(t) + \eta D_C^{\alpha}\epsilon(t)$$

where $\sigma(t)$ *is stress,* $\epsilon(t)$ *is strain, E is the elastic modulus,* η *is the viscosity coefficient, and* $0 < \alpha < 1$*. We can implement this model as follows:*

Algorithm 3 Fractional Kelvin-Voigt Model Simulation

```
procedure FRACTIONALKELVINVOIGT(ϵ, E, η, α, Δt, T)
    t ← LinSpace(0, T, T/Δt)
    Ω ← FrequencyGrid(t)
    E ← FFT(ϵ)
    S ← E · E + η · (iΩ)^α · E
    σ ← IFFT(S)
    return σ
end procedure
```

This implementation uses the frequency domain representation of D_C to efficiently compute the fractional derivative term.

41.4. Fractional Control Systems

The D_C operator can be used in control systems to design fractional-order controllers, which can offer improved performance over integer-order controllers in certain situations.

Example 4 (Fractional PID Controller)**.** *Consider a fractional PID controller with transfer function:*

$$C(s) = K_p + K_i s^{-\lambda} + K_d s^{\mu}$$

where $0 < \lambda, \mu < 1$*. We can implement this in the time domain using D_C:*

Algorithm 4 Fractional PID Controller

```
procedure FRACTIONALPID(e, K_p, K_i, K_d, λ, μ, Δt)
    E ← FFT(e)
    ω ← FrequencyGrid(E)
    I ← K_i · E · (iω)^(−λ)
    D ← K_d · E · (iω)^μ
    U ← K_p · E + I + D
    u ← IFFT(U)
    return u
end procedure
```

This implementation computes the fractional integral and derivative terms in the frequency domain using D_C, then combines them with the proportional term to produce the control signal.

41.5. Conclusion

These detailed examples demonstrate how the D_C operator can be practically implemented in various applications. The key advantages of using D_C in these contexts include:

1. Efficient computation using FFT algorithms. 2. Natural handling of periodic boundary conditions. 3. Flexibility in choosing fractional orders for fine-tuning model behavior. 4. Straightforward implementation in the frequency domain.

However, it's important to note that the use of D_C also comes with challenges, such as dealing with non-periodic boundary conditions and potential numerical instabilities for certain parameter

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ranges. These challenges often require careful consideration and possibly additional techniques to address effectively in practical applications.

42. Comprehensive Comparison of Fractional Derivative Operators

This section provides a detailed comparison between the Fourier Continuous Derivative (D_C) and other prominent fractional derivative operators.

42.1. Definitions of Fractional Derivative Operators

Definition 8 (Fourier Continuous Derivative)**.** *For a function $f(x)$ with Fourier transform $\hat{f}(\omega)$, the Fourier Continuous Derivative of order μ is defined as:*

$$D_C^\mu f(x) = \mathcal{F}^{-1}\{(i\omega)^\mu \hat{f}(\omega)\}(x)$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

Definition 9 (Riemann-Liouville Fractional Derivative)**.** *The Riemann-Liouville fractional derivative of order $\mu > 0$ is defined as:*

$$D_{RL}^\mu f(x) = \frac{1}{\Gamma(n-\mu)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\mu-1} f(t) dt$$

where $n = \lfloor \mu \rfloor + 1$ and Γ is the gamma function.

Definition 10 (Caputo Fractional Derivative)**.** *The Caputo fractional derivative of order $\mu > 0$ is defined as:*

$$D_C^\mu f(x) = \frac{1}{\Gamma(n-\mu)} \int_0^x (x-t)^{n-\mu-1} f^{(n)}(t) dt$$

where $n = \lfloor \mu \rfloor + 1$.

42.2. Comparative Analysis

Table 2. Comparison of fractional derivative operators

Property	D_C	Riemann-Liouville	Caputo	Riesz
Linearity	Yes	Yes	Yes	Yes
Semigroup property	Yes	Yes	No	Yes
Zero derivative of constants	Yes	No	Yes	Yes
Fourier transform	Simple	Complex	Complex	Simple
Initial conditions	Natural	Modified	Natural	Natural
Numerical implementation	FFT-based	Quadrature	Quadrature	FFT-based
Physical interpretation	Frequency	Time	Time	Space

42.3. Detailed Property Analysis

Property 5 (Linearity)**.** *All operators satisfy linearity: $D^\mu(af + bg) = aD^\mu f + bD^\mu g$ for constants a and b.*

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Proof. For D_C, this follows directly from the linearity of the Fourier transform. For Riemann-Liouville and Caputo, it follows from the linearity of integration and differentiation. For Riesz, it follows from the linearity of the Fourier transform and its inverse. □

Property 6 (Semigroup Property). *D_C, Riemann-Liouville, and Riesz satisfy $D^\alpha D^\beta = D^{\alpha+\beta}$, while Caputo does not in general.*

Proof. For D_C, this follows from $(i\omega)^\alpha (i\omega)^\beta = (i\omega)^{\alpha+\beta}$. For Riemann-Liouville and Riesz, it can be proven using their integral representations. Caputo does not satisfy this property due to its definition involving integer-order derivatives. □

Property 7 (Zero Derivative of Constants). *D_C, Caputo, and Riesz give zero when applied to constants, while Riemann-Liouville does not.*

Proof. For D_C, the Fourier transform of a constant is a delta function at $\omega = 0$, which vanishes when multiplied by $(i\omega)^\mu$ for $\mu > 0$. For Caputo, this follows from the vanishing of integer-order derivatives of constants. For Riemann-Liouville, $D^\mu 1 = \frac{x^{-\mu}}{\Gamma(1-\mu)} \neq 0$ for $\mu > 0$. □

Property 8 (Fourier Transform). *D_C and Riesz have simple representations in the Fourier domain, while Riemann-Liouville and Caputo have more complex representations.*

Proof. For D_C, $\mathcal{F}\{D_C^\mu f\}(\omega) = (i\omega)^\mu \hat{f}(\omega)$. For Riesz, $\mathcal{F}\{D_R^\mu f\}(\omega) = |\omega|^\mu \hat{f}(\omega)$. For Riemann-Liouville and Caputo, the Fourier transforms involve complex functions of ω. □

42.4. Comparative Advantages and Disadvantages

- D_C**:** Advantages include simple Fourier representation, natural handling of periodic functions, and efficient numerical implementation using FFT. Disadvantages include potential difficulties with non-smooth functions and edge effects in finite domains.
- **Riemann-Liouville:** Advantages include a clear relationship with integer-order calculus and well-developed theoretical foundations. Disadvantages include non-zero derivatives of constants and difficulties in physical interpretation of initial conditions.
- **Caputo:** Advantages include natural initial conditions and zero derivative of constants. Disadvantages include lack of semigroup property and more complex Fourier representation.
- **Riesz:** Advantages include symmetry in spatial variables and simple Fourier representation. Disadvantages include difficulties in handling boundary conditions in finite domains.

42.5. Conclusion

The choice of fractional derivative operator depends on the specific requirements of the problem at hand. D_C excels in problems involving periodic functions or where frequency domain analysis is natural. Riemann-Liouville and Caputo are often preferred in time-domain problems, especially those involving initial conditions. Riesz is particularly useful in spatial problems with symmetry requirements.

43. Definitions

43.1. Fourier Continuous Derivative (FCD)

$$D_C^\mu f(u) = \frac{1}{\Gamma(-\mu)} \int_0^\infty \frac{f(u+t) - f(u)}{t^{\mu+1}}\, dt$$

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43.2. Riemann-Liouville Fractional Derivative

$$D^{\mu}_{RL} f(x) = \frac{1}{\Gamma(n-\mu)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\mu-1} f(t)\, dt$$

where $n = \lfloor \mu \rfloor + 1$.

43.3. Caputo Fractional Derivative

$$D^{\mu}_{C} f(x) = \frac{1}{\Gamma(n-\mu)} \int_0^x (x-t)^{n-\mu-1} f^{(n)}(t)\, dt$$

where $n = \lfloor \mu \rfloor + 1$.

44. Comprehensive Comparison of Fractional Derivative Operators

1. **Structure:**
   - The FCD is defined as a single integral, similar to the Caputo derivative.
   - The Riemann-Liouville derivative involves both integration and differentiation.
2. **Memory Effect:**
   - The FCD integrates over $(0, \infty)$, potentially capturing long-term memory effects.
   - Both Riemann-Liouville and Caputo integrate from 0 to x, limiting the memory to the interval $[0, x]$.
3. **Singularity:**
   - The FCD kernel $t^{-\mu-1}$ has a singularity at $t = 0$, similar to Riemann-Liouville and Caputo.
   - The FCD's singularity is moderated by the difference $f(u+t) - f(u)$.
4. **Initial Conditions:**
   - The FCD naturally incorporates the function value at u, potentially simplifying the handling of initial conditions.
   - Caputo derivative is often preferred for initial value problems due to its use of integer-order derivatives.
   - Riemann-Liouville requires fractional-order initial conditions, which can be challenging to interpret physically.
5. **Derivative of Constants:**
   - For the FCD, if $f(u)$ is constant, the integral vanishes, giving zero.
   - Caputo derivative of a constant is also zero.
   - Riemann-Liouville derivative of a constant is generally non-zero.
6. **Computational Aspects:**
   - The FCD involves an improper integral, which may require special numerical techniques.
   - Riemann-Liouville and Caputo derivatives can be computed using finite-domain quadrature methods.
7. **Fourier Transform:**
   - The FCD has a simple representation in the Fourier domain: $(i\omega)^{\mu} \hat{f}(\omega)$.
   - Riemann-Liouville and Caputo have more complex Fourier representations.
8. **Physical Interpretation:**
   - The FCD can be interpreted as a weighted average of function differences, potentially offering intuitive physical meanings in certain applications.
   - Riemann-Liouville and Caputo derivatives have established interpretations in viscoelasticity and other fields.

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Characteristic	Fourier Continuous Derivative (D_C)	Riemann-Liouville Derivative	Caputo Derivative	Grünwald-Letnikov Derivative
Definition	$D_C^{\mu} f(u) = \frac{1}{\Gamma(-\mu)} \int_0^{\infty} \frac{f(u+t)-f(u)}{t^{\mu+1}} dt$	$D_{RL}^{\mu} f(x) = \frac{1}{\Gamma(n-\mu)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\mu-1} f(t)\, dt$	$D_C^{\mu} f(x) = \frac{1}{\Gamma(n-\mu)} \int_0^x (x-t)^{n-\mu-1} f^{(n)}(t)\, dt$	$D_{GL}^{\mu} f(x) = \lim_{h \to 0} \frac{1}{h^{\mu}} \sum_{k=0}^{\infty} (-1)^k \binom{\mu}{k} f(x-kh)$
Integration domain	$[0, \infty)$	$[0, x]$	$[0, x]$	Discrete
Frequency dependence	Independent	Dependent	Dependent	Dependent
Similarity to classical derivative	High	Moderate	Moderate	Low
Generalization to non-integer orders	Natural	Complex	Complex	Discrete approximation
Non-locality	Controlled global	Semi-local	Semi-local	Local
Flexibility for different orders	High	Moderate	Moderate	High
Convexity preservation	Yes	Not always	Not always	Not always
Applicability to non-periodic functions	High	Moderate	Moderate	High
Intuitive interpretation	Moderate	Low	Moderate	Low
Computational complexity	Moderate	High	High	Low

Table 3. Comparison of Fractional Derivative Operators

45. Conclusion

The Fourier Continuous Derivative offers a unique perspective on fractional differentiation, combining aspects of both local and global behavior. Its structure suggests potential advantages in problems involving long-range interactions or memory effects, while maintaining some of the desirable properties of classical fractional derivatives. However, its practical implementation and physical interpretation in various domains require further investigation.

46. Comprehensive Discussion on Limitations of the Fourier Continuous Derivative Operator

While the Fourier Continuous Derivative (D_C) operator offers several advantages in certain applications, it is crucial to understand its limitations and potential drawbacks. This section provides a detailed examination of these issues.

46.1. Theoretical Limitations

Limitation 1 (Non-locality)**.** *The D_C operator is inherently non-local, which can lead to difficulties in interpreting and applying it in contexts where local behavior is crucial.*

Example 5. *In boundary value problems where local conditions are important, such as in certain fluid dynamics applications, the non-locality of D_C can make it challenging to impose and interpret boundary conditions accurately.*

Limitation 2 (Periodicity Assumption)**.** *The D_C operator implicitly assumes periodicity of the function it operates on, which may not be appropriate for all applications.*

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Example 6. *When analyzing signals or data that are inherently non-periodic, such as financial time series or certain types of biomedical signals, the use of D_C may introduce artificial periodicities, leading to incorrect conclusions.*

Limitation 3 (Smoothness Requirements)**.** *The D_C operator may not be well-defined for functions that are not sufficiently smooth or do not have a well-behaved Fourier transform.*

Example 7. *For functions with discontinuities or sharp transitions, such as square waves or signals with abrupt changes, the D_C operator may produce unreliable results or exhibit Gibbs phenomenon-like artifacts.*

46.2. Computational Limitations

Limitation 4 (Computational Complexity)**.** *While efficient for periodic functions, the D_C operator can be computationally expensive for large datasets or high-dimensional problems.*

Example 8. *In image processing applications involving large, high-resolution images or video streams, the repeated application of D_C through FFT computations can become prohibitively time-consuming for real-time processing.*

Limitation 5 (Numerical Instability)**.** *For certain fractional orders or in the presence of noise, the D_C operator can exhibit numerical instability.*

Example 9. *When applied to noisy data with high-frequency components, D_C can amplify these high-frequency noise terms, potentially leading to unreliable or meaningless results.*

Limitation 6 (Finite Precision Effects)**.** *The reliance on FFT algorithms means that D_C is subject to finite precision effects, which can accumulate in iterative calculations.*

Example 10. *In long-term simulations of fractional differential equations using D_C, numerical errors can accumulate over time, potentially leading to significant deviations from the true solution.*

46.3. Application-Specific Limitations

Limitation 7 (Boundary Effects)**.** *The periodic assumption of D_C can lead to significant boundary effects when applied to finite domains.*

Example 11. *In image processing, applying D_C near the edges of an image can introduce artifacts due to the implicit assumption that the image wraps around at the boundaries.*

Limitation 8 (Physical Interpretability)**.** *In some physical systems, the meaning of fractional-order derivatives provided by D_C may not have a clear physical interpretation.*

Example 12. *While D_C can be used to model anomalous diffusion in porous media, the physical meaning of a fractional-order derivative in this context is not always clear, potentially making it difficult to connect model parameters to measurable physical quantities.*

Limitation 9 (Causality Issues)**.** *The non-local nature of D_C can lead to causality issues in time-dependent problems.*

Example 13. *In signal processing applications where causality is important, such as real-time filtering or prediction, the use of D_C may introduce non-causal effects, as it implicitly uses information from the entire time domain.*

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46.4. Comparison with Other Methods

Limitation 10 (Lack of Flexibility Compared to Time-Domain Methods)**.** *While D_C is efficient for certain types of problems, it may lack the flexibility of some time-domain fractional derivative methods.*

Example 14. *In problems involving variable-order fractional derivatives or mixed fractional derivatives, time-domain methods like the Grünwald-Letnikov approach may offer more flexibility than D_C.*

Limitation 11 (Difficulty with Non-uniform Sampling)**.** *D_C assumes uniform sampling in the frequency domain, which can be a limitation when dealing with non-uniformly sampled data.*

Example 15. *In applications such as astronomy or geophysics, where data may be collected at irregular intervals, the use of D_C may require additional preprocessing or interpolation steps, potentially introducing errors.*

46.5. Conclusions

While the Fourier Continuous Derivative operator is a powerful tool in many contexts, it is crucial to be aware of its limitations. The choice to use D_C should be made carefully, considering:

1. The nature of the problem (periodic vs. non-periodic, smooth vs. non-smooth) 2. Computational resources available 3. Required accuracy and stability 4. Physical interpretability of the results 5. Boundary conditions and causality requirements

In many cases, these limitations can be mitigated through careful problem formulation, pre-processing of data, or hybrid approaches combining D_C with other methods. However, in some situations, alternative fractional derivative operators or entirely different approaches may be more appropriate.

47. Distinctive Features of the D_C Operator

The Fourier Continuous Derivative, D_C, stands out due to several key attributes:

- It provides continuity and can be employed on smooth functions.
- As a linear operator, it enables differentiation of both sums and products of functions.
- It preserves invariance properties, ensuring consistency under transformations.
- Proves effective for fractional-order differential equations.

In conclusion, the Fourier Continuous Derivative offers a blend of versatility and precision, aligning with the classical definition of differentiation for integer orders and broadening the concept of differentiation to non-integer orders.

48. Conclusion

In conclusion, the Fourier Continuous Derivative is a versatile mathematical tool with potential applications across various domains. Its advantages lie in its ability to model convex systems, its mathematical consistency, and its extension to fractional-order differentiation. While the choice between classical fractional derivatives and D_C depends on context, the latter exhibits promise due to its advantageous properties and potential practical applications.

49. Current Research Directions

The Fourier Continuous Derivative (D_C) presents intriguing challenges and opportunities in various aspects of research. Here are some active research areas related to the D_C:

49.1. Numerical Implementation

The numerical implementation of the Fourier Continuous Derivative is a non-trivial task due to its complexity. Researchers are actively working on developing efficient and accurate numerical algorithms to compute the D_C for various applications. This area of research is essential for making the D_C more accessible and practical in real-world scenarios.

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49.2. Theoretical Understanding

The theoretical understanding of the Fourier Continuous Derivative is an ongoing endeavor. While its properties and invariance properties have been explored, a complete theoretical framework is still evolving. Researchers are delving into the mathematical foundations of the D_C to provide a deeper understanding of its behavior and properties.

49.3. Exploring New Applications

As a relatively new operator in the realm of fractional calculus, the Fourier Continuous Derivative continues to inspire the exploration of novel applications. Researchers are actively seeking new domains and problems where the D_C can offer unique insights or solutions. This dynamic field of research holds the potential for groundbreaking discoveries and innovative applications.

In summary, the Fourier Continuous Derivative represents an exciting and evolving area of research. The development of efficient numerical implementations, a deeper theoretical understanding, and the exploration of new applications are all contributing to the advancement of this mathematical tool. As researchers continue to push the boundaries of knowledge in these areas, the D_C's potential impact across various disciplines is expected to grow significantly.

50. Materials and Methods

This section describes the key materials, data sources, and procedures followed to perform the mathematical analysis and derivations supporting the properties of the Fourier Continuous Derivative (D_C) operator. The main methods include formal proofs, application examples on function archetypes, and comparisons with classical fractional differentiation techniques. Additionally, Python 3.8 and NumPy 1.23 were used for numerical validation and stability assessments under different conditions.

51. Results

This section presents the outcomes from formally defining and assessing properties of D_C including linearity, exponential function preservation, chain rule extension, and convexity retention on $\mathbb{Z}$ and $\mathbb{R}$. Specific highlights comprise the Fourier series representation enabling fractional differentiation via D_C, comparisons with Riemann-Liouville/Caputo derivatives, along with example functions where D_C enables efficient fractional differentiation. Tabulated results quantify the improved computational efficiency and accuracy attained by D_C over classical methods for the test functions.

52. Discussion

The key inferences from applying D_C are its mathematical consistency, well-defined behavior, ability to retain convexity, and fractional differentiation capability across smooth and non-smooth functions where classical derivatives can struggle or lose robustness. While further characterization is warranted, these initial results confirm the central hypotheses regarding D_C's properties and advantages. Of particular note is the connection with Fourier analysis at the crux of D_C's formulation, underscoring its aptness for frequency-domain representation and analysis.

53. Conclusions

In conclusion, the foundations and preliminary inspection presented in this work suggest that the Fourier Continuous Derivative has valuable mathematical attributes as a fractional differentiation operator, with wide-ranging possibilities across science and engineering problems dealing with frequency representations and convexity preservation.

54. Review of Axiomatic Basis

To ensure the rigorous foundation of the Fourier Continuous Derivative (FCD) operator and avoid any appearance of circular reasoning, we present a clear axiomatic basis:

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Axiom 1 (Existence)**.** *For any function $f(x)$ with a well-defined Fourier transform $\hat{f}(\omega)$, the FCD of order $\mu \in \mathbb{R}$ exists and is given by:*

$$D_C^{\mu} f(x) = \mathcal{F}^{-1}\{(i\omega)^{\mu} \hat{f}(\omega)\}(x)$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

Axiom 2 (Linearity)**.** *For any functions $f(x)$ and $g(x)$, and constants $a, b \in \mathbb{R}$:*

$$D_C^{\mu}(af(x) + bg(x)) = aD_C^{\mu} f(x) + bD_C^{\mu} g(x)$$

Axiom 3 (Consistency with Integer Order)**.** *For any non-negative integer n, the FCD reduces to the classical derivative:*

$$D_C^{n} f(x) = \frac{d^n}{dx^n} f(x)$$

Axiom 4 (Semigroup Property)**.** *For any $\mu, \nu \in \mathbb{R}$:*

$$D_C^{\mu}(D_C^{\nu} f(x)) = D_C^{\mu+\nu} f(x)$$

Axiom 5 (Zero Derivative of Constants)**.** *For any constant c and $\mu > 0$:*

$$D_C^{\mu} c = 0$$

These axioms form the foundation for the FCD operator. All properties and theorems presented in this work are derived from these fundamental axioms, ensuring a logically consistent and non-circular development of the theory.

Remark 2. *The integral representation of the FCD:*

$$D_C^{\mu} f(u) = \frac{1}{\Gamma(-\mu)} \int_0^{\infty} \frac{f(u+t) - f(u)}{t^{\mu+1}} dt$$

can be derived from the Fourier transform definition and is consistent with the axiomatic basis presented here.

By explicitly stating these axioms, we establish a clear and rigorous foundation for the FCD operator, avoiding any potential circular reasoning in subsequent proofs and derivations.

55. Conflict of Interests/Competing Interests

The author declares no conflict of interests.

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