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All Bi-Unitary Superperfect Polynomials over $\mathbb{F}_2$ with at Most Two Irreducible Factors

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Abstract: In this paper, we give all non splitting bi-unitary superperfect polynomials divisible by one or two irreducible polynomials over the prime field of two elements. We prove the nonexistence of odd bi-unitary superperfect polynomials over $\mathbb{F}_2$.

Keywords: sum of divisors; bi-unitary divisors; polynomials; finite fields; characteristic 2

1. Introduction

Let n and k be positive integers and let $\sigma(n)$ (resp. $\sigma^*(n)$) denotes the sum of positive (resp. unitary) divisors of the integer n. A divisor d of n is unitary if d and n/d are coprime. We call the number n a $k-$superperfect number if $\sigma^k(n) = \underbrace{\sigma(\sigma(...(\sigma(n))))}_{k-\text{times}} = 2n$. When $k = 1$, n is called a perfect number. An integer $M = 2^p - 1$, where p is a prime number, is called a Mersenne number. It is also well known that an even integer n is perfect if and only if $n = M(M+1)/2$ for some Mersenne prime number M. Suryanarayana [1] considered $k-$superperfect numbers in the case $k = 2$. Numbers of the form 2^{p-1} (p is prime) are 2-superperfect if $2^{p-1} - 1$ is a Mersenne prime. It is not known if there are odd k-superperfect numbers. Sitaramaiah and Subbarao [2] studied the unitary superperfect numbers, the integers n satisfying $\sigma^{*2}(n) = \sigma^*(\sigma^*(n)) = 2n$. They found all unitary superperfect numbers below 10^8. The first unitary superperfect numbers are $2, 9, 165$, and 238. A positive integer n has a bi-unitary divisor, d, if the greatest common unitary divisor of d and n/d is equal to 1. The arithmetic function $\sigma^{**}(n)$ denotes the sum of positive bi-unitary divisors of the integer n. Wall [3] proved that there are only three bi-unitary perfect numbers ($\sigma^{**}(n) = 2n$), namely 6, 60 and 90. Yamada [4] proved that 2 and 9 are the only bi-unitary superperfect numbers, that is $\sigma^{**2}(n) = 2n$ if and only if $n \in \{2, 9\}$.

Now, let A be a nonzero polynomial defined over the prime field $\mathbb{F}_2$. A divisor B of A is unitary (resp. bi-unitary) if $gcd(B, A/B) = 1$ (resp. $gcd_u(B, A/B) = 1$), where $gcd_u(A, A/B)$ denotes the greatest common unitary divisor of B and A/B. We denote by σ the sum of the monic divisors B of A, that is, $\sigma(A) = \sum_{B|A} B$. $\sigma^*(A)$ (resp. $\sigma^{**}(A)$) represents the sum of all unitary (resp. bi-unitary) monic divisors of A. Note that all the functions σ, σ^* and σ^{**} are multiplicative and degree preserving.

A is an even polynomial if it has a linear factor in $\mathbb{F}_2[x]$ else it is an odd polynomial. A polynomial M of the form $1 + x^a(x+1)^b$ is called Mersenne. The first five Mersenne polynomials over $\mathbb{F}_2$ are: $M_1 = 1 + x + x^2$, $M_2 = 1 + x + x^3$, $M_3 = 1 + x^2 + x^3$, $M_4 = 1 + x + x^2 + x^3 + x^4$, $M_5 = 1 + x^3 + x^4$. Note that all these polynomials are irreducible, so we call them Mersenne primes.

Let $\omega(A)$ denotes the number of distinct irreducible monic polynomials that divide A. The notion of perfect polynomials over $\mathbb{F}_2$ was introduced first by Canaday [5]. A polynomial A is perfect if $\sigma(A) = A$. Canaday studied the case of even perfect polynomials with $\omega(A) \leq 3$. In the past few years, Gallardo and Rahavandrainy [6–8] showed the non-existence of odd perfect polynomials over $\mathbb{F}_2$ with either $\omega(A) = 3$ or with $\omega(A) \leq 9$ in the case where all exponents of the irreducible factors of A are equal to 2. A polynomial A is said to be a unitary (resp. a bi-unitary) perfect if $\sigma^*(A) = A$ (resp. $\sigma^{**}(A) = A$). Also, A is called a unitary (resp. a bi-unitary) superperfect if $\sigma^{*2}(A) = \sigma^*(\sigma^*(A)) = A$

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(resp. $\sigma^{**2}(A) = \sigma^{**}(\sigma^{**}(A)) = A$).
Note that the function σ^{**2} is degree preserving but not multiplicative, and this is the main challenge in this work. So, working on bi-unitary superperfect polynomial over $\mathbb{F}_2$ is not an easy task especially when A is divisible by more than 2 irreducible factors.

Many researchers studied the unitary perfect polynomials over $\mathbb{F}_2$. The authors in [7,9,10] list the unitary perfect polynomials over $\mathbb{F}_2$ with $\omega(A) \leq 4$. They list others that are divisible by $x(x+1)P$, P is a Mersenne polynomial, raised to certain powers, see [7]). Beard [11] found many bi-unitary perfect polynomials over F_{p^d}, some of which are neither perfect nor unitary perfect. He conjectured a characterization of the bi-unitary perfect polynomials which splits over F_p when $p > 2$. Beard gave examples of non-splitting bi-unitary perfect polynomials over F_p when $p \in \{2, 3, 5\}$. Rahavandrainy [12] gave all bi-unitary perfect polynomials over the prime field $\mathbb{F}_2$, with at most four irreducible factors (Lemmas 12 and 13). Gallardo and Rahavandrainy [13] classified some unitary superperfect polynomials with a small number of prime divisors under some conditions on the number of prime factors of $\sigma^*(A)$.

Notations: We use the following notations throughout the article.

- $\mathbb{N}$ (resp. $\mathbb{N}^*$) represents the set of non-negative (resp. positive) integers.
- $deg(A)$ denotes the degree of the polynomial A.
- $\overline{A}$ is the polynomial obtained from A with x replaced by $x+1$, that is $\overline{A}(x) = A(x+1)$.
- P and Q are distinct irreducible non constant polynomials.
- P_i and Q_j are distinct odd irreducible non constant polynomials.

In this paper, we prove the non-existence of odd bi-unitary superperfect polynomials A when A is divisible by at least two irreducible factors (Corollary 4). We give a complete classification for all bi-unitary superperfect polynomials over $\mathbb{F}_2$ that are divisible by at most two distinct irreducible factors, (Theorem 1). Bi-unitary superperfect polynomials over $\mathbb{F}_2$ that are neither unitary perfect nor bi-unitary perfect are found. The polynomials $x^4(x+1)^4, x^9(x+1)^9, x^9(x+1)^{13}$, and $x^2(x+1)^{2^n-1}$ are such examples, n is a positive integer.

Our main result is given in the following theorem:

Theorem 1. *If $\omega(A) \leq 2$ and A is a bi-unitary superperfect over $\mathbb{F}_2$ if and only if $A, \overline{A} \in \{x^2, x^{2^d-1}, x^2(x+1)^2, x^4(x+1)^4, x^9(x+1)^9, x^9(x+1)^{13}, x^2(x+1)^{2^d-1}, x^{2^m-1}(x+1)^{2^n-1}\}$, where $m, n \in \mathbb{N}^*$.*

2. Preliminaries

The following two lemmas are helpful.

Lemma 1. *Let A be a polynomial in $\mathbb{F}_2[x]$, then $\sigma^*\left(A^{2^n}\right) = (\sigma^*(A))^{2^n}$, n is a non-negative integer.*

Proof. The result follows since σ^* is multiplicative and $\sigma^*(p^{2^n}) = 1 + p^{2^n} = (1+p)^{2^n} = (\sigma^*(p))^{2^n}$. □

Lemma 2. *If A is a unitary superperfect polynomial over $\mathbb{F}_2$, then A^{2^n} is also a unitary superperfect polynomial over $\mathbb{F}_2$ for all non-negative integers n.*

Proof. Let A be a unitary superperfect and let $B = \sigma^*(A)$. By Lemma 1, we have $\sigma^{*2}\left(A^{2^n}\right) = \sigma^*\left(\sigma^*\left(A^{2^n}\right)\right) = \sigma^*\left(B^{2^n}\right) = (\sigma^*(B))^{2^n} = (\sigma^*(\sigma^*(A)))^{2^n} = A^{2^n}$. □

Lemma 3. [Lemma 2.4 in [13]] Let A be a polynomial in $\mathbb{F}_2[x]$.

1) If P is an odd prime factor of A, then $x(x+1)$ divides $\sigma^*(A)$.
2) If $x(x+1)$ divides A, then $x(x+1)$ divides $\sigma^*(A)$.
3) If A is unitary superperfect that has an odd prime factor, then $x(x+1)$ divides A.

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The following results are needed, and they are a result of Beard [11], and Rahavandrainy [12] works.

Lemma 4. [Theorem 1 and its Corollary in [11]] If A is a non-constant bi-unitary perfect polynomial, then $x(x+1)$ divides A and $\omega(A) \geq 2$.

Lemma 5. [Lemma 2.2 in [12]]

1) $\sigma^{**}(P^{2a+1}) = \sigma(P^{2a+1})$.
2) $\sigma^{**}(P^{2a}) = (1+P^{a+1})\sigma(P^{a-1}) = (1+P)\sigma(P^{a})\sigma(P^{a-1})$.

Corollary 1. [Corollary 2.3 in [12]] Let $T \in \mathbb{F}_2[x]$ be irreducible. Then
i) If $a \in \{4r, 4r+2\}$, where $2r-1$ or $2r+1$ is of the form $2^{\alpha}u-1$, u odd, then $\sigma^{**}(P^a) = (1+P)^{2^{\alpha}} \cdot \sigma(P^{2r}) \cdot (\sigma(P^{u-1}))^{2^{\alpha}}$, $\gcd(\sigma(P^{2r}), \sigma(P^{u-1})) = 1$.
ii) If $a = 2^{\alpha}u - 1$ is odd, with u odd, then $\sigma^{**}(P^a) = (1+P)^{2^{\alpha}-1} \cdot (\sigma(P^{u-1}))^{2^{\alpha}}$.

The proof of the below lemma follows from Lemma 5 and the binomial formula. Table 6 shows some values of $\sigma^{**}(A)$ when A is a power of the first five Merssene primes.

Lemma 6. *Let the polynomial M_i be Mersenne prime and Q_j be an irreducible polynomial over $\mathbb{F}_2$ and let $a, c \in \mathbb{N}^*$. If $\alpha_j \in \mathbb{N}$, then*

1) $x(x+1)$ *divides* $\sigma^{**}(M_i^c)$.
2) $\sigma^{**}(M_1^c) = x^a(x+1)^a \prod_j Q_j^{\alpha_j}$.
3) $\sigma^{**}(M_2^c) = x^a(x+1)^{2a} \prod_j Q_j^{\alpha_j}$.
4) $\sigma^{**}(M_3^c) = x^{2a}(x+1)^{a} \prod_j Q_j^{\alpha_j}$.
5) $\sigma^{**}(M_4^c) = x^{a}(x+1)^{3a} \prod_j Q_j^{\alpha_j}$.
6) $\sigma^{**}(M_5^c) = x^{3a}(x+1)^{a} \prod_j Q_j^{\alpha_j}$.

Lemma 7. [Corollary 2.4 in [12]]

1) $\sigma^{**}(x^a)$ splits over F_2 if and only if $a = 2$ or $a = 2^d - 1$, for some $d \in \mathbb{N}^*$.
2) $\sigma^{**}(P^c)$ splits over F_2 if and only if P is Mersenne and $c = 2$ or $c = 2^d - 1$ for some $d \in \mathbb{N}^*$.

Lemma 8 summarizes key results taken from Canaday's paper [5].

Lemma 8. *Let T be irreducible in $\mathbb{F}_2[x]$ and let $n, m \in \mathbb{N}$.*
i) *If T is a Mersenne prime and if $T = T^*$, then $T \in \{M_1, M_4\}$.*
ii) *If $\sigma(x^{2n}) = PQ$ and $P = \sigma((x+1)^{2m})$, then $2n = 8$, $2m = 2$, $P = M_1$ and $Q = P(x^3) = 1 + x^3 + x^6$.*
iii) *If any irreducible factor of $\sigma(x^{2n})$ is a Mersenne prime, then $2n \leq 6$.*
iv) *If $\sigma(x^{2n})$ is a Mersenne prime, then $2n \in \{2, 4\}$.*

Lemma 9. [Lemma 2.6 in [14]] *Let $m \in \mathbb{N}^*$ and M be a Mersenne prime. Then, $\sigma(x^{2m})$, $\sigma((x+1)^{2m})$ and $\sigma(M^{2m})$ are all odd and squarefree.*

3. Bi-unitary superperfect Polynomials

Recall that A is a bi-unitary superperfect polynomial in $\mathbb{F}_2[x]$ if $\sigma^{**2}(A) = \sigma^{**}(\sigma^{**}(A)) = A$. The polynomial $A = x^4(1+x)^4$ is a bi-unitary superperfect polynomial over $\mathbb{F}_2$.
The proof of the following lemmas follow directly.

Lemma 10. *If A is a bi-unitary perfect polynomial over $\mathbb{F}_2$, then A is also a bi-unitary superperfect polynomial.*

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Lemma 11. *If A is a bi-unitary superperfect polynomial over $\mathbb{F}_2$, then $B = \sigma^{**}(A)$ is also a bi-unitary superperfect polynomial.*

Rahavandrainy (Lemma 2.6 in [12]) proved that if A is a bi-unitary perfect polynomial over $\mathbb{F}_2$ where $A = A_1A_2$ such that $gcd(A_1, A_2) = 1$, then A_1 is a bi-unitary perfect polynomial if and only if A_2 is a bi-unitary perfect polynomial. Rahavandrainy's previous result is not valid in the case of bi-unitary superperfect polynomials because the bi-unitary superperfect polynomial $A = x^2(1+x)^2(1+x+x^2)^2$ is a counterexample over $\mathbb{F}_2$. In fact, $A_1 = x^2(1+x)^2$ is a bi-unitary superperfect but $A_2 = (1+x+x^2)^2$ is not a bi-unitary superperfect.

The following polynomials are considered over $\mathbb{F}_2$:

$C = 1+x+x^4,$	$B_1 = x^3(x+1)^4M_1,$	$B_2 = x^3(x+1)^5M_1^2,$
$B_3 = x^4(x+1)^4M_1^2,$	$B_4 = x^6(x+1)^6M_1^2,$	$B_5 = x^4(x+1)^5M_1^3,$
$B_6 = x^7(x+1)^8M_5,$	$B_7 = x^7(x+1)^9M_5^2,$	$B_8 = x^8(x+1)^8M_4M_5,$
$B_9 = x^8(x+1)^9M_4M_5^2,$	$B_{10} = x^7(x+1)^{10}M_1^2M_5,$	$B_{11} = x^7(x+1)^{13}M_2^2M_3^2,$
$B_{12} = x^9(x+1)^9M_4^2M_5^2,$	$B_{13} = x^{14}(x+1)^{14}M_2^2M_3^2,$	$R_1 = x^4(x+1)^5M_1^4C,$
$R_2 = x^4(x+1)^5M_1^5C^2.$		

Lemma 12. [Theorem 1.1 in [12]] Let $A \in \mathbb{F}_2[x]$ be bi-unitary perfect polynomial such that $\omega(A) = 3$. Then $A, \overline{A} \in \{B_j : j \leq 7\}$.

Lemma 13. [Theorem 1.2 in [12]] Let $A \in \mathbb{F}_2[x]$ be bi-unitary perfect polynomial such that $\omega(A) = 4$. Then $A, \overline{A} \in \{B_j : 8 \leq j \leq 13\} \bigcup \{R_1, R_2\}$.

Lemma 14. *If $A(x)$ is a bi-unitary superperfect polynomial over $\mathbb{F}_2$, then so is $\overline{A}(x)$.*

4. Proof of Theorem 1

We start this section by the following corollary.

Corollary 2. *If a is a positive integer, then*

1) *1+x divides $\sigma^{**}(x^a)$.*
2) *x divides $\sigma^{**}((1+x)^a)$.*

Proof. An immediate result of Lemma 5. □

Lemma 15. *$x(x+1)$ divides $\sigma^{**}(P^a)$, a is a positive integer.*

Proof. Since P is odd, then $P(0) = P(1) = 1$. If $a = 2n+1$, then $\sigma^{**}(P^{2n+1})(0) = \underbrace{1 + P(0) + \ldots + P^{2n+1}(0)}_{(2n+1)\text{-times}} = 1 + 2n + 1 = 0$. If $a = 2n$, then $1 + P^{n+1}(0) = 0$. So, x divides $\sigma^{**}(P^a)$ for every $a \in \mathbb{N}$. Similarly, $x+1$ divides $\sigma^{**}(P^a)$. Hence, $x(x+1)$ divides $\sigma^{**}(P^a)$. □

Lemma 16. *Let A be a polynomial in $\mathbb{F}_2[x]$.*

1) *If P is an odd prime factor of A, then $x(x+1)$ divides $\sigma^{**}(A)$.*
2) *If $x(x+1)$ divides A, then $x(x+1)$ divides $\sigma^{**}(A)$.*

Proof. 1) We write $A = P^aB$ where $a \in \mathbb{N}^*$ and $B \in \mathbb{F}_2[x]$ such that $gcd(P, B) = 1$. But, $1 + P$ divides $\sigma^{**}(A)$ and the result follows since $x(x+1)$ divides $1 + P$.
2) In a similar manner, we write $A = x^a(x+1)^bB$ where $a, b \in \mathbb{N}^*$.
□

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Corollary 3. *If $A \in \mathbb{F}_2[x]$ and $\omega(A) \geq 2$, then $x(x+1)$ divides $\sigma^{**}(A)$.*

Proof. Let $\omega(A) \geq 2$. If $x(x+1)$ divides A, then we are done by Corollary 2. If $x(x+1)$ does not divide A, then A is divisible by an irreducible polynomial $P \notin \{x, 1+x\}$ and the result follows by Lemma 15. □

Corollary 4. *Let A be a polynomial in $\mathbb{F}_2[x]$ with $\omega(A) \geq 2$. If A is a bi-unitary superperfect, then $x(x+1)$ divides A.*

Proof. Let $A = \sigma^{**2}(A) = \sigma^{**}(B)$, where $B = \sigma^{**}(A)$. Since $\omega(A) \geq 2$, then either P or $x(x+1)$ divides A. In both cases, $x(x+1)$ divides $\sigma^{**}(A) = B$ (Lemma 16). So, $x(x+1)$ divides $\sigma^{**}(B) = \sigma^{**2}(A)$. □

The following lemma is similar to Lemma 7.

Lemma 17. *Let $a, b \in \mathbb{N}^*$, then*

1) *If a is even, then $\sigma^{**2}(x^a)$ and $\sigma^{**2}((x+1)^a)$ splits over $\mathbb{F}_2$ if and only if $a \in \{2, 4, 10, 12\}$.*
2) *If a is odd, then $\sigma^{**2}(x^a)$ and $\sigma^{**2}((x+1)^a)$ splits over $\mathbb{F}_2$ if and only if $a \in \left\{5, 9, 13, 2^d - 1\right\}$ for some $d \in \mathbb{N}^*$.*

Proof. 1) If $\sigma^{**}(x^a)$ splits, the $a = 2$ (Lemma 7) and $\sigma^{**2}(x^a) = (x+1)^2$. Suppose, $\sigma^{**}(x^a)$ does not split with $a = 4r, 2r - 1 = 2^\alpha u - 1$, (resp. $a = 4r + 2, 2r + 1 = 2^\alpha u - 1$), u is odd, $r \geq 1$. But $\sigma^{**2}(x^a) = \sigma^{**}\left((1+x)^{2^\alpha} \cdot \sigma(x^{2r}) \cdot (\sigma(x^{u-1}))^{2^\alpha}\right)$, so $\sigma^{**}\left((1+x)^{2^\alpha}\right)$ must split. Hence, $\alpha = 1$ and since $\sigma(x^{2r})$ is odd and square free (Lemma 9), then $\sigma(x^{2r})$ has a Mersenne factor. So, $2r \leq 6$ and hence $u \leq 3$.

2) Assume $a = 2^\alpha u - 1$, with u is odd. If $\sigma^{**}(x^a)$ splits, then $a = 2^d - 1$, d is positive (Lemma 7). If $\sigma^{**}(x^a)$ does not split, then $a \neq 2^d - 1$ and since $\sigma^{**2}(x^a) = x^{2^\alpha - 1} \cdot \sigma^{**}\left((\sigma(x^{u-1}))^{2^\alpha}\right)$ splits, $u > 1$. Again, by Lemma 9, $\sigma(x^{2r})$ has a Mersenne factor. So, $u - 1 \leq 6$ and hence $u \in \{3, 5, 7\}$. For $u = 3$, $\sigma^{**2}(x^a) = x^{2^\alpha - 1} \cdot \sigma^{**}\left((\sigma(x^2))^{2^\alpha}\right) = x^{2^\alpha - 1} \cdot \sigma^{**}\left(M_1^{2^\alpha}\right)$. Hence, $\alpha = 1$ and the same result is obtained when $u \in \{5, 7\}$.
The same proof is done for $\sigma^{**2}((x+1)^a)$ and the proof is compete. □

Lemma 18. *Let a and b have the form $2^n - 1$ where $n \in \mathbb{N}^*$ and let the polynomial $A = 1 + x^a(x+1)^b$ be Mersenne prime over $\mathbb{F}_2$, then $\sigma^{**2}(A) = x^b(x+1)^a$.*

Proof. Let $a = 2^{n_1} - 1$ and $b = 2^{n_2} - 1$, then

$$\begin{aligned}
\sigma^{**2}(A) &= \sigma^{**2}\left(1 + x^a(x+1)^b\right) \\
&= \sigma^{**}(\sigma(1 + x^a(x+1)^b)) \\
&= \sigma^{**}\left(x^a(x+1)^b\right) \\
&= x^b(x+1)^a.
\end{aligned}$$

□

4.1. Case w(A)=1

We prove that $\sigma^{**}(A)$ can not have more than one prime factor when A is a prime power.

Lemma 19. *If $A \in \{x, x+1\}$ and $\sigma^{**2}(A^a)$ splits over $\mathbb{F}_2$, then A is a bi-unitary superperfect polynomial.*

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Proof. Follows from part 1) of Lemma 17. □

Lemma 20. *If $A = P^\alpha \in \mathbb{F}_2[x]$, then A is not a bi-unitary superperfect polynomial.*

Proof. Assume $A = P^a$ is a bi-unitary superperfect. Since P divides A, then $x(x+1)$ divides $\sigma^{**}(A)$ and by Lemma 16 we have $x(x+1)$ divides $\sigma^{**2}(A) = P^a$. A contradiction. □

In particular, if M is a Mersenne prime polynomial over $\mathbb{F}_2$, then M^c (c is a positive integer) is never a bi-unitary superperfect polynomial.

Corollary 5. *Let $a \in \mathbb{N}^*$ and let $A = P^a$ be a bi-unitary superperfect polynomial over $\mathbb{F}_2$, then $P \in \{x, x+1\}$.*

It is clear from the preceding two corollaries that a bi-unitary superperfect polynomial must be even.

Theorem 2. *Let A be a polynomial over $\mathbb{F}_2$ with $\omega(A) = 1$, then A is a bi-unitary superperfect polynomial if and only if $A, \overline{A} \in \{x^2, x^{2^d-1}\}$, where $d \in \mathbb{N}^*$.*

Proof. By Corollary 5, $A = x^\alpha$ or $(x+1)^\alpha$. Assume $A = x^\alpha$ and $\alpha = 2m$, then $\sigma^{**2}(A) = \sigma^{**}\left((x^{m+1}+1)\dfrac{x^m-1}{x-1}\right)$. Both $x^{m+1}+1$ and x^m+1 split over $\mathbb{F}_2$ only when $m = 1$. Thus, $\sigma^{**2}(A) = \sigma^{**}\left(x^2+1\right) = x^2$. If $\alpha = 2m+1$, then $\sigma^{**2}(A) = \sigma^{**}\left(\dfrac{x^{2(m+1)}-1}{x-1}\right)$. The expression $x^{2(m+1)}+1$ splits over $\mathbb{F}_2$ when $2m+2 = 2^d$, $d \in \mathbb{N}^*$. Then, $\sigma^{**2}(A) = \sigma^{**}\left(\dfrac{x^{2^d}-1}{x-1}\right) = A = x^{2^d-1}$. The sufficient condition follows by a direct computation and the result follows since if A is a bi-unitary superperfect, then so is $\overline{A}$. □

4.2. Case w(A)=2

We consider the polynomial $A = P^a Q^b$ and $a, b \in \mathbb{N}^*$. Note that $A = x^2(1+x)^2$ and $A = x^{2^\alpha-1}(1+x)^{2^\alpha-1}$ are bi-unitary superperfect polynomials over $\mathbb{F}_2$, see Lemma 10 and (Theorem 5 in [11]).

Corollary 6. *If A is a bi-unitary superperfect polynomial over $\mathbb{F}_2$, then $A = x^a(x+1)^b$.*

Proof. Follows directly from Corollary 4. □

Lemma 21. [Lemma 3.1 in [12]] If the polynomial $\sigma^{**}(x^a(x+1)^b)$ does not split, then ($a \geq 3$ or $b \geq 3$) and ($a \neq 2^n - 1$ or $b \neq 2^m - 1$ for any $n, m \geq 1$).

Lemma 22. *Let $a, b, d \in \mathbb{N}^*$. The polynomial $A = x^a(x+1)^b$ is a bi-unitary superperfect over $\mathbb{F}_2$ if and only if one of the following is true.*

1) *If a and b are odd and $\sigma^{**}(x^a(x+1)^b)$ splits, then a and b are of the form $2^d - 1$.*
2) *If a and b are odd and $\sigma^{**}(x^a(x+1)^b)$ doe not split, then $(a, b) \in \{(9, 9), (9, 13), (13, 9)\}$.*
3) *If a and b are even, then $a = b \in \{2, 4\}$.*
4) *If a is odd and b is even, then $(a, b) \in \left\{\left(2, 2^d - 1\right), \left(2^d - 1, 2\right)\right\}$.*

Proof. 1) If $a = 2m+1$ and $b = 2n+1$, then $\sigma^{**2}(A) = \sigma^{**}\left(\sigma^{**}(x^a)(1+x)^b\right)$. But $\sigma^{**}(x^{2m+1})$ and $\sigma^{**}(x+1)^{2n+1}$ split over $\mathbb{F}_2$ when $2m+1$ and $2n+1$ are of the form $2^d - 1$ (Lemma 7).
2) If $a = 2^\alpha u - 1$ and $b = 2^\beta v - 1$, u, v are odd. We have $u > 1$ and $v > 1$, since $\sigma^{**}(x^a(x+1)^b)$

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does not split. $\sigma^{**}(x^a(x+1)^b) = \sigma^{**}\left((1+x)^{2^\alpha-1}\left(\sigma(x^{u-1})\right)^{2^\alpha} x^{2^\beta-1}\sigma\left((x+1)^{v-1}\right)^{2^\beta}\right)$. By Lemma 21 $(u-1 \geq 3$ and $\alpha = 1)$ or $(v-1 \geq 3$ and $\beta = 1)$. Also, $\sigma\left(x^{u-1}\right)$ and $\sigma\left((x+1)^{v-1}\right)$ does not split since $\sigma^{**}(x^a(x+1)^b)$ does not split. So, there exit Merssene primes M (resp. M') that divides $\sigma\left(x^{u-1}\right)$ (resp. $\sigma\left((x+1)^{v-1}\right)$. Hence, $(u-1 \leq 6)$ or $(v-1 \leq 6)$ and we have $u, v \in \{5,7\}$. If $u = v = 5$, then $a = b = 9$. If $u = 5$ and $v = 7$, then $a = 9$ and $b = 13$. If $u = v = 7$, then $a = b = 13$ is dismissed.
3) If a, b even, then $a \in \{4r, 4r+2\}$ such that $2r-1, 2r+1$ is of the form $2^\alpha u - 1$ with u is odd and $b \in \{4r', 4r'+2\}$ such that $2r'-1, 2r'+1$ is of the form $2^\beta v - 1$, v odd. Thus,

$$\sigma^{**}(A) = (1+x)^{2^\alpha-1}\sigma(x^{2r})\left(\sigma(x^{u-1})\right)^{2^\alpha} x^{2^\beta-1}\sigma\left((x+1)^{2r'}\right)\left(\sigma\left((x+1)^{v-1}\right)\right)^{2^\beta}.$$

If $\sigma(x^{2r})$, $\sigma\left((x+1)^{2r'}\right)$, $\sigma(x^{u-1})$ and $\sigma\left((x+1)^{v-1}\right)$ are Mersenne, then $2r, 2r', u-1, v-1 \in \{2,4\}$. So, $a = b = 4$. If $\sigma(x^{2r})$, $\sigma(x^{u-1})$, $\sigma\left((x+1)^{2r'}\right)$ and $\sigma\left((x+1)^{v-1}\right)$ are not Mersenne, then $r, r', u-1, v-1 > 2$ and $\omega(\sigma^{**2}(A)) > 2$, a contradiction. For $a = b = 2$, A is bi-unitary perfect and hence A is a bi-unitary superperfect.

4) Now, let $a = 2m+1$ and $b = 2n$. Since $\sigma^{**}\left((x+1)^{2n}\right)$ splits over $\mathbb{F}_2$ only when $n = 1$, then $\sigma^{**2}(A) = \sigma^{**}\left(\sigma^{**}(x^{2m+1})\sigma^{**}\left((x+1)^2\right)\right)$. But $\sigma^{**}(x^{2m+1})$ splits over $\mathbb{F}_2$ if $2m+1$ is of the form $2^d - 1$. If $a = 2m$ and $b = 2n+1$, then $a = 2$ and $b = 2^d - 1$. The sufficient condition can be easily verified. □

The proof of Theorem 1 is now complete.

5. Conclusion

In conclusion, a non constant bi-unitary superperfect polynomial A over $\mathbb{F}_2$ can be divisible by one irreducible polynomial x or $x+1$ and its exponent is 2 or $2^n - 1$ for a positive integer n. Moreover, the only bi-unitary superperfect polynomials over $\mathbb{F}_2$ with exactly two prime factors are $x^a(x+1)^b$ with $a, b \in \{2, 4, 9, 13, 2^d - 1\}$, d is a positive integer and $a = b$ if and only if $a, b \in \{2,4\}$.

6. Table

Consider the polynomials $C_1 = x^4 + x + 1$, $C_2 = x^6 + x^5 + x^4 + x^2 + 1$, $C_3 = x^6 + x^5 + x^4 + x + 1$, and $C_4 = x^{10} + x^9 + x^8 + x^7 + x^2 + x + 1$. The below table lists the values of $\sigma^{**}(A)$ and $\sigma^{**2}(A)$ for $A \in \{x^a, (x+1)^a, M_i^b\}$ with $1 \leq$ a ≤ 13, $1 \leq b \leq 7$.

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A	a	σ^{**}	σ^{**2}
x^a	1	x	$x+1$
	2	x^2	$(x+1)^2$
	3	x^3	$(x+1)^3$
	4	x^2M_1	$x(x+1)^3$
	5	xM_1^2	$x^2(x+1)^3$
	6	x^4M_1	$x(x+1)^3M_1$
	7	x^7	$(x+1)^7$
	8	x^4M_5	$x^3(x+1)^3M_1$
	9	xM_5^2	$x^6(x+1)^3$
	10	$x^2M_1^2M_5$	$x^5(x+1)^5$
	11	$x^3M_1^4$	$x^2(x+1)^5C_1$
	12	$x^2M_1^2M_2M_3$	$x^5(x+1)^7$
	13	$xM_2^2M_3^2$	$x^6(x+1)^7$
....	....	....	....
$(1+x)^a$	1	x	$x+1$
	2	x^2	$(x+1)^2$
	3	x^3	$(x+1)^3$
	4	x^2M_1	$x(x+1)^3$
	5	xM_1^2	$x^2(x+1)^3$
	6	x^4M_1	$x(x+1)^3M_1$
	7	x^7	$(x+1)^7$
	8	x^4M_5	$x^3(x+1)^3M_1$
	9	xM_5^2	$x^6(x+1)^3$
	10	$x^2M_1^2M_5$	$x^5(x+1)^5$
	11	$x^3M_1^4$	$x^2(x+1)^5C_1$
	12	$x^2M_1^2M_2M_3$	$x^5(x+1)^7$
	13	$xM_2^2M_3^2$	$x^6(x+1)^7$
....	....	....	....
M_1^a	1	$x(x+1)$	$x(x+1)$
	2	$x^2(x+1)^2$	$x^2(x+1)^2$
	3	$x^3(x+1)^3$	$x^3(x+1)^3$
	4	$x^2(x+1)^2C_1$	$x^3(x+1)^3M_1$
	5	$x(x+1)C_1^2$	$x^3(x+1)^3M_1^2$
	6	$x^4(x+1)^4C_1$	$x^3(x+1)^3M_1^3$
	7	$x^7(x+1)^7$	$x^7(x+1)^7$
....	....	....	....
M_2^a	1	$x(x+1)^2$	$x^2(x+1)$
	2	$x^2(x+1)^4$	$x^2(x+1)^2M_1$
	3	$x^3(x+1)^6$	$x^4(x+1)^3M_1$
	4	$x^2(x+1)^4M_1M_5$	$x^6(x+1)^4M_1$
	5	$x(x+1)^2M_1^2M_5^2$	$x^{10}(x+1)^5$
	6	$x^4(x+1)^8M_1M_5$	$x^8(x+1)^4M_1M_5$
	7	$x^7(x+1)^{14}$	$x^8(x+1)^7M_2M_3$

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M_3^a	1	$x^2(x+1)$	$x(x+1)^2$
	2	$x^4(x+1)^2$	$x^2(x+1)^2M_1$
	3	$x^6(x+1)^3$	$x^3(x+1)^4M_1$
	4	$x^4(x+1)^2M_1M_4$	$x^4(x+1)^6M_1$
	5	$x^2(x+1)M_1^2M_4^2$	$x^5(x+1)^{10}$
	6	$x^8(x+1)^4M_1M_4$	$x^4(x+1)^8M_1M_4$
	7	$x^{14}(x+1)^7$	$x^7(x+1)^8M_2M_3$
....	....	....	....
M_4^a	1	$x(x+1)^3$	$x^3(x+1)$
	2	$x^2(x+1)^6$	$x^4(x+1)^2M_1$
	3	$x^3(x+1)^9$	$x(x+1)^3\,(M_5)^2$
	4	$x^2(x+1)^6M_1C_2$	$x^7(x+1)^4M_1M_2$
	5	$x(x+1)^3M_1^2C_2^2$	$x^9(x+1)^5M_2^2$
	6	$x^4(x+1)^{12}M_1C_2$	$x^5(x+1)^4M_1^3M_2^2M_3$
	7	$x^7(x+1)^{21}$	$x(x+1)^7$ C_4^2
....	....	....	....
$(M_5)^a$	1	$x^3(x+1)$	$x(x+1)^3$
	2	$x^6(x+1)^2$	$x^2(x+1)^4M_1$
	3	$x^9(x+1)^3$	$x^3(x+1)M_4^2$
	4	$x^6(x+1)^2M_1C_3$	$x^4(x+1)^7M_1M_3$
	5	$x^3(x+1)M_1^2C_3^2$	$x^5(x+1)^9M_3^2$
	6	$x^{12}(x+1)^4M_1C_3$	$x^4(x+1)^5M_1^3M_2M_3^2$
	7	$x^{21}(x+1)^7$	$x^7(x+1)\left(\sigma\left(x^{10}\right)\right)^2$

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