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Abstract: The aim of this paper is to deep into the dynamic study of the well known Chebyshev-Halley family of iterative methods for solving nonlinear equations. Our objetives are twofold: one the one hand, we are interested in characterize the existence of extraneous attracting fixed points when the methods in the family are applied to polynomial equations. On the other hand, we are also interested in study the free critical points of the methods in the family, as a previous step to determine the existence of attracting cycles. In both cases, we want to identify situations where the methods in the family have a bad behavior from the root-finding point of view. Finally, and joining these two studies, we look for polynomials for which there are methods in the family where this two situations happen simultaneously. That is, the rational map obtained by applying a method in the Chebyshev-Halley family to a polynomial has both super-attracting extraneous fixed point and super-attracting cycles different to the roots of the polynomial.

Keywords: Chebyshev-Halley family of iterative methods; extraneous fixed points; critical points; parameter plane; dynamical systems

MSC: Primary 37F10; Secondary 65H05

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1. Introduction

The well known Chebyshev-Halley family of iterative methods ($\mathcal{CH}$ from now on) for solving nonlinear scalar equations $f(z) = 0$ was firstly introduced by Werner [27] in 1980. This family, initially defined for real valued functions, has been extended to complex variables, systems of equations or even equations defined in Banach spaces, as can be seen, for instance, in [1,2,15] or [20]. In this work we focus our interest in functions defined on the complex plane $f : \mathbb{C} \to \mathbb{C}$.

Each member in $\mathcal{CH}$ is given by an iteration map, that can be seen as a modification of the Newton iteration map

$$N(z) = z - \frac{f(z)}{f'(z)}, \tag{1}$$

with the inclusion of a parameter $\alpha \in \mathbb{C}$ and new evaluations of $f(x)$ an its derivatives up to order 2. Specifically, we build a sequence $z_{n+1} = G_\alpha(z_n)$, $n \geq 0$, where

$$G_\alpha(z) = z - \left(1 + \frac{1}{2}\frac{L_f(z)}{1 - \alpha L_f(z)}\right)\frac{f(z)}{f'(z)}, \tag{2}$$

and

$$L_f(z) = \frac{f(z)f''(z)}{f'(z)^2}. \tag{3}$$

All methods in $\mathcal{CH}$ have cubic order of convergence for simple roots. In the case of multiple roots, there exist several variants of $\mathcal{CH}$ that makes it possible to recover the cubic order of convergence, as the given by Osada [23].

One of the reasons for considering the $\mathcal{CH}$ family is because it allows to study in a unified way the most famous third order iterative methods, such as Chebyshev's method ($\alpha = 0$), Halley's method ($\alpha = 1/2$) or super-Halley method ($\alpha = 1$). In general, it is not possible to establish a classification of the members on $\mathcal{CH}$ in terms of efficiency, velocity of convergence and other similar numerical criteria because the behavior of $G_\alpha(z)$ depends on the considered function $f(z)$, through the asymptotic error constant (see [12] for more details). However, for particular problems it is possible to find the optimal method in $\mathcal{CH}$ to approximate the solution. This is the case of the calculus of n-th roots, (see Dubeau and Gnang [13] or Gutiérrez, Hernández and Salanova [17] for a more detailed study), the computation of the matrix sign (Cordero et al. [11]) or the simultaneous calculus of all the roots of a polynomial (Osada [23]).

The dynamical study of the iteration maps arising from the application of the methods in $\mathcal{CH}$ (2) to polynomial equations is a problem that has attracted many researchers. For instance, Cordero, Torregrosa and Vindel [8] study the dynamics of the methods $\mathcal{CH}$ applied to quadratic polynomials. In the analysis of the corresponding parameter planes appears a singular set, baptized as "the cat" by the authors, with curious similarities with the Mandelbrot set. In this same paper, it is emphasized the existence of members in $\mathcal{CH}$ with a pathological behavior as root-finding methods: they can be attracted by limits that are not roots of the considered equation $f(z) = 0$, such as periodic orbits (cycles) or extraneous fixed points. In [9,10] it is proved that there exist methods in $\mathcal{CH}$ with attracting 2-cycles; Campos et al. ([6,7]) study the behavior of $\mathcal{CH}$ for polynomials in the form $z^n + c$, where c is a complex parameter. In particular, they characterize methods withs Fatou components that are simply connected and hence the Julia set is connected; Gutiérrez, Magreñán and Varona [18] characterize the universal Julia sets for methods in $\mathcal{CH}$ applied to quadratic polynomials; in this same line, Babajee, Cordero and Torregrosa [3] introduce the *Cayley Quadratic Test* as a first step in the study of the stability of families of iterative processes for solving nonlinear equations. In brief, this test allows to check if the universal Julia set of an iterative process is conjugated with the unit circle or not.

In the rest of this work we continue with the dynamic study of the methods of the $\mathcal{CH}$-family. Specifically, in Section 2 we characterize the existence of super-attracting fixed points for the methods in $\mathcal{CH}$. We make a spacial emphasis in the case of Chebyshev and super-Halley methods. For these methods we even prove the existence of polynomials with both super-attracting fixed points and super-attracting cycles. In Section 3 we study the number of critical points of the methods in $\mathcal{CH}$. In particular, we show that the graphical tool known as parameter plane is useful only for two methods in the $\mathcal{CH}$-family: $\alpha = 0$ (Chebyshev's method) and $\alpha = 1/2$ (Halley's method). These two cases have been profusely studied by Gutiérrez and Varona [19] and by Roberts and Horgan-Kobelski [24] respectively. For the rest of methods in the family the high number of free critical points and the difficulty for obtaining them discourages the use of the parameter plane.

2. Fixed Points in the Family $\mathcal{CH}$

In this section we apply the methods introduced in (2) to a polynomial equation

$$p(z) = 0, \quad p(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_1 z + a_0. \tag{4}$$

We can assume, without loss of generality, that $p(z)$ is a monic polynomial. We use the notation $R_{\alpha,p}(z)$ for the rational map obtained in this case

$$R_{\alpha,p}(z) = z - \left(1 + \frac{1}{2}\frac{L_p(z)}{1 - \alpha L_p(z)}\right)\frac{p(z)}{p'(z)}, \tag{5}$$

where $L_p(z)$ is defined (3). In addition, we consider the rational map related to Newton's method (1) in the polynomial case

$$R_p(z) = z - \frac{p(z)}{p'(z)}. \tag{6}$$

The first thing to do for investigating the dynamics of the rational maps defined in (5) is to study its degree. Following Nayak and Pal [22] we can generalize the result they give for Chebyshev's method to all the methods in the family (5). The exact degree is given in terms of the number of distinct roots of p and in terms of certain types of critical points introduced by these authors. Indeed, given a polynomial $p(z)$, a critical point $\omega \in \mathbb{C}$ is called special if $p(\omega) \neq 0$ and $p''(\omega) = 0$.

Theorem 1. *Let $p(z)$ be the polynomial defined in* (4). *Let m, n and r denote the number of its distinct simple roots, double roots and roots of multiplicity bigger than 2 respectively. Let s be the number of distinct special critical points of $p(z)$. Then*

$$\deg(R_{\alpha,p}(z)) = 3(m+n+r) - 2 - B + s$$

where B is the sum of multiplicities of all the special critical points. If $p(z)$ has no special critical points then $\deg(R_{\alpha,p}(z)) = 3(m+n+r) - 2$. If $p(z)$ has no special critical points neither multiple roots, then $\deg(R_{\alpha,p}(z)) = 3d - 2$.

Proof. The proof mimics the one given by Nayak and Pal [22] for the case of Chebyshev's method, $R_{0,p}(z)$ given by $\alpha = 0$ in (5). □

As a root-finder method, it would be desirable that each attracting fixed point of $R_{\alpha,p}(z)$ would be a root of $p(z)$. However, this "ideal behavior" is disturbed by the appearance of other attracting phenomena, such as periodic orbits (cycles) or extraneous fixed points, that is fixed points of $R_{\alpha,p}(z)$ that are not roots of $p(z)$.

It is well known that the only fixed points of Newton's method in the complex plane are the roots of $p(z)$ (see the classical book of Traub [25], for instance). If we consider the extended complex plane, it is also known that the infinity point is a repelling fixed point for Newton's method, with multiplier $d/(d-1) > 1$, where d is the degree of $p(z)$. In addition, we have that simple roots of $p(z)$ are super-attracting fixed points of $R_p(z)$, whereas roots with multiplicity $m > 1$ are attracting fixed points with multiplier $(m-1)/m < 1$.

For the case of Halley's method (obtained for $\alpha = 1/2$ in (5)) it is known (Kneisl [21, Theorem 2.6.3]) that the only attracting fixed points are the roots of $p(z)$. There exists extraneous fixed points in the complex plane, but all of them are repelling with multiplier $1 + 2/j$ for an adequate $j \in \mathbb{N}$. The infinity point is a repelling fixed point for Halley's method, with multiplier $(d+1)/(d-1) > 1$, where d is the degree of $p(z)$.

For the case of Chebyshev method ($\alpha = 0$ in (5)) the existence of attracting extraneous fixed points has been proven (see [21, Theorem 2.6.4] or Vrscay-Gilbert [26, pp. 12]). In addition, in [19] the existence of super-attracting extraneous fixed points has been established in terms of the quotients $L_p(z)$, defined in (4) and $L_{p'}(z)$, defined by

$$L_{p'}(z) = \frac{p'(z)p'''(z)}{p''(z)^2}. \tag{7}$$

In particular, ω is a extraneous fixed point of Chebyshev's method if $L_p(z) = -2$. ω is attracting if $\left|6 - 2L_{p'}(z)\right| < 1$ and super-attracting if $L_{p'}(z) = 3$.

In this section we are going to generalize this result, by determining sufficient conditions for the existence of super-attracting extraneous fixed points for the methods in $\mathcal{CH}$. First, we estate a preliminary technical result that will help us in the further theoretical development.

Lemma 1. *Let $p(z)$ be a d-degree polynomial and $L_p(z)$, $L_{p'}(z)$ the rational functions defined in* (4) *and* (7) *respectively. Then*

$$L'_p(z)\frac{p(z)}{p'(z)} = L_p(z) - 2L_p(z)^2 + L_{p'}(z)L_p(z)^2. \tag{8}$$

Proof. The proof simply requires a process of derivation and grouping of terms in an appropriate way. Firstly

$$L'_p(z) = \left(\frac{p(z)p''(z)}{p'(z)^2} \right)' = \frac{(p'(z)p''(z) + p(z)p'''(z))p'(z)^2 - 2p(z)p'(z)p''(z)^2}{p'(z)^4},$$

next

$$L'_p(z)\frac{p(z)}{p'(z)} = L_p(z) + \frac{p(z)^2 p'''(z)}{p'(z)^3} - 2L_p(z)^2.$$

Just by multiplying and dividing by $p''(z)^2/p'(z)$ in the second term, and taking (7) into account, we arrive at the result. $\square$

Theorem 2. *For $\alpha \neq 1/2$, let $\omega \in \mathbb{C}$ be a point such $p(\omega) \neq 0$, $p'(\omega) \neq 0$, $p''(\omega) \neq 0$. Then, if*

$$L_p(\omega) = \frac{2}{2\alpha - 1}, \quad L_{p'}(\omega) = 3 - \alpha, \tag{9}$$

ω is a super-attracting extraneous fixed point of the rational map $R_{\alpha,p}(z)$ defined in (5) and therefore of the iterative method corresponding to the parameter α in $\mathcal{CH}$.

Proof. First, as $p(\omega) \neq 0$, ω is not a root of $p(z)$. In addition, ω is a fixed point of $R_{\alpha,p}(z)$ because

$$1 + \frac{1}{2}\frac{L_p(\omega)}{1 - \alpha L_p(\omega)} = 0, \tag{10}$$

just by taking into account the first equation in (9). So ω is a extraneous fixed point of $R_{\alpha,p}(z)$.

For ω to be a super-attractor, it must be true that $R'_{\alpha,p}(\omega) = 0$. Then, taking into account (10), we have

$$R'_{\alpha,p}(\omega) = 1 - \left(1 + \frac{1}{2}\frac{L_p(z)}{1 - \alpha L_p(z)} \right)'\Big|_{z=\omega} \frac{p(\omega)}{p'(\omega)}.$$

Note that

$$\left(1 + \frac{1}{2}\frac{L_p(z)}{1 - \alpha L_p(z)} \right)' = \frac{1}{2}\frac{L'_p(z)}{(1 - \alpha L_p(z))^2},$$

and then, by Lemma 1,

$$R'_{\alpha,p}(\omega) = 1 - \frac{1}{2}\frac{L_p(\omega) - 2L_p(\omega)^2 + L_{p'}(\omega)L_p(\omega)^2}{(1 - \alpha L_p(\omega))^2}.$$

As $L_p(\omega) = 2/(2\alpha - 1)$, we obtain

$$R'_{\alpha,p}(\omega) = 2(3 - \alpha - L_{p'}(\omega)).$$

Finally, as $L_{p'}(\omega) = 3 - \alpha$, the condition $R'_{\alpha,p}(\omega) = 0$ also holds and, as a consequence, ω is a super-attracting extraneous fixed point of the rational map $R_{\alpha,p}(z)$
$\square$

The previous theorem allows us to find, as long as its conditions are met, extraneous super-attractor fixed points for all methods in the family $\mathcal{CH}$, with the exception of Halley's method ($\alpha = 1/2$). Note that conditions $p(\omega) \neq 0$, $p'(\omega) \neq 0$, $p''(\omega) = 0$ imply $L_p(\omega) = 0$, that makes the existence of extraneous super-attractor fixed points impossible. If $\alpha \neq 1/2$, $p(\omega) \neq 0$ and $p'(\omega) = 0$, the expression of the corresponding method in (5) is not well defined.

For Halley's method, we can give the following result, which was also proven by Kneisl ([21, Theorem 2.6.3]). Previously, we write the corresponding iteration function with an alternative expression:

$$R_{1/2,p}(z) = z - \left(1 + \frac{1}{2}\frac{L_p(z)}{1 - L_p(z)/2}\right)\frac{p(z)}{p'(z)} = z - \frac{2p(z)p'(z)}{2p'(z)^2 - p(z)p''(z)}. \tag{11}$$

Theorem 3. *Let us consider Halley's method, obtained for $\alpha = 1/2$ in $\mathcal{CH}$ and whose iteration function is shown in (11). The extraneous fixed points of the rational function $R_{1/2,p}(z)$ are solutions of $p'(z) = 0$, with $p(z) \neq 0$. All of them are all repulsors.*

Proof. The fixed points of $R_{1/2,p}(z)$ are the roots of $p(z)$ and the solutions of $p'(z) = 0$. Then, ω is a extraneous fixed point of $R_{1/2,p}(z)$ if and only if $p'(\omega) = 0$ and $p(\omega) \neq 0$. Let $m \in \mathbb{N}$ be the multiplicity of ω as a root of $p'(z)$. We can write

$$p'(z) = (z - \omega)^m g(z),$$

with $g(\omega) \neq 0$. After a few calculus in (11) we obtain

$$R'_{1/2,p}(\omega) = 1 + \frac{2}{m} > 1.$$

Therefore ω is a repelling fixed point. $\square$

Remark 1. *In the extended complex plane, the infinity point is a fixed point for the methods in $\mathcal{CH}$, with multiplier*

$$\frac{2d(d(\alpha - 1) - \alpha)}{(d - 1)(2d(\alpha - 1) - 2\alpha + 1)}.$$

In the case of the most famous methods in the family, infinity is a repelling fixed point. Indeed, for $\alpha = 0, 1/2, 1$, that is, Chebyshev, Halley and super-Halley methods, the multiplier of the infinity point is $2d^2/((2d - 1)(d - 1))$, $(d + 1)/(d - 1)$ and $2d/(d - 1)$ respectively. However, we can see that infinity is not always a repelling fixed point. Even more, for each degree $d \geq 2$ we can obtain a method, corresponding with the value of

$$\alpha = \frac{d}{d - 1},$$

for which infinity is a super-attracting fixed point. In Figure 1 we show the basins of attraction related to the method in $\mathcal{CH}$ with $\alpha = 2$ applied to $p(z) = z^2 - 1$ and related to the method in $\mathcal{CH}$ with $\alpha = 3/2$ applied to $p(z) = z^3 - 1$. Together with the basins of the roots of these polynomials we have plot in white the basin of attraction of the infinity point.
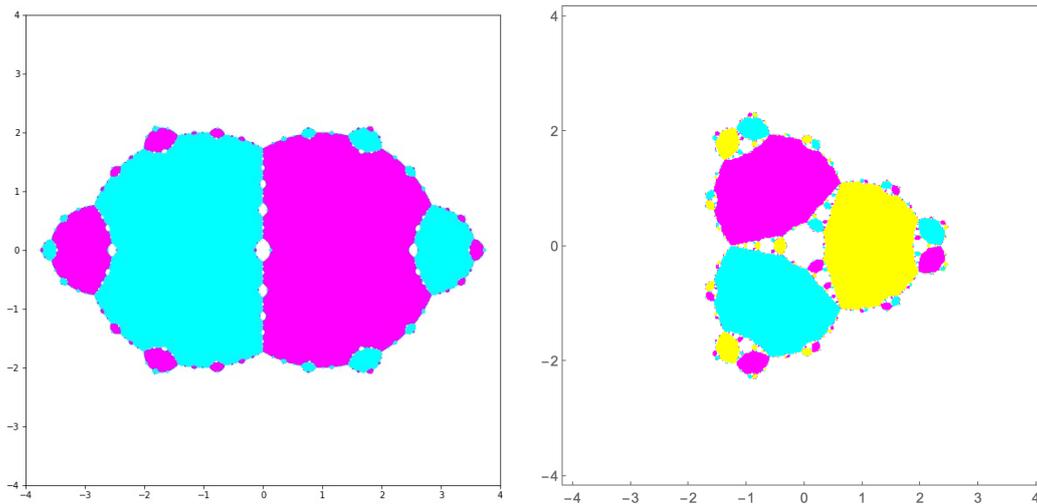
Figure 1. On the left, basins of attraction of the method corresponding to $\alpha = 2$ applied to the polynomial $p(z) = z^2 - 1$. On the right, basins of the method corresponding to $\alpha = 3/2$ applied to the polynomial $p(z) = z^3 - 1$. In both cases the basin of attraction of the infinity point can be seen in white.

We are now going to characterize polynomials for which $z = 0$ is a strange super-attractor fixed point for the methods of the (5) family. To do this, we consider the generic polynomial of degree d defined in (4).

Theorem 4. *Let $p(z)$ be the polynomial defined in(4) and $R_{\alpha,p}(z)$ the rational map defined in (5), with $\alpha \neq 1/2$. Then if*

$$a_1 = \beta, \quad a_2 = \frac{\beta^2}{2\alpha - 1}, \quad a_3 = \frac{2(3 - \alpha)}{3} \frac{\beta^3}{(2\alpha - 1)^2},$$

with $\beta \in \mathbb{C} \setminus \{0\}$, $z = 0$ is a super-attracting extraneous fixed point of $R_{\alpha,p}(z)$.

Proof. Note that, since $z = 0$ must not be a root of $p(z)$, we can consider, without loss of generality, that $a_0 = 1$. Furthermore, for the conditions of the Theorem 2 to be fulfilled, $p'(0) \neq 0$ and $p''(0) \neq 0$, it is also necessary that $a_1 \neq 0$ and $a_2 \neq 0$. The proof continues simply by solving the system given by the equations (9) which, in this case, are

$$L_p(0) = \frac{2a_2}{a_1^2} = \frac{2}{2\alpha - 1}, \quad L_{p'}(0) = \frac{3a_1 a_3}{2a_2^2} = \frac{2}{2\alpha - 1}.$$

□

As particular cases, we can obtain polynomials with $z = 0$ as an extraneous super-attracting fixed point for the Chebyshev's method (already known by García-Olivo et al. [14]). In this case,

$$a_1 = \beta \neq 0, \quad a_2 = -\beta^2, \quad a_3 = 2\beta^3. \tag{12}$$

Super-Halley method is another well-known method in $\mathcal{CH}$ ([4], [16]). However, its dynamical properties have been less studied. We can obtain polynomials with $z = 0$ as an extraneous super-attracting fixed point for the super-Halley method

$$a_1 = \beta \neq 0, \quad a_2 = \beta^2, \quad a_3 = \frac{4}{3}\beta^3. \tag{13}$$

In the left side of Figure 2 we show the basins of attraction of the Chebyshev's method applied to the polynomial

$$p(z) = z^3 + z^2 + 2z - 4. \tag{14}$$

It is obtained by taking $\beta = 1/2$ in (12) and next by multiplying by 4. The basin of the root $z = 1$ appears colored in cyan, that of the root $z = -1 + \sqrt{3}i$ in yellow and that of the root $z = -1 - \sqrt{3}i$ in magenta. The basin of attraction of the extraneous fixed point appears in white.

In the right part of Figure 2 we show the basins of attraction of the super-Halley applied to the polynomial

$$p(z) = 4z^3 + 3z^2 + 3z + 3. \tag{15}$$

It is obtained by taking $\beta = 1$ in (13) and next by multiplying by 3. The basin of the three roots $z = -0.873873$, $z = 0.0619363 - 0.924344i$ and $z = 0.0619363 + 0.924344i$ are colored in cyan, yellow and magenta respectively. The basin of attraction of the extraneous fixed point $z = 0$ appears in white.
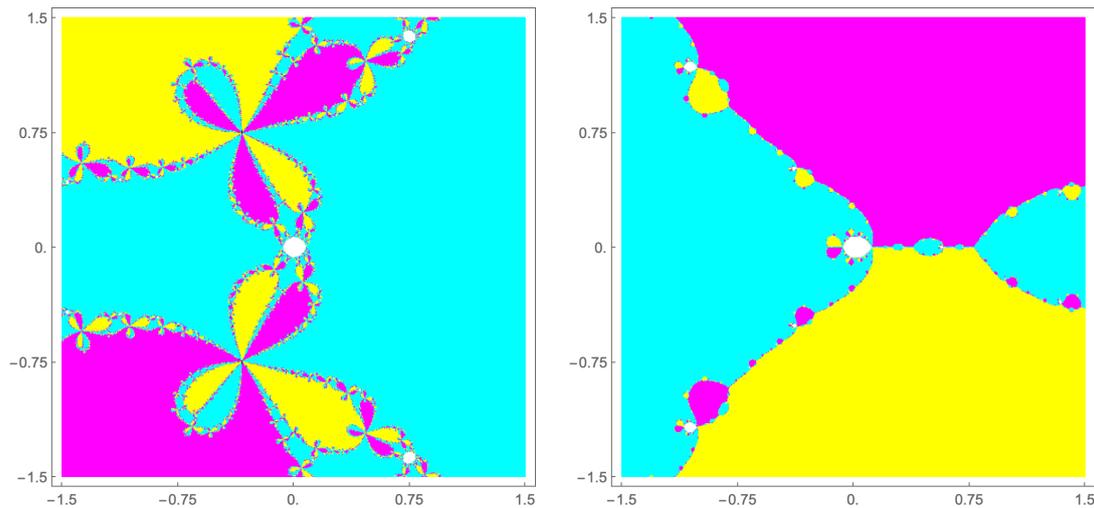
Figure 2. On the left, basins of attraction of the Chebyshev method applied to the polynomial (14). On the right, basins of attraction of the super-Halley method applied to the polynomial (15). In both cases the basin of attraction of the extraneous fixed point at $z = 0$ can be seen in white.

In general, Chebyshev's method applied to polynomials in the form

$$p(z) = -1 + az + a^2z^2 + 2a^3z^3 + \sum_{j=4}^{d} a_j z^j \tag{16}$$

has a super-attracting extraneous fixed point at $z = 0$. Actually, we have

$$p(0) \neq 0, \quad L_p(0) = -2, \quad L_{p'}(0) = 3,$$

so the conditions in Theorem 2 are fulfilled.

In a similar way, super-Halley method applied to polynomials in the form

$$p(z) = 1 + az + a^2z^2 + 4/3a^3z^3 + \sum_{j=4}^{d} a_j z^j \tag{17}$$

has a super-attracting extraneous fixed point at $z = 0$. Indeed, we have

$$p(0) \neq 0, \quad L_p(0) = L_{p'}(0) = 2,$$

so the conditions in Theorem 2 are satisfied.

3. Critical Points in the Family $\mathcal{CH}$

The parameter plane (space) is a very powerful graphical tool for better understanding the dynamic behavior of an iterative method for solving a family of nonlinear equations depending ion a complex parameter. It is based on the Fatou-Julia Theorem [5] which says that the immediate basin of attraction of a (super) attractor cycle contains at least one critical point. Consequently, to determine the existence of attracting behaviors (fixed points, cycles), we must study the iterations of the critical points of the iteration function in question.

Let us restrict our interest to the case of iterative methods applied to polynomial equations (4). In this case, a free critical point of an iterative method is a critical point of the corresponding iteration map that is not a root of the polynomial $p(z)$. Taking into account that the roots of $p(z)$ are (super) attracting fixed points of the iteration map, all of them have their own basin of attraction that is related to a critical point (the same root). Therefore to detect attracting behaviors different from the root we must follow the orbits of the free critical point.

For example, G. Roberts and J. Horgan-Kobelski [24] characterize cubic polynomials in the form

$$p_\lambda(z) = (z-1)(z+1)(z-\lambda), \quad \lambda \in \mathbb{C}, \tag{18}$$

for which Newton's method has super-attracting n-cycles. Specifically, they obtain (numerically) some values of the parameters $\lambda_n = \beta_n i$, with β_n given in Table 1, for which Newton's method applied to the polynomial $p_{\lambda_n}(z)$ defined in (18) with $\lambda_n = \beta_n i$ has a super-attracting n-cycle.

Table 1. Approximate values of the parameter β_n for which Newton's method applied to the polynomial $p_{\lambda_n}(z)$ defined in (18) with $\lambda_n = \beta_n i$ has a super-attracting n-cycle (see [24]).

n	β_n	n	β_n	n	β_n
2	4.500116	6	1.982100	10	1.777227
3	2.938069	7	1.892459	20	1.732819
4	2.396385	8	1.836401	25	1.732152
5	2.131089	9	1.800522	50	1.732051

The strategy consists of coloring the parameter space $\lambda \in \mathbb{C}$ according to the convergence of the only free critical point $\lambda/3$, as done in Figure 3. If the orbit of $\lambda/3$ converges to 1, -1 or λ, the value of the corresponding parameter λ is colored in cyan, magenta or yellow respectively.
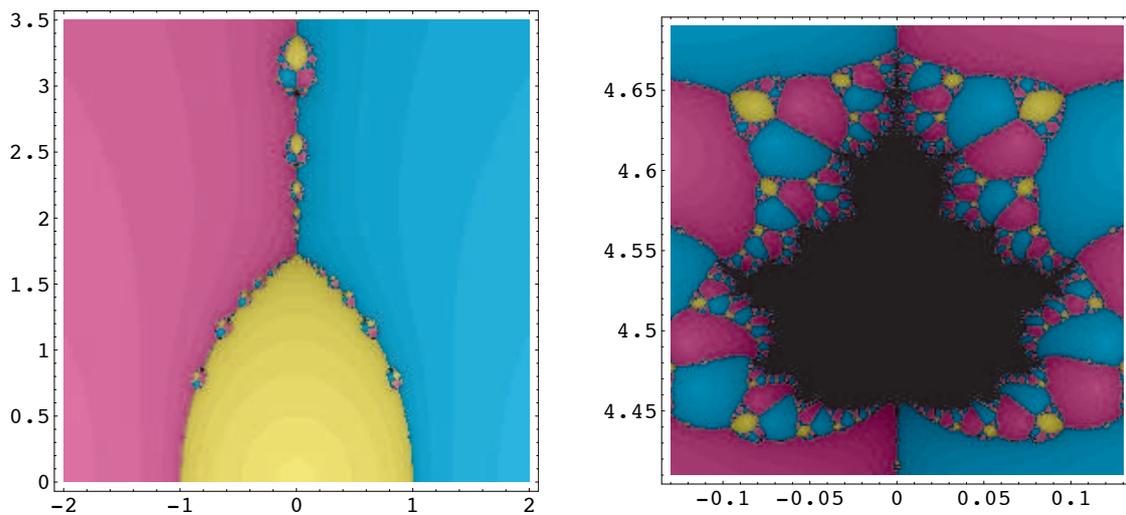
Figure 3. Graphical representation of the parameter plane of Newton's method applied to polynomials in the form (18). The figure on the right shows an enlargement of a black area where a Mandelbrot-type set can be seen.

The black colored regions in the parameter space are formed by the values of λ for which Newton's method applied to the corresponding polynomial $p_\lambda(z)$, has an attracting cycle that does not contain any root of the polynomial. The appearance of Mandelbrot-type sets in the black areas of the parameter plane is a notable phenomenon.

Continuing in this line of work, Roberts and Horgan-Kobelski themselves [24] or, previously, E. R. Vrscay and W. J. Gilbert [26], prove the existence of polynomials with attracting cycles for Halley's method. Specifically, the table of values of β_n for which Halley's method applied to the polynomial $p_{\lambda_n}(z)$ with $\lambda_n = \beta_n i$, has a super-attracting n-cycle is shown in Table 2.

Table 2. Approximate values of the parameters β_n for which Halley's method applied to the polynomial $p_{\lambda_n}(z)$ defined in (18) with $\lambda_n = \beta_n i$ has super-attracting n-cycle(see [24]).

| n	β_n	n	β_n	n	β_n
2	1.342232060	6	1.018656160	10	1.0011575040
3	1.158338303	7	1.009291550	20	1.0000011298
4	1.076647075	8	1.004637455	25	1.0000000353
5	1.037611090	9	1.002316127	50	1.0000000011

The strategy to graphically represent the parameter space associated with Halley's method applied to polynomials of the form (18) changes slightly, due to the appearance of two free critical points:

$$\rho_\pm(\lambda) = \frac{2\lambda \pm \sqrt{-2\lambda^2 - 6}}{6}.$$

As can be seen in [24], the range of colors in the parameter plane is expanded, according to the criterion shown in Table 3. For example, λ is colored blue if the orbit of $\rho_-(\lambda)$ converges to the root -1 and the orbit of $\rho_+(\lambda)$ converges to the root 1. The result of coloring the parameter plane of Halley's method in this way is shown in 4.

Table 3. Color scheme for drawing the parameter plane associated with Halley's method applied to the polynomials defined in (18).

Color	$(\rho_-(\lambda), \rho_+(\lambda)) \rightarrow$	Color	$(\rho_-(\lambda), \rho_+(\lambda)) \rightarrow$
Yellow	$(-1, -1)$	Blue	$(-1, 1)$
Green	$(1, -1)$	Red	(λ, λ)
Brown	$(-1, \lambda)$	Pink	$(\lambda, -1)$
Orange	$(1, 1)$	Cyan	$(\lambda, 1)$
Purple	$(1, \lambda)$		
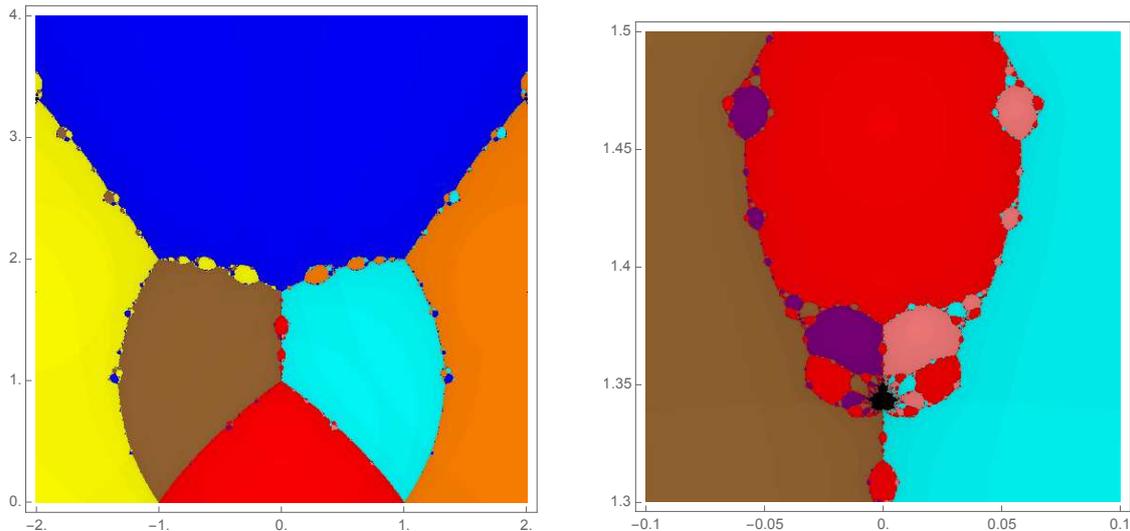
Figure 4. Parameter plane of Halley's method applied to the family of polynomials (18). In the figure on the right we can see a Mandelbrot type set that appears in an enlargement of a black area.

The richness of the dynamic study of the methods of the family $\mathcal{CH}$ increases when we consider Chebyhev's method, as evidenced in the work of Gutiérrez and Varona [19]. With techniques similar to those used for the Newton or Halley methods, values of the parameter β_n can be given for which Chebyshev's method applied to the polynomial $p_{\lambda_n}(z)$ defined in (18) with $\lambda_n = \beta_n i$ has a super-attracting n-cycle (see Table 4).

Table 4. Approximate values of the parameter β_n for which Chebyshev's method applied to the polynomial $p_{\lambda_n}(z)$ defined in (18) with $\lambda_n = \beta_n i$ has a super-attracting n-cycle (see [19]).

n	β_n	n	β_n	n	β_n
2	1.28657	6	1.48369	10	1.62056
3	1.34015	7	1.52557	20	1.72078
4	1.38943	8	1.56245	25	1.72856
5	1.43776	9	1.59405	50	1.72856

The strategy for coloring the parameter plane associated with Chebyshev's method applied to polynomials of the family (18) is the same as that followed for Halley's method (see [19] for more details), studying the orbits of the two fixed points which, in this case are:

$$\rho_\pm(\lambda) = \frac{5\lambda \pm \sqrt{-5\lambda^2 - 15}}{15}.$$

The result is shown in Figure 5.

Now, when analyzing the parameter plane associated with Chebyshev's method applied to the polynomials of the family (18), we have that the "black holes" that appear in it are caused by two reasons: attracting cycles or attracting extraneous fixed points.

On the left side of Figure 5 the parameter plane of Chebyshev's method applied to the family of polynomials (18) is shown. The figure on the right shows an enlargement, around the imaginary axis, where a sort of channel of black holes can be seen. The black hole shown below is associated with a strange fixed point at $z = 2\sqrt{3}/3i$. However, the rest of the black holes in this channel are associated with attractor cycles.

Figure 6 shows two details of black holes, one associated with an extraneous fixed point (on the left) and another with an attracting 2-cycle attractor (on the right) .
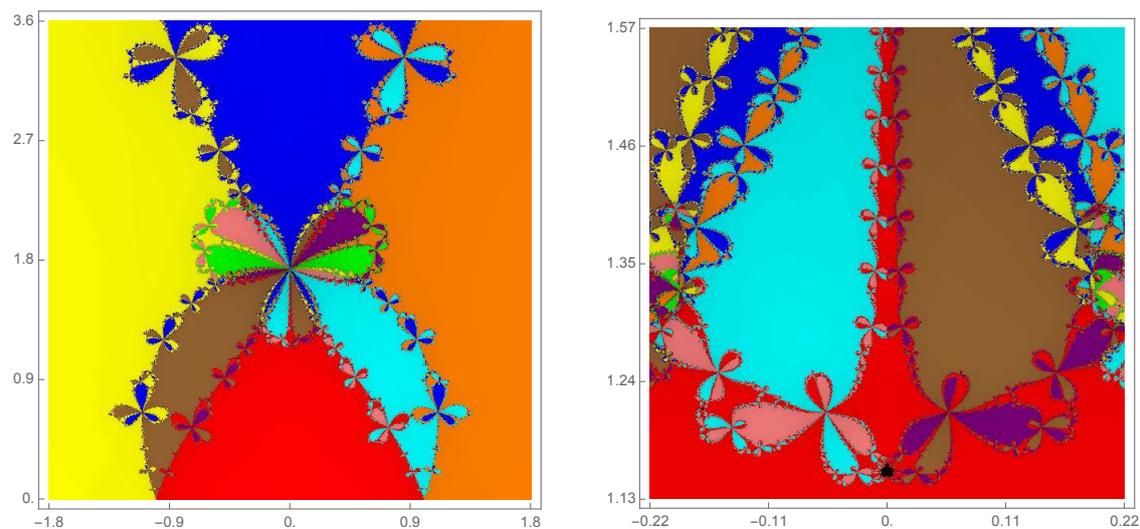
Figure 5. Parameter plane of Chebyshev's method applied to the family of polynomials (18). In the figure on the right a channel of black holes around the imaginary axis can be seen.
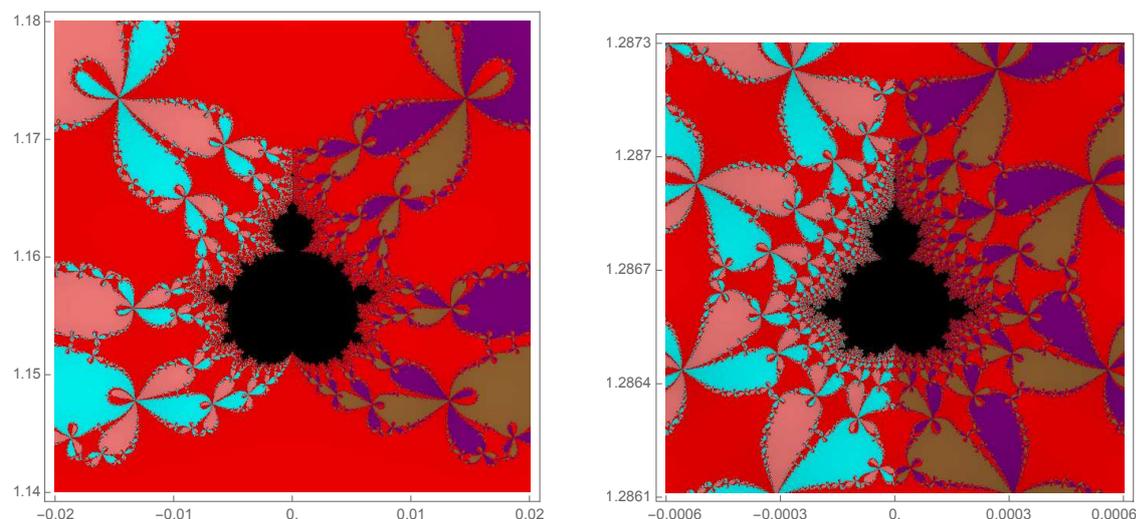
Figure 6. Two details of black holes of different nature: on the left, one associated with an extraneous fixed point and on the right one associated with an attracting 2-cycle.

What happens when we want to draw the parameter plane associated with other methods of the family $\mathcal{CH}$? The situation becomes considerably more complicated. The following results explain the reason.

Lemma 2. *Let $p(z)$ be a polynomial and let $L_p(z)$ and $L_{p'}(z)$ be the rational functions defined in (4) and (7) respectively. Suppose that $p'(z) \neq 0$ and that $p''(z) \neq 0$. Then, the free critical points of the rational function $R_{\alpha,p}(z)$ defined in (5) are solutions of the equation*

$$L_{p'}(z) = 3(1 - \alpha) + \alpha(2\alpha - 1)L_p(z). \tag{19}$$

Proof. To calculate the free critical points associated with a method in $\mathcal{CH}$, we derive the rational function $R_{\alpha,p}(z)$ that appears in (5). Taking into account (7) and Lemma 1, we obtain

$$R'_{\alpha,p}(z) = \frac{L_p(z)^2 \left(3(1 - \alpha) + \alpha(2\alpha - 1)L_p(z) - L_{p'}(z)\right)}{2(1 - \alpha L_p(z))^2}. \tag{20}$$

Note that $L_p(z) \neq 0$ because we are assuming that $p''(z) \neq 0$ and $p(z) \neq 0$ because z is a free critical point (is not the root of $p(z)$). The result follows by simply solving for $L_{p'}(z)$ in the equation $R'_{\alpha,p}(z) = 0$. $\square$

The analysis of Lemma 2 allows us to obtain some interesting conclusions about the number of free critical points in the family $\mathcal{CH}$. First, we note that there are two situations for which the equation (19) has particularly simple solutions. These are the cases $\alpha = 0$ (Chebyshev method) and $\alpha = 1/2$ (Halley method). In both cases we arrive at an equation of the type $L_{p'}(z) = C$ with C a constant, which leads us to a polynomial equation of degree $2d - 4$, with d being the degree of the polynomial $p(z)$. In the case of cubic polynomials, such as those given in (18), the equation to be solved is quadratic. Consequently, to draw the parameter plane associated with Chebyshev's and Halley's methods it is necessary to analyze the orbits of the two free critical points obtained, as has been done, for example, in [19] and [24] respectively.

The number of free critical points in other methods in $\mathcal{CH}$ increases, which complicates their analysis from a dynamic point of view. For example, in another of the named methods of the family, such as the super-Halley method ($\alpha = 1$), the equation $L_{p'}(z) = L_p(z)$ is obtained. This equation leads to a polynomial equation of degree $4d - 6$. In the case of cubic polynomials (18), an equation of degree 6 must be solved, which greatly complicates the process. To be more specific, there is no formula that provides in an analytic way the roots in terms of the coefficients, as occurs in the cases of Chebyshev's and Halley's method, where a quadratic equation must be solved. Furthermore, with 6 possible free critical points and 3 roots, the range of colors to be handled is $3^6 = 729$, which makes cumbersome to use in the case of the super-Halley method the strategy developed for Chebyshev's and Halley's methods. The situation is similar for the rest of the methods of the $\mathcal{CH}$ family, where the degree of the polynomial equation obtained to find the free critical points is also $4d - 6$.

4. Polynomials with a Double Misbehavior

We have seen that some methods in $\mathcal{CH}$ can have a pathological behavior when they are applied to polynomial equations. By pathological we mean the convergence to points or cycles that are not the roots of the polynomial. In particular, we have analyzed this bad behavior for Chebyshev and super-Halley method. We have found polynomial for which each of these methods has extraneous fixed points or super-attracting cycles not including the roots. Now, we face the following question: is it possible to find polynomials and methods in $\mathcal{CH}$ so that the rational map obtained by applying the method to the polynomial has both extraneous fixed points and super-attracting cycles? Halley's method must be exclude from this search, because it has not attracting extraneous fixed points. Our first "candidate" is Chebyshev's method.

We follow a numerical strategy to find a polynomial in the form (16) (it has a super-attracting extraneous fixed point at 0) such that Chebyshev's method applied to it has a super-attracting 2-cycle

in $\{-1, 1\}$. To do this, let $R_{0,p}(z)$ the iteration map related to Chebyshev's method (see (5), with $\alpha = 0$). We look for a solution of the system of equations

$$\begin{cases} R_{0,p}(1) & = & -1, \\ R_{0,p}(-1) & = & 1, \\ R'_{0,p}(1) & = & 0, \\ R'_{0,p}(-1) & = & 0. \end{cases} \tag{21}$$

Taking into account (see (20) in Lemma 2)

$$R'_{0,p}(z) = \frac{1}{2}(3 - L_{p'}(z))L_p(z)^2,$$

the last two equations in (21) can be substituted by $L_{p'}(1) = 3$ and $L_{p'}(-1) = 3$. As we have 4 equations, we take four parameters a, a_4, a_5 and a_6 in (16). So we solve numerically the nonlinear system (21) and we obtain 2 solutions with real coefficients:

$$a = 0.115238, \quad a_4 = 0.00106173, \quad a_5 = 0.000786477, \quad a_6 = 0.000284636,$$

$$a = -0.115238, \quad a_4 = 0.00106173, \quad a_5 = -0.000786477, \quad a_6 = 0.000284636.$$

In Figure 7 we show the basins of attraction of Chebyshev's method applied to the polynomial

$$p(z) = -1 + az + a^2 z^2 + 2a^3 z^3 + a_4 z^4 + a_5 z^5 + a_6 z^6, \tag{22}$$

where a, a_4, a_5 and a_6 are the first of the above solutions. Together with the basins of the 6 roots of the polynomial, we can see (in white) the basin of the super-attracting extraneous fixed point $z = 0$. In yellow, we can see the basin of the super-attracting 2-cycle $\{-1, 1\}$.
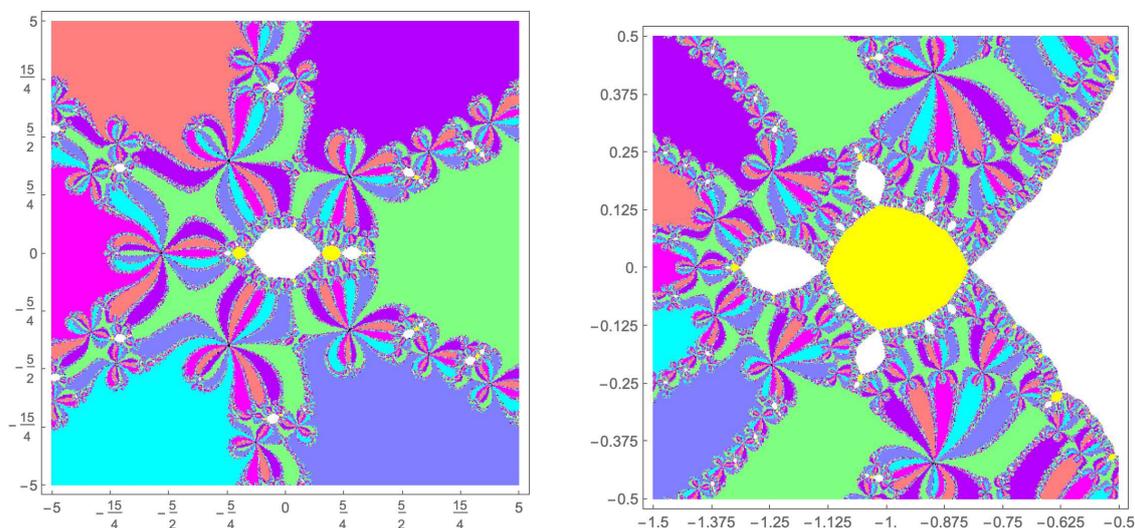
Figure 7. Two details of the basins of attraction of Chebyshev's method applied to the polynomial defined in (22) where we can see the basin of attraction of super-attracting extraneous fixed point (in white) and a super-attracting 2-cycle (in yellow).

In the case of super-Halley method, we proceed in a similar way. In this case we find a polynomial in the form (17) (it has a super-attracting extraneous fixed point at 0) such that super-Halley method

applied to it has a super-attracting 2-cycle in $\{-1, 1\}$. To do this, let $R_{1,p}(z)$ the iteration map related to super-Halley method (see (5), with $\alpha = 1$). We look for a solution of the system of equations

$$\begin{cases} R_{1,p}(1) &=& -1, \\ R_{1,p}(-1) &=& 1, \\ R'_{1,p}(1) &=& 0, \\ R'_{1,p}(-1) &=& 0. \end{cases} \tag{23}$$

Taking into account (see (20) in Lemma 2)

$$R'_{1,p}(z) = \frac{L_p(z) - L_{p'}(z)}{2(1 - L_p(z))^2} L_p(z)^2,$$

the last two equations in (23) can be substituted by $L_p(1) = L_{p'}(1)$ and $L_p(-1) = L_{p'}(-1)$. As we have 4 equations, we take four parameters a, a_4, a_5 and a_6 in (17). So we solve numerically the nonlinear system (23) and we obtain the solution (not the only one)

$$a = 0.325178, \quad a_4 = -0.009199, \quad a_5 = -0.029499, \quad a_6 = -0.011043,$$

In Figure 8 we show the basins of attraction of super-Halley method applied to the polynomial

$$p(z) = -1 + az + a^2z^2 + 2a^3z^3 + a_4z^4 + a_5z^5 + a_6z^6, \tag{24}$$

where a, a_4, a_5 and a_6 are the above parameters. Together with the basins of the 6 roots of the polynomial, we can see (in white) the basin of the super-attracting extraneous fixed point $z = 0$. In yellow, we can see the basin of the super-attracting 2-cycle $\{-1, 1\}$.
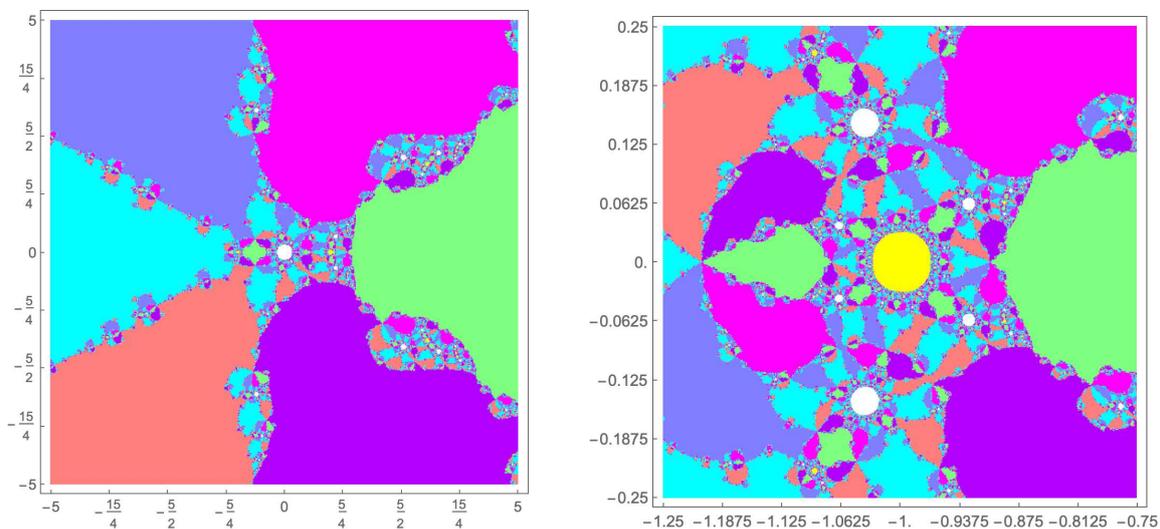
Figure 8. Two details of the basins of attraction of super-Halley method applied to the polynomial defined in (24) where we can see the basin of attraction of super-attracting extraneous fixed point (in white) and a super-attracting 2-cycle (in yellow).

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