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On Extending L^r-Norm Based Derivatives to Intuitionistic Fuzzy Set

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Abstract: Study of differential equation theory has come a long way with applications in the various fields. In 1961, Zygmund and Calderón introduced the notion of derivatives on metric L^r which proved to be better in applications than approximate derivatives. But most of the studies involved are on fuzzy set theory, so it seems likely that intuitionistic fuzzy L^r - norm - based derivatives deserve study. In the study, the fuzzy derivative is extended to intuitionistic fuzzy derivatives with respect.to L^r - norm - based derivatives using intuitionistic fuzzy number valued functions. The efficacy of the proposed method is established by proving the associated theorems along with numerical examples. Further, the Cauchy problem is also studied with respect to intuitionistic fuzzy setting.

Keywords: fuzzy set; intuitionistic fuzzy number; Hukuhara differentiable; generalized Hukuhara differentiable

MSC: 34A07; 54A40

1. Introduction

Fuzzy set theory introduced by L A Zadeh in 1965 [1], paved ways in studies to deal with vagueness or uncertainty in mathematical models where, each element $u \in U$ (universal set) is defined in terms of A(u) [membership function] $\{A: U \to [0,1]\}$. Following that, many researchers expanded the theory [2–5], one of which is Intuitionistic fuzzy set (IFS) introduced by Atanassov, in which each element is defined in terms of A(u), membership and B(u), non-membership functions such that $A + B \leq 1$ and $1 - (A + B)$ is the degree of hesitation [2]; this is a generalisation of Zadeh's [1] fuzzy set theory. Atanassov later conducted additional research to improve on the IFS theory [2,6–8]. Intuitionistic fuzzy set has better applications than that of classical fuzzy set theory in real life situations since IFS includes an additional information i.e., non-membership function and it has been utilised in numerous fields such as decision-making problems [9–11], medical diagnosis [12,13], software selection [14], environmental management [15], transport problem [16]. Susanto et al. [17] generated fuzzy interval data from crisp data using Cheng et al. [18] correlation method to determine the relationship between students' anxiety and mathematical self-efficacy, based on the concept of α-cut from a fuzzy set.

Fuzzy differential equation (FDE) is one of many extensions of fuzzy set theory to classical mathematics. Many researchers studied FDE with different approaches such as differential inclusions [19], Hukuhara derivatives [20], Zadeh's extension principle [21,22]. In 1967, [20] Hukuhara put forth the idea of Hukuhara difference to eliminate the problem of the inverse element of Minkowski's sum [23]. $\alpha -$ cut being a compact and convex set, Madan et al. [24] proposed an approach to solve fuzzy differential equations using the concept of Hukuhara derivative [20]. "However, this approach has a weak point which is that the solution becomes fuzzier as time passes by and hence the length of the support of the fuzzy solution increases" [25]. So, in order to overcome the disadvantages, Bede at al. [26] introduced the concept of generalized Hukuhara derivatives. Later, Stefanini [27] proposed a generalized Hukuhara difference which was applied to Bede's generalized Hukuhara derivative [28].

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This yielded a better result as it made problem solving less complicated and since then, the study of GH-derivative has been expanded further by the authors [29–31]. Akin et al. [32] proposed to solve second order initial value problem with intuitionistic fuzzy initial values under intuitionistic Zadeh's extension principle. In [22] Akin et al. extended GH-difference to intuitionistic fuzzy set by using the properties of α *and* β cuts and proposed the concept of strongly generalized Hukuhara derivatives for intuitionistic fuzzy environment. They have further extended the concept to solve differential equations in intuitionistic fuzzy environment with the use of (i)-GH and (ii)-GH differentiability [33].

The concept of fuzzy metric is used in many applications such as in image processing, in [34] Gregori et al. proposed on how to apply fuzzy metric in engineering methods such as evaluating the proximity of two pixels in a colour image using a fuzzy metric. The image processing filter created as a result performs better than other traditional ones. A very similar existing example is of eliminating noise in colour image using fuzzy metric [35]. Park [36] introduced the concept of intuitionistic fuzzy metric based on [37] fuzzy metric with possible application of fuzzy topology to quantum physics, particularly in connection with both string and $\epsilon^{(\infty)}$ theory [38]. In [38], the authors claimed that the topology generated by Intuitionistic fuzzy metric space coincides with the topology generated by fuzzy metric space with respect to the analysis of the probability of two slit experiment. Yadav et al. [39] proved two fixed point theorems in intuitionistic fuzzy metric space based on the outcomes of Tripathy et al. [40] which is the improved results of Alaca et al. [41].

Differential equations in metric spaces by [42] was the introduction of a new concept called metric derivative. This concept was further developed to solve Cauchy problem for fuzzy as well as intuitionistic fuzzy differential equations [42,43] and set valued functions [44]. Khastan et. al [45] introduced a new metric - based derivatives and compared with other existing metric - based derivatives. L^r - norm - based derivatives were introduced by Calderón and Zygmund in 1961 [46] in order to solve elliptic partial differential equations. Accordingly, Shao et al. [47] introduced fuzzy L^r- derivatives and fuzzy L^r-Henstock-Kurzweil integral based on the derivative. The objective of this paper is to extend the derivative to intuitionistic fuzzy set valued functions based on L^r- norm, which was prposed in [47]. Secondly, we shall study its properties such as continuity, uniqueness and many more. Next, we shall check that GH-differentiability implies $IFL^r -$ differentiable but the converse is always not true. Finally, we propose to solve Intuitionistic fuzzy differential equation using the extended derivative.

This paper is prepared as follows. In Section 2, some fundamental definitions and theorems of intuitionistic fuzzy sets and definitions of L^r-norm based derivatives on fuzzy sets are given. In Section 3, some definitions and theorems related to L^r- norm based derivatives are extended from fuzzy case to intuitionistic fuzzy case by using the definitions and theorems in Section 2 along with numerical examples. In Section 4, we conclude the paper by giving summary and results.

2. Preliminaries

Definition 2.1 [2] Let $P = \{(u, A_P(u), B_P(u)) \mid u \in U\}$ be an intuitionistic fuzzy set (IFS)

where $A_P, B_P: U \to [0, 1]$,

$A_P(u)$ is the membership function of u,

$B_P(u)$ is the non-membership function of u

and the condition $0 \le A_P(u) + B_P(u) \le 1$ holds true.

Atanassov's intuitionistic fuzzy set is the generalization of Zadeh's fuzzy set. Then Zadeh's fuzzy set can be written as $P = \{(u, A_P(u), 0) \mid u \in U\}$ where the non-membership function $B_P(u) = 0$.

IF(U) will be used to denote the set of all intuitionistic fuzzy sets in U.

Definition 2.2 [2] The α-cut of $P \in IF(\mathbb{R}^n)$ is defined as

$$P(\alpha) = \{ u: u \in \mathbb{R}^n, A_P(u) \ge \alpha\}, \text{ for } 0 < \alpha \le 1, \text{ and}$$
$$P(0) = cl\left(\bigcup_{\alpha \in (0,1]} P(\alpha)\right), \text{ for } \alpha = 0$$

Definition 2.3 [2] The β-cut of $P \in IF(\mathbb{R}^n)$ is defined as

$$P^*(\beta) = \{ u: u \in \mathbb{R}^n, B_P(u) \le \beta\}, \text{ for } 0 < \beta < 1, \text{ and}$$
$$P^*(1) = cl\left(\bigcup_{\beta \in [0,1)} P^*(\beta)\right), \text{ for } \beta = 1$$

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Definition 2.4 [22] If P ∈ IF ($\mathbb{R}^n$) satisfies the following conditions, then it is called an intuitionistic fuzzy number in $\mathbb{R}^n$:

1. P is a normal set, i.e., ∃ u_0 ∈ $\mathbb{R}^n$ such that A_P (u_0) = 1 (hence, B_P (u_0) = 0).
2. P (0) and P*(1) are bounded sets in $\mathbb{R}^n$.
3. A_P: $\mathbb{R}^n$ → [0, 1] is upper semi-continuous: ∀ k ∈ [0, 1], the set { u: u ∈$\mathbb{R}^n$, A_P (u) < k} is open.
4. B_P: $\mathbb{R}^n$ → [0, 1] is lower semi-continuous: ∀ t ∈ [0, 1], the set { u: u ∈ $\mathbb{R}^n$, B_P (u) > k} is open.
5. The membership function A_P is quasi-concave:

$$A_P(\pi u + (1-\pi) v) \geq \min\{A_P(u), A_P(v)\}, \forall\ u, v \in \mathbb{R}^n,\ \pi \in [0, 1].$$

6. The non-membership function B_P is quasi-convex:

$$B_P(\pi u + (1-\pi) v) \leq \max\{B_P(u), B_P(v)\}, \forall\ u, v \in \mathbb{R}^n,\ \pi \in [0, 1].$$

We will denote the set of intuitionistic fuzzy number of $\mathbb{R}^n$ by IF_N ($\mathbb{R}^n$)

Theorem 2.1 [23] The family of all compact and convex subsets of $\mathbb{R}^n$ is closed under Minkowski's addition and scalar multiplication.

Definition 2.5 [22] Let P, Q ∈ IF_N ($\mathbb{R}^n$) and c ∈ ℝ − {0}. Then, addition and scalar multiplication of fuzzy numbers in IF_N ($\mathbb{R}^n$) are defined as follows:

(i) P+ Q = D ⇔ D(α) = P(α) + Q(α) and D*(β) = P* (β) + Q*(β).

(ii) c(P) = D ⇔ D(α) = cP(α) and D*(β) = cP*(β)

Theorem 2.2 [22] Let P, Q ∈ IF_N ($\mathbb{R}^n$). Let us define the following distance functions as

$$\mathfrak{D}_1(P, Q) = \sup\{d_H(P(\alpha), Q(\alpha)): \alpha \in [0, 1]\}$$

$$\mathfrak{D}_2(P, Q) = \sup\{d_H(P^*(\beta) + Q^*(\beta)): \beta \in [0, 1]\},$$

Here d_H is Hausdorff metric. The function

$$\mathfrak{D}\ (P, Q) = \max\{\ \mathfrak{D}_1(P, Q),\ \mathfrak{D}_2(P, Q)\}$$

defines a metric on IF_N ($\mathbb{R}^n$). Hence, (IF_N ($\mathbb{R}^n$), $\mathfrak{D}$) is a metric space.

Definition 2.6 [22] Let P, Q ∈ IF_N ($\mathbb{R}^n$) then

- Hukuhara difference of P and Q, if it exists is given by

$$P \ominus_H Q = R \Leftrightarrow P = Q + R.$$

- Generalized Hukuhara difference of P and Q, if it exists is given by

$$P \ominus_{GH} Q = R \Leftrightarrow P = Q + R \text{ or } Q = P + (-1) R.$$

Definition 2.8 [22] Let $\mathfrak{F}$: (m, n) → IF_N (ℝ) be an intuitionistic fuzzy number valued function and u, u+h ∈ (m, n). $\mathfrak{F}$ is called Hukuhara differentiable at u if there exists an element $\mathfrak{F}'_H(u)$ ∈ IF_N (ℝ) such that for all h > 0 the following is satisfied

$$\lim_{\hbar\to0^+}\frac{\mathfrak{F}(u+\hbar)\ominus_H\mathfrak{F}(u)}{\hbar} = \lim_{\hbar\to0^+}\frac{\mathfrak{F}(u)\ominus_H\mathfrak{F}(u-\hbar)}{\hbar} = \mathfrak{F}'_H(u)$$

Definition 2.9 [22] The intuitionistic fuzzy number valued function $\mathfrak{F}$ is called generalized Hukuhara differentiable at u if there exists an element $\mathfrak{F}'_{GH}(u)$ ∈ IF_N (ℝ) such that for all h > 0 at least one of the following conditions is satisfied:

i)

$$\lim_{h\to0^+}\frac{\mathfrak{F}(u)\ominus_H\mathfrak{F}(u+h)}{-h} = \lim_{h\to0^+}\frac{\mathfrak{F}(u-h)\ominus_H\mathfrak{F}(x)}{-h} = \mathfrak{F}'_{GH}(u)$$

ii)

$$\lim_{h\to0^+}\frac{\mathfrak{F}(u+h)\ominus_H\mathfrak{F}(u)}{h} = \lim_{h\to0^+}\frac{\mathfrak{F}(u-h)\ominus_H\mathfrak{F}(u)}{-h} = \mathfrak{F}'_{GH}(u)$$

iii)

$$\lim_{h\to0^+}\frac{\mathfrak{F}(u+h)\ominus_H\mathfrak{F}(u)}{h} = \lim_{h\to0^+}\frac{\mathfrak{F}(u)\ominus_H\mathfrak{F}(u-h)}{h} = \mathfrak{F}'_{GH}(u)$$

iv)

$$\lim_{h\to0^+}\frac{\mathfrak{F}(u)\ominus_H\mathfrak{F}(u+h)}{-h} = \lim_{h\to0^+}\frac{\mathfrak{F}(u)\ominus_H\mathfrak{F}(u-h)}{h} = \mathfrak{F}'_{GH}(u)$$

Definition 2.10 [47] Let F_N(ℝ) be a set of fuzzy subsets, X ∈ F_N(ℝ) is said to be a fuzzy number if X is normal, convex, upper semi-continuous and

X_α = { u: u ∈ R, A_X (u) ≥ α}, for 0 < α ≤ 1, and X_0 = $cl\left(\cup_{\alpha\in(0,1]} X(\alpha)\right)$, for α = 0 is bounded.

The distance $d_H(X, Y)$ is defined as

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$$d_H(X,Y) = sup_{\alpha \in [0,1]}\{|X_\alpha{}^- - Y_\alpha{}^-|, |X_\alpha{}^+ - Y_\alpha{}^+|\}$$

Obviously, ($F_N(\mathbb{R})$; d_H) is a complete metric space.

Definition 2.12 [47] $\mathfrak{F}$ is fuzzy L^r- differentiable (FL^r- differentiable) at $u \in$ [m, n], if there exists $\mathfrak{F}' \in F_N(\mathbb{R})$ such that the following four situation holds:

i)

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [d_H(\mathfrak{F}(u+\hbar),\mathfrak{F}(u)+\mathfrak{F}'(u)\hbar)]^r dh\right\}^{\frac{1}{r}}$$

$$= \lim_{l\to 0^-}\left\{\frac{1}{l}\int_l^0 [d_H(\mathfrak{F}(u),\mathfrak{F}(u-\hbar)+\mathfrak{F}'(u)\hbar)]^r dh\right\}^{\frac{1}{r}} = 0$$

ii)

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [d_H(\mathfrak{F}(u),\mathfrak{F}(u-\hbar)+\mathfrak{F}'(u)\hbar)]^r dh\right\}^{\frac{1}{r}}$$

$$= \lim_{l\to 0^-}\left\{\frac{1}{l}\int_l^0 [d_H(\mathfrak{F}(u+\hbar),\mathfrak{F}(u)+\mathfrak{F}'(u)\hbar)]^r dh\right\}^{\frac{1}{r}} = 0$$

iii)

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [d_H(\mathfrak{F}(u+\hbar),\mathfrak{F}(u)+\mathfrak{F}'(u)\hbar)]^r dh\right\}^{\frac{1}{r}}$$

$$= \lim_{l\to 0^-}\left\{\frac{1}{l}\int_l^0 [d_H(\mathfrak{F}(u+\hbar),\mathfrak{F}(u)+\mathfrak{F}'(u)\hbar)]^r dh\right\}^{\frac{1}{r}} = 0$$

iv)

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [d_H(\mathfrak{F}(u),\mathfrak{F}(u-\hbar)+\mathfrak{F}'(u)\hbar)]^r dh\right\}^{\frac{1}{r}}$$

$$= \lim_{l\to 0^-}\left\{\frac{1}{l}\int_l^0 [d_H(\mathfrak{F}(u),\mathfrak{F}(u-\hbar)+\mathfrak{F}'(u)\hbar)]^r dh\right\}^{\frac{1}{r}} = 0$$

If $\mathfrak{F}$ satisfies case (i) then $\mathfrak{F}$ is (i)-FL^r- differentiable. Similarly, for the other cases as well.

3. Proposed Definitions

Definition 3.1 The distance of intuitionistic fuzzy numbers P, Q $\in IF_N(\mathbb{R}^n)$ with respect to their α-cut and β-cut is denoted by $\mathfrak{D}$(P, Q) and is defined as

$\mathfrak{D}_1(P, Q) = sup_{\alpha \in [0,1]} max\{|P(\alpha)^- - Q(\alpha)^-|, |P(\alpha)^+ - Q(\alpha)^+|\}$

$\mathfrak{D}_2(P, Q) = sup_{\beta \in [0,1]}\, max\, \{|P^*(\beta)^- - Q^*(\beta)^-|,\ |P^*(\beta)^+ - Q^*(\beta)^+|\}$

$\mathfrak{D}(P, Q) = max\, \{\mathfrak{D}_1(P,Q), \mathfrak{D}_2(P,Q)\}$

Lemma 3.1 For P, Q, R, S $\in IF_N(\mathbb{R}^n)$

(i) $\mathfrak{D}(P + R, Q + R) = \mathfrak{D}(P, Q)$

(ii) $\mathfrak{D}(c \cdot P, c \cdot Q) = |c|\mathfrak{D}(P,Q)$, $c \in \mathbb{R}$

(iii) $\mathfrak{D}(P + Q,\ R + S) \le \mathfrak{D}(P,R) + \mathfrak{D}(Q,S)$

(iv) $\mathfrak{D}(\tau \cdot P, \omega \cdot P) = |\tau - \omega|\mathfrak{D}(P, 0)$, for $\tau\omega > 0$

(v) P $\le$ Q iff P(α) $\le$ Q(α), $\alpha \in [0, 1]$ iff $P(\alpha)^+ \le Q(\alpha)^+$; $P(\alpha)^- \le Q(\alpha)^-$ and $P^*(\beta)^+ \ge Q^*(\beta)^+$; $P^*(\beta)^- \ge Q^*(\beta)^-$; $\alpha, \beta \in [0, 1]$.

Definition 3.2 For $1 \le r \le \infty$

(1) $\mathfrak{F}$ is right-hand upper Intuitionistic fuzzy L^r- differentiable (IFL^r- differentiable), if there exists $\mathfrak{F}' \in$ IFN $(\mathbb{R})$ such that

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+\hbar),\mathfrak{F}(u)+\mathfrak{F}'(u)\hbar)]^r dh\right\}^{\frac{1}{r}} = 0$$

Similarly, $\mathfrak{F}$ is left-hand upper Intuitionistic fuzzy L^r- differentiable (IFL^r- differentiable), if there exists $\mathfrak{F}' \in$ IFN $(\mathbb{R})$ such that

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u),\mathfrak{F}(u-\hbar)+\mathfrak{F}'(u)\hbar)]^r dh\right\}^{\frac{1}{r}} = 0$$

(2) $\mathfrak{F}$ is right-hand lower Intuitionistic fuzzy L^r- differentiable (IFL^r- differentiable), if there exists $\mathfrak{F}' \in$ IFN $(\mathbb{R})$ such that

$$\lim_{l\to 0^-}\left\{\frac{1}{l}\int_l^0 [\mathfrak{D}(\mathfrak{F}(u),\mathfrak{F}(u-\hbar)+\mathfrak{F}'(u)\hbar)]^r dh\right\}^{\frac{1}{r}} = 0$$

Similarly, $\mathfrak{F}$ is left-hand lower Intuitionistic fuzzy L^r- differentiable (IFL^r- differentiable), if there exists $\mathfrak{F}' \in$ IFN $(\mathbb{R})$ such that

$$\lim_{l\to 0^-}\left\{\frac{1}{l}\int_l^0 [\mathfrak{D}(\mathfrak{F}(u+\hbar),\mathfrak{F}(u)+\mathfrak{F}'(u)\hbar)]^r dh\right\}^{\frac{1}{r}} = 0$$

$\mathfrak{F}$ is upper IFL^r- differentiable if two Intuitionistic fuzzy L^r-derivatives in (1) exists and are equal. Similarly, $\mathfrak{F}$ is lower IFL^r- differentiable if two Intuitionistic fuzzy L^r-derivatives in (2) exists and are equal.

Definition 3.3 Let $\mathfrak{F}$ be an Intuitionistic fuzzy L^r- differentiable (IFL^r- differentiable) at $u \in$ [m, n], then there exists $\mathfrak{F}' \in$ IFN $(\mathbb{R})$ such that the following four situation holds:

i)

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+\hbar),\mathfrak{F}(u)+\mathfrak{F}'(u)\hbar)]^r dh\right\}^{\frac{1}{r}}$$

$$= \lim_{l\to 0^-}\left\{\frac{1}{l}\int_l^0 [\mathfrak{D}(\mathfrak{F}(u),\mathfrak{F}(u-\hbar)+\mathfrak{F}'(u)\hbar)]^r dh\right\}^{\frac{1}{r}} = 0$$

ii)

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u),\mathfrak{F}(u-h)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} = \lim_{l\to 0^-}\left\{\frac{1}{l}\int_l^0 [\mathfrak{D}(\mathfrak{F}(u+h),\mathfrak{F}(u)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} = 0$$

iii)

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h),\mathfrak{F}(u)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} = \lim_{l\to 0^-}\left\{\frac{1}{l}\int_l^0 [\mathfrak{D}(\mathfrak{F}(u+h),\mathfrak{F}(u)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} = 0$$

iv)

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u),\mathfrak{F}(u-h)+\mathfrak{F}(u)h)]^r dh\right\}^{\frac{1}{r}} = \lim_{l\to 0^-}\left\{\frac{1}{l}\int_l^0 [\mathfrak{D}(\mathfrak{F}(u),\mathfrak{F}(u-h)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} = 0$$

If $\mathfrak{F}$ satisfies case (i), then $\mathfrak{F}$ is (i)-IFL^r- differentiable. Similarly, for the rest cases.

Theorem 3.1 If $\mathfrak{F}$ is IFL^r- differentiable, then the derivative is unique.

Proof. Without loss of generality, let us assume $\mathfrak{F}$ is (ii)- IFL^r- differentiable at $u \in$ [m, n]. Let $\mathbb{D}_1$ and $\mathbb{D}_2$ be the derivatives of $\mathfrak{F}$. Then, $\forall\epsilon > 0\ \exists\ \delta > 0$ such that

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$$\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u), \mathfrak{F}(u-\hbar) + \mathbb{D}_1\hbar)]^r dh\right\}^{\frac{1}{r}} < \frac{\epsilon}{2} \tag{3.1}$$

$$\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u), \mathfrak{F}(u-\hbar) + \mathbb{D}_2\hbar)]^r dh\right\}^{\frac{1}{r}} < \frac{\epsilon}{2} \tag{3.2}$$

Then by lemma 3.1 and Minkowski's inequality, we get

$$\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathbb{D}_1 h, \mathbb{D}_2 h)]^r dh\right\}^{\frac{1}{r}}$$

$$= \left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\ \mathfrak{F}(u-h) + \mathbb{D}_1 h, \mathfrak{F}(u-h)\right.$$

$$\left. +\ \mathbb{D}_2 h)]^r dh\right\}^{\frac{1}{r}}$$

$$\leq \left\{\frac{1}{l}\int_0^l \left[\mathfrak{D}\big(\mathfrak{F}(u-h) + \mathbb{D}_1 h, \mathfrak{F}(u)\big)\right.\right.$$

$$\left.\left. + \mathfrak{D}(\mathfrak{F}(u), \mathfrak{F}(u-h) +\ \mathbb{D}_2 h)\right]^r dh\right\}^{\frac{1}{r}}$$

$$\leq \left\{\frac{1}{l}\int_0^l \left[\mathfrak{D}\big(\mathfrak{F}(u-h) + \mathbb{D}_1 h, \mathfrak{F}(u)\big)\right]^r dh\right\}^{\frac{1}{r}}$$

$$+ \left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u), \mathfrak{F}(u-h) +\ \mathbb{D}_2 h)]^r dh\right\}^{\frac{1}{r}} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$= \epsilon \tag{3.3}$$

Therefore, the derivatives $\mathbb{D}_1$ and $\mathbb{D}_2$ are equal and hence unique.

Similar result follows for cases (i), (iii) and (iv).

Theorem 3.2 If $\mathfrak{F}$ is IFL^r - differentiable, then it is continuous.

Proof. Assume that $\mathfrak{F}$ is (i)-IFL^r- differentiable. Then, $\forall\epsilon > 0 \ \exists\ \delta > 0$, for $|l| < \delta$

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} < \frac{\epsilon}{2} \tag{3.4}$$

Then by **Lemma 3.1** and Minkowski's inequality we get,

$$\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h),\mathfrak{F}(u))]^r dh\right\}^{\frac{1}{r}}$$

$$=\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h)+\mathfrak{F}'(u)h,\mathfrak{F}(u)$$

$$+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}}$$

$$\leq\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h)+\mathfrak{F}(u)h,\mathfrak{F}(u))$$

$$+\mathfrak{D}(\mathfrak{F}(u),\mathfrak{F}(u)+\mathfrak{F}'(u)h))]^r dh\right\}^{\frac{1}{r}}$$

$$\leq\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h)+\mathfrak{F}'(u)h,\mathfrak{F}(u))]^r\right\}^{\frac{1}{r}}$$

$$+\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u),\mathfrak{F}(u)+\mathfrak{F}'(u)h)]^r\right\}^{\frac{1}{r}}$$

$$\leq\epsilon+\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}'(u)h,0)]^r\right\}^{\frac{1}{r}}$$

$$\leq\epsilon+\left\{\frac{l^r}{r+1}\left\|\mathfrak{F}'^{(u)}\right\|^r\right\}^{\frac{1}{r}} \qquad (3.5)$$

$$\Rightarrow\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h),\mathfrak{F}(u))]^r dh\right\}^{\frac{1}{r}}=0 \qquad (3.6)$$

Similarly,

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u),\mathfrak{F}(u-h))]^r dh\right\}^{\frac{1}{r}}=0 \qquad (3.7)$$

Therefore, f is continuous.

Similar result follows for cases (ii), (iii) and (iv).

Example 3.1 Suppose T(u) = T₀ is a constant function, then T(u) is IFL^r–differentiable and T′(u) = 0.

Solution. From the case (i) of **Definition 3.3**

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h),\mathfrak{F}(u)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}}$$

$$=\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(T(u+h),T(u)+0h)]^r dh\right\}^{\frac{1}{r}}$$

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$$= \lim_{l \to 0^+} \left\{ \frac{1}{l} \int_0^l [\mathfrak{D}(T_0, T_0 + 0)]^r dh \right\}^{\frac{1}{r}}$$

$$= \lim_{l \to 0^+} \left\{ \frac{1}{l} \int_0^l [\mathfrak{D}(T_0, T_0)]^r dh \right\}^{\frac{1}{r}} = 0 \qquad (3.8)$$

And

$$\lim_{l \to 0^-} \left\{ \frac{1}{l} \int_l^0 [\mathfrak{D}(\mathfrak{F}(u), \mathfrak{F}(u-h) + \mathfrak{F}'(u)h)]^r dh \right\}^{\frac{1}{r}}$$

$$= \lim_{l \to 0^-} \left\{ \frac{1}{l} \int_l^0 [\mathfrak{D}(T(u), T(u-h) + 0h)]^r dh \right\}^{\frac{1}{r}}$$

$$= \lim_{l \to 0^-} \left\{ \frac{1}{l} \int_l^0 [\mathfrak{D}(T_0, T_0 + 0)]^r dh \right\}^{\frac{1}{r}} = 0 \qquad (3.9)$$

Therefore, fuzzy constant function is (i)- IFL^r – differentiable.

Similarly, for cases (ii), (iii) and (iv).

Theorem 3.3 Let $\mathfrak{F}$, $\mathfrak{G}: [m, n] \to$ IFN $(\mathbb{R})$, $1 \leq r < \infty$ Suppose $\mathfrak{F}$ and $\mathfrak{G}$ are upper IF L^r – differentiable then

i) $\mathfrak{F} + \mathfrak{G}$ is upper IFL^r – differentiable and $(\mathfrak{F} + \mathfrak{G})' = \mathfrak{F}' + \mathfrak{G}'$.

ii) $(\omega\mathfrak{F})'$ (u) $= \omega\mathfrak{F}'$(u), for all $\omega \in \mathbb{R}$.

Proof. i) Let $\mathfrak{F}$ *and* $\mathfrak{G}$ be upper IFL^r – differentiable.

Then, $\forall \epsilon > 0, \exists\, \delta > 0$ s. t. for $|l| < \delta$

$$\left\{ \frac{1}{l} \int_0^l [\mathfrak{D}(\mathfrak{F}(u), \mathfrak{F}(u-h) + \mathfrak{F}'(u)h)]^r dh \right\}^{\frac{1}{r}} < \frac{\epsilon}{2} \qquad (3.10)$$

$$\left\{ \frac{1}{l} \int_0^l [\mathfrak{D}(\mathfrak{G}(u), \mathfrak{G}(u-h) + \mathfrak{G}'(u)h)]^r dh \right\}^{\frac{1}{r}} < \frac{\epsilon}{2} \qquad (3.11)$$

Then, by **Lemma 3.1** and Minkowski's inequality we get

$$\left\{ \frac{1}{l} \int_0^l [\mathfrak{D}(\mathfrak{G}(u) + \mathfrak{F}(u), \mathfrak{G}(u-h) + \mathfrak{F}(u-h) + \mathfrak{G}'(u)h\, \mathfrak{F}'(u)h)]^r dh \right\}^{\frac{1}{r}}$$

$$\leq \left\{ \frac{1}{l} \int_0^l [\mathfrak{D}(\mathfrak{G}(u), \mathfrak{G}(u-h) + \mathfrak{G}'(u)h) \right.$$

$$\left. + \mathfrak{D}(\mathfrak{F}(u), \mathfrak{F}(u-h) + \mathfrak{F}'(u)h)]^r dh \right\}^{\frac{1}{r}}$$

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$$\leq \left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{G}(u), \mathfrak{G}(u-h) + \mathfrak{G}'(u)h)]^r dh\right\}^{\frac{1}{r}}$$

$$+ \left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u), \mathfrak{F}(u-h) + \mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}}$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad (3.12)$$

Similarly, $\forall \epsilon > 0, \exists\, \delta > 0$ s. t. for $|l| < \delta$

$$\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} < \frac{\epsilon}{2} \qquad (3.13)$$

$$\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{G}(u+h), \mathfrak{G}(u) + \mathfrak{G}'(u)h)]^r dh\right\}^{\frac{1}{r}} < \frac{\epsilon}{2} \qquad (3.14)$$

Then, by **Lemma 3.1** and Minkowski's inequality we get

$$\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{G}(u+h) + \mathfrak{F}(u+h), \mathfrak{G}(u) + \mathfrak{F}(u) + \mathfrak{G}'(u)h\, \mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}}$$

$$\leq \left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{G}(u+h), \mathfrak{G}(u) + \mathfrak{G}'(u)h)\right.$$

$$\left. + \mathfrak{D}(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}}$$

$$\leq \left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{G}(u+h), \mathfrak{G}(u) + \mathfrak{G}'(u)h)]^r dh\right\}^{\frac{1}{r}}$$

$$+ \left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}}$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad (3.15)$$

Thus, $\mathfrak{F} + \mathfrak{G}$ is upper IFL^r – differentiable at x and $(\mathfrak{F} + \mathfrak{G})'_{GH} = \mathfrak{F}' + \mathfrak{G}'$.
ii) Let $\mathfrak{F}$ be upper IFL^r – differentiable.
Then, $\forall \epsilon > 0, \exists\, \delta > 0$ s. t. for $|l| < \delta$

$$\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u), \mathfrak{F}(u-h) + \mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} < \frac{\epsilon}{2} \qquad (3.16)$$

Then, by **Lemma 3.1** and Minkowski's inequality we get

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$$\left\{\frac{1}{l}\int_0^l [\mathfrak{D}((\omega\mathfrak{F})(u),(\omega\mathfrak{F})(u-h)+(\omega\mathfrak{F})'(u)h)]^r dh\right\}^{\frac{1}{r}}, \lambda \in \mathbb{R}$$

$$=\left\{\frac{1}{l}\int_0^l \omega[\mathfrak{D}(\mathfrak{F}(u),\mathfrak{F}(u-h)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}}$$

$$=\omega\left\{\frac{1}{l}\int_0^l \left[\mathfrak{D}\left(\mathfrak{F}(u),\mathfrak{F}(u-h)+\mathfrak{F}'^{(u)}h\right)\right]^r dh\right\}^{\frac{1}{r}} < \frac{\epsilon}{2} < \epsilon$$

(3.17)

Similarly,

$$\left\{\frac{1}{l}\int_0^l [\mathfrak{D}((\omega\mathfrak{F})(u+h),(\omega\mathfrak{F})(u)+(\omega\mathfrak{F})'(u)h)]^r dh\right\}^{\frac{1}{r}}$$

$$=\left\{\frac{1}{l}\int_0^l \omega[\mathfrak{D}(\mathfrak{F}(u+h),\mathfrak{F}(u)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}}$$

$$=\omega\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h),\mathfrak{F}(u)+\mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} < \frac{\epsilon}{2}$$

$$< \epsilon \qquad (3.18)$$

Thus, $(\omega\mathfrak{F})' = \omega\mathfrak{F}'$, at u for all $\omega \in \mathbb{R}$.

Remark 3.1

i) If $\mathfrak{F}$ and $\mathfrak{G}$ are lower IFL^r– differentiable then **Theorem 3.3** holds.

ii) Also, if $\mathfrak{F}$ is upper IFL^r– differentiable and $\mathfrak{G}$ is lower IFL^r– differentiable, then $\mathfrak{F}$ and $\mathfrak{G}$ are IFL^r – differentiable and **Theorem 3.3** still holds.

Theorem 3.4 Let $\mathfrak{G} : [m,n] \to \mathbb{R}$ be differentiable ($\mathfrak{G}^*$ *be its derivative*), $\mathfrak{F}: [m,n] \to IF_N(\mathbb{R})$ be GH-differentiable. If $H: [m,n] \to IF_N(\mathbb{R})$ s. t. $\mathrm{H}(u) = g\,(u)\,\mathfrak{F}(u)$, then $\mathrm{H}(u)$ is IFL^r – differentiable and $\mathrm{H}'(u) = \mathfrak{G}^*\,(u)\,\cdot\mathfrak{F}(u) + \mathfrak{G}(u)\,\cdot\mathfrak{F}'_{GH}(u)$.

Proof. Let us assume that $\mathrm{H}'(u) \in IF_N(\mathbb{R})$ exists, we shall prove that $\mathrm{H}(u)$ is IFL^r – differentiable

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(H(u+h),H(u)+\mathrm{H}'(u)h)]^r dh\right\}^{\frac{1}{r}}$$

$$=\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\,\mathfrak{G}(u+h)\,\mathfrak{F}(u+h),\mathfrak{G}(u)\,\mathfrak{F}(u)\right.$$

$$\left.+\mathrm{H}'(u)h)]^r dh\right\}^{\frac{1}{r}}$$

$$=\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\,\mathfrak{G}(u+h)\,\mathfrak{F}(u+h) - \mathfrak{G}(u)\mathfrak{F}(u+h)\right.$$

$$\left.+\ \mathfrak{G}(u)\mathfrak{F}(u+h),\mathfrak{G}(u)\,\mathfrak{F}(u)+H'(u)h)]^r dh\right\}^{\frac{1}{r}}$$

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$$= \lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\,\mathfrak{G}(u+h)\,\mathfrak{F}(u+h) - \,\mathfrak{G}(u)\mathfrak{F}(u+h) + \,\mathfrak{G}(u)\mathfrak{F}(u+h)\,\ominus_H\,\mathfrak{G}(u)\,\mathfrak{F}(u), \mathrm{H}'(u\,)h)]^r dh\right\}^{\frac{1}{r}}$$

$$= \lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l \left[\mathfrak{D}\big((\,\mathfrak{G}(u+h) - \,\mathfrak{G}(u))\mathfrak{F}(u+h) + \,\mathfrak{G}(u)(\mathfrak{F}(u+h)\ominus_H \mathfrak{F}(u)), \mathrm{H}'(u)\big)\right]^r dh\right\}^{\frac{1}{r}}$$

$$= \lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l \left[\mathfrak{D}\big(\mathfrak{G}^*\,(u)\mathfrak{F}(u+h) + \,\mathfrak{G}(u)\mathfrak{F}'_{GH}(u), \mathrm{H}'(u)\big)\right]^r dh\right\}^{\frac{1}{r}}$$

as when h→ 0^+, $\mathrm{H}'(u) = \mathfrak{G}^*\,(u)\mathfrak{F}(u) + \mathfrak{G}(u)\mathfrak{F}'_{GH}(u)$ (3.19)

Therefore,

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(H(u+h), H(u) + \mathrm{H}'(u)h)]^r dh\right\}^{\frac{1}{r}}$$

$$= \lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l \left[\mathfrak{D}\big(\mathfrak{G}^*\,(u)\mathfrak{F}(u+h) + \,\mathfrak{G}(u)\mathfrak{F}'_{GH}(u), \mathrm{H}'(u)\big)\right]^r dh\right\}^{\frac{1}{r}}$$

$$= \lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l \left[\mathfrak{D}\big(H'(u), \mathrm{H}'(u)\big)\right]^r dh\right\}^{\frac{1}{r}} = 0 \qquad (3.20)$$

Similarly,

$$\lim_{l\to 0^-}\left\{\frac{1}{l}\int_l^0 [\mathfrak{D}(H(u), H(u-h) + \mathrm{H}'(u)h)]^r dh\right\}^{\frac{1}{r}} = 0 \qquad (3.21)$$

Thus, $\mathrm{H}(u)$ is (i)-IFL^r – differentiable at u and $\mathrm{H}(u) = \mathfrak{G}^*\,(u)\,\cdot\mathfrak{F}(u) + \mathfrak{G}(u)\,\cdot\mathfrak{F}'_{GH}(u)$.
Similar results for cases (ii), (iii) and (iv).

Theorem 3.5 If $\mathfrak{F}$ is GH-differentiable, then it is IFL^r – differentiable

Proof. Without loss of generality, let us assume that $\mathfrak{F}$ is (iii)-GH-differentiable, then for $l > 0$, we have,

$$\lim_{h\to 0^+}\frac{\mathfrak{F}(u+h)\ominus_H \mathfrak{F}(u)}{h} = \lim_{h\to 0^+}\frac{\mathfrak{F}(u)\ominus_H \mathfrak{F}(u\,-\,h)}{h}$$
$$= \mathfrak{F}'_{GH}(u)$$

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(3.22)

Thus, $\lim_{h\to 0^+}\frac{1}{h}\mathfrak{D}(\mathfrak{F}(u+h)\ominus_H \mathfrak{F}(u), \mathfrak{F}'_{GH}(u)h) = \lim_{h\to 0^+}\frac{1}{h}\mathfrak{D}(\mathfrak{F}(u)\ominus_H \mathfrak{F}(u-h), \mathfrak{F}'_{GH}(u)h)$

$= 0$

Then, $\lim_{h\to 0^+}\frac{1}{h}\mathfrak{D}(\mathfrak{F}(u)\ominus_H \mathfrak{F}(u-h), \mathfrak{F}'_{GH}(u)) = \lim_{h\to 0^+}\frac{1}{-h}\mathfrak{D}(\mathfrak{F}(u-h)\ominus_H \mathfrak{F}(u), -\mathfrak{F}'_{GH}(u)h)$

$= \lim_{h'\to 0^-}\frac{1}{h'}\mathfrak{D}(\mathfrak{F}(u+h')\ominus_H \mathfrak{F}(u), \mathfrak{F}'_{GH}(u)h')$

(putting $h' = -h$)

$= \lim_{h\to 0^+}\frac{1}{h}\mathfrak{D}(\mathfrak{F}(u+h)\ominus_H \mathfrak{F}(u), \mathfrak{F}'_{GH}(u)h)$

Therefore, $\forall\, \epsilon > 0,\ \exists\, T_1 > 0\ s.t.\ 0 < h < T_1,$

$$\mathfrak{D}(\mathfrak{F}(u+h)\ominus_H \mathfrak{F}(x), \mathfrak{F}'_{GH}(x)h) < \epsilon$$

i.e.,

$$D(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'_{GH}(u)h) < \epsilon \qquad (3.23)$$

Furthermore, $\forall\, \epsilon > 0,\ \exists\, T_1 > 0$, we restrict $0 < l < T_1$, then

$$\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'_{GH}(u)h)]^r dh\right\}^{\frac{1}{r}} < \left\{\frac{1}{l}\int_0^l \epsilon^r dh\right\}^{\frac{1}{r}}$$

$$= \epsilon \qquad (3.24)$$

Similarly, $\forall\, \epsilon > 0,\ \exists\, T_2 < 0$, s. t. $T_2 < l < 0$, then

$$\left\{\frac{1}{l}\int_l^0 [\mathfrak{D}(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'_{GH}(u)h)]^r dh\right\}^{\frac{1}{r}}$$

$$< \epsilon \qquad (3.25)$$

This implies that

$$\lim_{l\to 0^+}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'_{GH}(u)h)]^r dh\right\}^{\frac{1}{r}}$$

$$= \lim_{l\to 0^-}\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'_{GH}(u)h)]^r dh\right\}^{\frac{1}{r}}$$

$$= 0 \qquad (3.26)$$

Therefore, $\mathfrak{F}$ is IFL^r – differentiable at u.

Similar result follows for cases (i), (ii) and (iv).

Remark 3.2 The inverse of the above theorem may or may not be true. We shall prove this by an example.

Example 3.2 Let $\mathfrak{F}$: [-2, 2] → IFN (ℝ)

$$[\mathfrak{F}(u)]_\alpha = \left[2\alpha - 2 + \left(\frac{1}{2} - \frac{3}{4}\alpha\right)|u|, 2 - 2\alpha + \left(\frac{3}{4}\alpha - \frac{1}{2}\right)|u|\right]$$

$$[\mathfrak{F}(u)]_\beta = \left[-2\beta + \left(\frac{3}{4}\beta - \frac{1}{3}\right)|u|, 2\beta + \left(\frac{1}{3} - \frac{3}{4}\beta\right)|u|\right]$$

$$0 \le \alpha + \beta \le 1$$

In the case of $\boldsymbol{\alpha}$, $\mathfrak{F}(0)$ = (-2, 0, 2) which is a triangular fuzzy number

In particular, $\mathfrak{F}(u) = \left(-1, -\frac{1}{2}, \frac{1}{2}, 1\right)$, when $u = \pm 2$

Since, $[\mathfrak{F}(0)]_\alpha = [2\alpha - 2, 2 - 2\alpha]$ and $[\mathfrak{F}(2)]_\alpha = \left[\frac{1}{2}\alpha - 1, 1 - \frac{1}{2}\alpha\right]$,

it is easy to see that for, $\delta > 0,\ \forall\, h \in B(0,\delta)$, $\mathfrak{F}(0)\ominus_H \mathfrak{F}(h)$, $\mathfrak{F}(0)\ominus_H \mathfrak{F}(-h)$, $\mathfrak{F}(h)\ominus_H \mathfrak{F}(0)$ and $\mathfrak{F}(-h)\ominus_H \mathfrak{F}(0)$ don't exist. Thus, $\mathfrak{F}$ isn't GH-differentiable.

For $u = 0$

$$\mathfrak{D}_1(\mathfrak{F}(0+h), \mathfrak{F}(0) + \mathfrak{F}'(0)\mathrm{h})$$
$$= \mathfrak{D}_1\left(\left[2\alpha - 2 + \left(\frac{1}{2} - \frac{3}{4}\alpha\right)|h|, 2 - 2\alpha + \left(\frac{3}{4}\alpha - \frac{1}{2}\right)|h|\right], [2\alpha - 2, 2 - 2\alpha]\right)$$
$$= sup_{\alpha \in [0,1]} max\left\{\left|\left(\frac{1}{2} - \frac{3}{4}\alpha\right)h\right|, \left|\left(\frac{3}{4}\alpha - \frac{1}{2}\right)h\right|\right\}$$
$$= \frac{1}{2}|h| \qquad (3.27)$$

In the case of $\boldsymbol{\beta}$, $\mathfrak{F}(0) = \{0\}$

In particular, $\mathfrak{F}(u) = \left(-\frac{7}{6}, -\frac{2}{3}, \frac{2}{3}, \frac{7}{6}\right)$, when $u = \pm 2$

Since, $[\mathfrak{F}(0)]_\beta = [-2\beta, 2\beta]$ and $[\mathfrak{F}(2)]_\beta = \left[-\frac{1}{2}\beta - \frac{2}{3}, \frac{1}{2}\beta - \frac{2}{3}\right]$,

it is easy to see that for, $\delta > 0$, $\forall\, h \in B(0,\delta)$, $\mathfrak{F}(0) \ominus_H \mathfrak{F}(h)$, $\mathfrak{F}(0) \ominus_H \mathfrak{F}(-h)$, $\mathfrak{F}(h) \ominus_H \mathfrak{F}(0)$ and $\mathfrak{F}(-h) \ominus_H \mathfrak{F}(0)$ don't exist. Thus, $\mathfrak{F}$ isn't GH-differentiable.

For $u = 0$

$$\mathfrak{D}_2(\mathfrak{F}(0+h), \mathfrak{F}(0) + \mathfrak{F}'(0)h)$$
$$= \mathfrak{D}_2\left(\left[-2\beta + \left(\frac{3}{4}\beta - \frac{1}{3}\right)|h|, 2\beta + \left(\frac{1}{3} - \frac{3}{4}\beta\right)|h|\right], [-2\beta, 2\beta]\right)$$
$$= sup_{\alpha \in [0,1]} max\left\{\left|\left(\frac{3}{4}\beta - \frac{1}{3}\right)h\right|, \left|\left(\frac{1}{3} - \frac{3}{4}\beta\right)h\right|\right\}$$
$$= \frac{5}{12}|h| \qquad (3.28)$$

Next, we can find that $\mathfrak{F}$ is (i)-IFL^r- differentiable and $\mathfrak{F}'(0) = 0$.

For $u = 0$

$$\lim_{\tau\to 0} \mathfrak{D}(\mathfrak{F}(0+h), \mathfrak{F}(0) + \mathfrak{F}'(0)h)$$
$$= \lim_{h\to 0} max\{\mathfrak{D}_1(\mathfrak{F}(0+h), \mathfrak{F}(0) + \mathfrak{F}'(0)h), \mathfrak{D}_2(\mathfrak{F}(0+h), \mathfrak{F}(0) + \mathfrak{F}'(0)h)\}$$
$$= \lim_{h\to 0} max\left\{\frac{1}{2}|h|, \frac{5}{12}|h|\right\} = \lim_{h\to 0}\frac{1}{2}|h| = 0$$

Therefore, $\forall\, \epsilon > 0$, $\exists\, T_1 > 0$ $s.t.$ $0 < h < T_1$,

$$\mathfrak{D}(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'(u)h) < \epsilon \qquad (3.29)$$

Furthermore, $\forall\, \epsilon > 0$, $\exists\, T_1 > 0$, we restrict $0 < l < T_1$, then

$$\left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} < \left\{\frac{1}{l}\int_0^l \epsilon^r dh\right\}^{\frac{1}{r}} = \epsilon \qquad (3.30)$$

Similarly, $\forall\, \epsilon > 0$, $\exists\, T_2 < 0$, s. t. $T_2 < l < 0$, then

$$\left\{\frac{1}{l}\int_l^0 [\mathfrak{D}(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{F}'(u)h)]^r dh\right\}^{\frac{1}{r}} < \epsilon \qquad (3.31)$$

Therefore, $\mathfrak{F}$ is (i)-IFL^r- differentiable at $u = 0$ with $\mathfrak{F}'(0) = 0$ but not GH-differentiable.
Similarly result follows for cases (i), (ii) and (iv).

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4. Intuitionistic fuzzy Cauchy problem

Lemma 4.1: Let $\mathfrak{F}_1$, $\mathfrak{F}_2$: [m, n] → IF_N be differentiable and assume that its derivatives $\mathfrak{G}_1$, $\mathfrak{G}_2$: [m, n] → IF_N is integrable over [m, n] then,

$\mathfrak{D}(\mathfrak{F}_1(u), \mathfrak{F}_2(u)) \le \mathfrak{D}(\mathfrak{F}_1(u_0), \mathfrak{F}_2(u_0)) + \int_{u_0}^{u} \mathfrak{D}(\mathfrak{G}_1(s), \mathfrak{G}_2(s))\, ds$ for all s ∈ [m, n] (4.1)

Proof: $\forall \epsilon > 0, \exists T_1 > 0$ s.t. $0 < h < T_1$, $\mathfrak{D}(\mathfrak{F}(u+h), \mathfrak{F}(u) + \mathfrak{G}(u)h) < \epsilon$ (4.2)

Define $\zeta(u) = \mathfrak{D}(\mathfrak{F}_1(u), \mathfrak{F}_2(u))$

Then, $\zeta(u+h) - \zeta(u) = \mathfrak{D}(\mathfrak{F}_1(u+h), \mathfrak{F}_2(u+h)) - \mathfrak{D}(\mathfrak{F}_1(u), \mathfrak{F}_2(u))$

$= \mathfrak{D}(\mathfrak{F}_1(u+h), \mathfrak{F}_1(u) + h\,\mathfrak{G}_1(u)) + \mathfrak{D}(\mathfrak{F}_1(u) + h\,\mathfrak{G}_1(u), \mathfrak{F}_2(u) + h\,\mathfrak{G}_1(u)) +$
$\mathfrak{D}(\mathfrak{F}_2(u) + h\,\mathfrak{G}_1(u), \mathfrak{F}_2(u)) + h\,\mathfrak{G}_2(u)) + \mathfrak{D}(\mathfrak{F}_2(u) + h\,\mathfrak{G}_2(u), \mathfrak{F}_2(u+h)) - \mathfrak{D}(\mathfrak{F}_1(u), \mathfrak{F}_2(u))$

$= \mathfrak{D}(\mathfrak{F}_1(u+h), \mathfrak{F}_1(u) + h\,\mathfrak{G}_1(u)) + \mathfrak{D}(\mathfrak{F}_1(u), \mathfrak{F}_2(u)) + h\,\mathfrak{D}(\mathfrak{G}_1(u), \mathfrak{G}_2(u)) +$
$\mathfrak{D}(\mathfrak{F}_2(u) + h\,\mathfrak{G}_2(u), \mathfrak{F}_2(u+h)) - \mathfrak{D}(\mathfrak{F}_1(u), \mathfrak{F}_2(u))$

$\zeta(u+h) - \zeta(u) = \mathfrak{D}(\mathfrak{F}_1(u+h), \mathfrak{F}_1(u) + h\,\mathfrak{G}_1(u)) + h\,\mathfrak{D}(\mathfrak{G}_1(u), \mathfrak{G}_2(u)) + \mathfrak{D}(\mathfrak{F}_2(u) + h\,\mathfrak{G}_2(u), \mathfrak{F}_2(u+h))$

$$\frac{\zeta(u+h) - \zeta(u)}{h} = \frac{\mathfrak{D}(\mathfrak{F}_1(u+h), \mathfrak{F}_1(u) + h\mathfrak{G}_1(u))}{h} + \mathfrak{D}(\mathfrak{G}_1(u), \mathfrak{G}_2(u) + \frac{\mathfrak{D}\big(\mathfrak{F}_2(u) + h\mathfrak{G}_2(u), \mathfrak{F}_2(u+h)\big)}{h} \quad (4.3)$$

Now, ∀ epsilon > 0, ∃ $T_1 > 0$, we restrict $0 < l < T_1$,

$$\frac{\zeta(u+h) - \zeta(u)}{h} \le \left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}_1(u+h), \mathfrak{F}_1(u) + \mathfrak{G}(u)h)]^r dh\right\}^{\frac{1}{r}} + \mathfrak{D}(\mathfrak{G}_1(u), \mathfrak{G}_2(u)) + \left\{\frac{1}{l}\int_0^l [\mathfrak{D}(\mathfrak{F}_2(u+h), \mathfrak{F}_2(u) + \mathfrak{G}_2(u)h)]^r dh\right\}^{\frac{1}{r}}$$

as lim tends to 0^+ $\zeta'(u) \le \mathfrak{D}(\mathfrak{G}_1(u), \mathfrak{G}_2(u))$

which implies $\zeta(u) \le \int_{u_0}^{u} \mathfrak{D}(\mathfrak{G}_1(s), \mathfrak{G}(s))\, ds$, (4.4)

$$\mathfrak{D}\big(\mathfrak{F}_1(u), \mathfrak{F}_2(u)\big) \le \mathfrak{D}\big(\mathfrak{F}_1(u_0), \mathfrak{F}_2(u_0)\big) + \int_{u_0}^{u} \mathfrak{D}\big(\mathfrak{G}(s), \mathfrak{G}_2(s)\big) ds \quad (4.5)$$

Similar results follow for the left end limit.

Consider an initial value problem for the intuitionistic fuzzy differential equation

$\mathfrak{F}'(u) = \mathfrak{G}(u, \mathfrak{F}(u))$

$\mathfrak{F}(u_0) = (A(u_0), B(u_0)))$, I = [m, n] and $\mathfrak{G}$: I × IF_N → IF_N. (4.6)

Let C(I × IF_N, IF_N) the set of all continuous mappings from I × IF_N to IF_N.

Definition 4.2: [48] $\mathfrak{F}$: I → IF_N is a solution of the initial value problem iff it is continuous and satisfies the integral solution

$$\mathfrak{F}(u) = \mathfrak{F}(u_0) \oplus \int_{u_0}^{u} \mathfrak{G}(s, \mathfrak{F}(s)) ds \quad (4.7)$$

Theorem 4.3: Let $\mathfrak{G}$ ∈ C (I × IFN, IF_N) such that there exists a constant k≥ 0 satisfying

$\mathfrak{D}(\mathfrak{G}(u, \mathfrak{F}_1), \mathfrak{G}(u, \mathfrak{F})) \le k\,\mathfrak{D}(\mathfrak{F}_1, \mathfrak{F}_2)$; $(u, \mathfrak{F}_1), (u, \mathfrak{F}_2) \in I \times IF_N$ (4.8)

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Then the initial intuitionistic fuzzy problem (4.6) has a unique solution

Proof For $\mathfrak{F} \in C(I, IF_N)$, let us consider a mapping G: $X_0 \rightarrow X_0$, where $X_0 = C(I, IF_N)$ defined by

$$G\mathfrak{F}_1(u) = \mathfrak{F}(u_0) \oplus \int_{u_0}^{u} \mathfrak{G}(s, \mathfrak{F}(s))ds \quad (4.9)$$

Let, $\varphi(u) = G\mathfrak{F}_1(u + h)$, $\psi(u) = G\mathfrak{F}_1(u)$

Then, $\mathfrak{D}(\varphi(u), \psi(u)) = \mathfrak{D}(G\mathfrak{F}_1(u + h), G\mathfrak{F}_1(u))$

$$= \mathfrak{D}(G\mathfrak{F}_1(u + h), G\mathfrak{F}_1(u))$$

$$= \mathfrak{D}\left(\mathfrak{F}(u_0) \oplus \int_{u_0}^{u+\hbar} \mathfrak{G}(s, \mathfrak{F}_1(s))ds, \mathfrak{F}(u_0) \oplus \int_{u_0}^{u} \mathfrak{G}(s, \mathfrak{F}_1(s))ds\right)$$

$$= \mathfrak{D}\left(\int_{u_0}^{u+\hbar} \mathfrak{G}(s, \mathfrak{F}(s))ds, \int_{u_0}^{u} \mathfrak{G}(s, \mathfrak{F}(s))ds\right)$$

$$= \mathfrak{D}\left(\int_{u_0}^{u+\hbar} \mathfrak{G}(s, \mathfrak{F}_1(s))ds, \int_{u_0}^{u} \mathfrak{G}(s, \mathfrak{F}_1(s))ds\right)$$

$$= max\left(\mathfrak{D}_1\left(\int_{u_0}^{u+\hbar} \mathfrak{G}(s, \mathfrak{F}(s))ds, \int_{u_0}^{u} \mathfrak{G}(s, \mathfrak{F}_1(s))ds\right), \mathfrak{D}_2\left(\int_{u_0}^{u+\hbar} \mathfrak{G}(s, \mathfrak{F}_1(s))ds, \int_{u_0}^{u} \mathfrak{G}(s, \mathfrak{F}_1(s))ds\right)\right)$$

$$= max\left(sup_{\alpha \in [0,1]} max\left\{\left|\int_{u_0}^{u+\hbar} \mathfrak{G}(s, \mathfrak{F}_1(s))(\alpha)^- ds - \int_{u_0}^{u} \mathfrak{G}(s, \mathfrak{F}_1(s))(\alpha)^- ds\right|, \left|\int_{u_0}^{u+\hbar} \mathfrak{G}(s, \mathfrak{F}_1(s))(\alpha)^+ ds - \int_{u_0}^{u} \mathfrak{G}(s, \mathfrak{F}_1(s))(\alpha)^+ ds\right|\right\},\right.$$
$$\left. sup_{\alpha \in [0,1]} max\left\{\left|\int_{u_0}^{u+\hbar} \mathfrak{G}^*(s, \mathfrak{F}_1(s))(\beta)^- ds - \int_{u_0}^{u} \mathfrak{G}^*(s, \mathfrak{F}_1(s))(\beta)^- ds\right|, \left|\int_{u_0}^{u+\hbar} \mathfrak{G}^*(s, \mathfrak{F}(s))(\beta)^+ ds - \int_{u_0}^{u} \mathfrak{G}^*(s, \mathfrak{F}_1(s))(\beta)^+ ds\right|\right\}\right)$$

$$= max\left(sup_{\alpha \in [0,1]} max\left\{\left|\int_{u}^{u+\hbar} \mathfrak{G}(s, \mathfrak{F}_1(s))(\alpha)^- ds\right|, \left|\int_{u}^{u+\hbar} \mathfrak{G}(s, \mathfrak{F}_1(s))(\alpha)^+ ds\right|\right\},\right.$$
$$\left. sup_{\alpha \in [0,1]} max\left\{\left|\int_{u}^{u+\hbar} \mathfrak{G}^*(s, \mathfrak{F}_1(s))(\beta)^- ds\right|, \left|\int_{u}^{u+\hbar} \mathfrak{G}^*(s, \mathfrak{F}_1(s))(\beta)^+ ds\right|\right\}\right)$$

when $h \rightarrow 0^+$, $\mathfrak{D}(\varphi(u), \psi(u)) \rightarrow 0^+$. Therefore, $G\mathfrak{F}_1 \in C(I, IF_N)$ (4.10)

Now, let $\mathfrak{F}_1, \mathfrak{F}_2 \in C(I, IF_N)$ and by **Lemma 4.1**

$$\mathfrak{D}(G\mathfrak{F}_1(u), G\mathfrak{F}_2(u)) \leq \mathfrak{D}\left(\int_{u_0}^{u} (\mathfrak{G}(s, \mathfrak{F}_1(s), \mathfrak{G}(s, \mathfrak{F}_2(s))ds\right)$$

$$\leq \left(\mathfrak{D}\int_{u_0}^{u} (\mathfrak{G}(s, \mathfrak{F}_1(s), \mathfrak{G}(s, \mathfrak{F}(s))ds\right)$$

$$\leq k(u - u_0)\mathfrak{D}(\mathfrak{F}_1, \mathfrak{F}_2) \ [Lipschitz\ condition] \quad (4.11)$$

$$\therefore G\ is\ a\ contraction$$

and hence the initial value problem (4.6) has a unique solution.

5. Conclusions

In this paper, we have extended fuzzy L^r – norm based derivative (FL^r – derivative) to intuitionistic fuzzy number valued function. We have proposed a definition of intuitionistic fuzzy L^r – norm based derivative (IFL^r – derivative). Next, we proved some properties such that IFL^r – derivative is unique, continuous. Also, IFL^r – derivative can be written as a product of derivative and GH-derivative and moreover, GH-differentiability implies IFL^r – differentiable but the converse is not always true i.e., the existence of GH-difference is not a necessary condition for the existence of IFL^r – derivative. Lastly, we solved the Cauchy Problem of initial value problem for intuitionistic fuzzy differential equation with the extended IFL^r – derivative. We can further conclude that IFL^r – derivative is a generalization of GH-derivative.

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