Preprints.org

Article

Not peer-reviewed version

Queueing-Inventory Systems with Catastrophes under Various Replenishment Policies

Serife Ozkar , Agassi Melikov , Janos Sztrik *

Posted Date: 15 November 2023

doi: 10.20944/preprints202311.0961.v1

Keywords: Queueing-inventory system; Catastrophe; Negative customer; (s,S)-type policy; (s,Q)-type policy; Matrix geometric method; MAP arrival; Phase-type distribution

Article

Queueing-Inventory Systems with Catastrophes under Various Replenishment Policies

Serife Ozkar [1], Agassi Melikov [2] and Janos Sztrik [3,*]

1 Department of International Trade and Logistics, Balikesir University, 10145 Balikesir, Turkey; serife.ozkar@balikesir.edu.tr
2 Department of Mathematics, Baku Engineering University, AZ 0101, Absheron, Azerbaijan; agassi.melikov@gmail.com or agasimelikov@isi.az
3 Faculty of Informatics, University of Debrecen, 4032 Debrecen, Hungary
* Correspondence: sztrik.janos@inf.unideb.hu

Abstract: We discuss two queueing-inventory systems with catastrophes in the warehouse. Catastrophes occur according to Poisson process and upon arrival of a catastrophe all inventory in the system is instantly destroyed. But consumer customers in the system (in the server or in the buffer) continue still waiting for the replenishment of the stock. The arrivals of the consumer customers follow a Markovian Arrival Process (MAP) and they can be queued in an infinite buffer. Service time of a consumer customer follows a phase-type distribution. The system receives negative customers whose have Poisson flows to service facility and upon arrival of a negative customer one consumer customer is pushed out from the system, if any. One of two replenishment policies can be used in the system: either (s, S) or (s, Q). If upon arrival of the consumer customer, the inventory level is zero, then according to the Bernoulli scheme, this customer is either lost (lost sale scheme) or join the queue (backorder sale scheme). The system is formulated by a four-dimensional continuous-time Markov chain. Steady state distribution is obtained using the matrix-geometric method. A comprehensive numerical study is performed on the performance measures under various replenishment policies. Finally, an optimization study is presented.

Keywords: queueing-inventory system; catastrophe; negative customer; (s, S)-type policy; (s, Q)-type policy; Matrix geometric method; MAP arrival; phase-type distribution

MSC: 60J28; 60K25; 90B05; 90B22

---

1. Introduction

Until the early 90s of the last century, in the theory of operations research, models of queuing systems (QS) and models of inventory control systems (ICS) were studied separately. In other words, it was believed that in ICS there is no server for releasing items to consumers (i.e., a self-service rule is used), and in QS, only an idle server is required to service customers (i.e., no additional items are required). However, in real ICSs, the release of items to consumer customers (c-customers) requires the presence of a service station in which the incoming c-customer is processed, and the processing time is often a positive random variable. A classic example of such systems is the widespread systems of gas stations. These ICSs with positive service time can also be considered as QSs, in which in order to service c-customers, in addition to an idle server, a positive level of certain inventory is required.

Note that ICSs with positive service time are called queuing-inventory systems (QIS) in [1,2]. However, QIS models were first proposed earlier in [3,4] and have been intensively studied by various authors over the past three decades. For a detailed overview of known results on QIS models, see [5–7].

To classify QISs models, their various properties can be taken as a basis. Based on the type of QIS model being studied, the lifetime of the system's inventory is taken as the basis for the classification. The vast majority of work on QIS assumes that the system's inventory never deteriorates.

However, in real situations, system inventories often lose their quality over time and after a certain time (deterministic or random) they become unsuitable for use. Such systems are called systems with perishable inventory and have been studied in detail in numerous works, see, for example, [8–16]. Note that inventory damage can occur instantly as a result of some accidents, like power outage, equipment failures, staff negligence, etc. A sequence of accidents can be considered as a flow of destructive customers (d-customers).

Note that QIS models with d-customers have been hardly studied, although, as indicated above, they are accurate models of systems in real life. In papers [17–20], it was assumed that upon arrival of d-customers, the inventory level was instantly reduced only by one. However, there are many realistic QISs in which upon arrival of d-customers all items damage together. Below this type of systems is called QISs with catastrophes in warehouse. It is necessary to distinguish between models of QIS with catastrophes in the warehouse and models of QIS with common lifetime (e.g., foods with the same expiry date, medicines manufactured with the same expiry date and so on), see [21–24]. In models of QISs with common lifetime, it is assumed that, at any given time, all items in the warehouse have the same age; in other words, it is considered that all items of inventory arrived as a result of execution of one batch of orders. However, in the model of QIS with catastrophes in the warehouse, this assumption is not required.

Note that similar models of QS (but not QIS) with catastrophes are widely investigated in available literature. In lieu of reviewing work related to models of QS with catastrophes, we highlight representative papers [25–31] and refer readers to their reference lists. In QS a disaster events immediately wipe out the system in that all customers waiting in the queue as well as the ones getting service are removed from the system.

To increase the adequacy of the QIS model under study to real situations, we also take into account the possibility of negative customers (n-customers) arriving to the service station. Negative customer can be interpreted as customer that agitate c-customers in the system so that they do not buy the inventory in that system. In other words, n-customers do not require the inventory, but upon arrival they force one c-customer out of the system, i.e. they can be considered as d-customers in the service station of QIS.

One of the main shortcomings of the known works devoted to QIS is that they analyze models with either backorders or lost sales, i.e. QIS models that simultaneously use both backorders and lost sales are practically not considered. However, in realistic QIS an arrived c-customer either join the queue (backorder) or lost the system without inventory (lost sale) if upon its arrival an inventory level is zero, i.e. hybrid sale rule is frequently used in realistic QISs. Regardless of popularity, models of QISs with hybrid sales are poorly understood due to their complexity.

The model of single-server perishable QIS (without d-customers) with finite waiting room for c-customers under (s, Q), $Q = S - s > s + 1$, replenishment policy for the first time was considered in [32]. It was assumed that both types of c-customers and n-customers arrive according to a Markovian arrival process (MAP) and the service time of c-customers, lead time and life time of each item have exponential distributions with finite means; a n-customer at an arrival epoch removes random number of waiting c-customers. The joint probability distribution of the number of c-customers in the system and the inventory level is obtained and key performance measures of the system are calculated. Similar double sources model of QIS was considered in a recent paper [33].

The motivation for this study is that models of QIS with warehouse catastrophes under realistic assumptions have been practically unstudied. To our best knowledge, only in recent paper [34] assuming the all kind of customers are arrived according to an independent Poisson processes and all other underlying random variables to be exponentially distributed (Poisson/exponential assumptions), authors study the such kind of models in steady-state under various replenishment policies. This paper is a continuation of the research begun in [34] under more realistic assumptions related to system operation, i.e. here we assume that c-customers arrive according to MAP, c-customers and n-customers arrives according to an independent Poisson processes, the service times to be

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 November 2023 doi:10.20944/preprints202311.0961.v1

of phase-type distribution (PH-distribution), and lead times to be exponentially distributed. Under these assumptions we use matrix-analytic methods to study the QISs models with catastrophes in warehouse in steady-state under two replenishment policies: (s, S) and (s, Q) policies.

More specifically, the main differences between our model and the model considered in known works are as follows: (i) we consider model of QISs with catastrophes in warehouse; (ii) the model with infinite queue for c-customers is investigated; (iii) service time of c-customers have phase time (PH) distribution; (iv) only c-customers represents MAP flow; (v) hybrid sale rule is used, i.e. some customers may join the queue (backorder scheme) or be lost (lose sale scheme) according to the Bernoulli scheme if the inventory level is zero at the time of their arrival.

The paper is organized as follows. In Section 2 the proposed queueing-inventory system is thoroughly described. Section 3 demonstrates the construction of the generator matrices for the underlying processes and provides the steady-state analysis of the systems. That is, Subsection 3.1 includes matrices and analysis for the model-1 under (s, S)-policy, and Subsection 3.2 includes ones for the model-2 under (s, Q)-policy. Expressions for various essential performance measures to assess the both system's efficiency are formulated in Section 4. Section 5 presents numerical analysis to highlight separately the qualitative behaviour of the queueing-inventory system under each inventory policy; the effect of the system parameters on the performance measures under various arrival process and service time distribution in Subsection 5.1 and optimization study for the each inventory policy in Subsection 5.2. Finally, concluding remarks are given in Section 6.

At this point, we define some notation for use in sequel. $\boldsymbol{e}$ is a unit column vector; $\boldsymbol{e}_j$ is a unit column vector is of dimension j; $\boldsymbol{e}_j(i)$ is a unit column vector with 1 in the i^{th} position and 0 elsewhere; and $\boldsymbol{I}_k$ is an identity matrix of order k. The symbols $\otimes$ and $\oplus$ represent the Kronecker product and the Kronecker sum, respectively. If $\boldsymbol{A}$ is a matrix of order $m \times n$ and if $\boldsymbol{B}$ is a matrix of order $p \times q$, then the Kronecker product of the two matrices is given by $\boldsymbol{A} \otimes \boldsymbol{B}$, a matrix of order $mp \times nq$; the Kronecker sum of two square matrices, say, $\boldsymbol{G}$ of order g and $\boldsymbol{H}$ of h, is given by $\boldsymbol{G} \oplus \boldsymbol{H} = \boldsymbol{G} \otimes \boldsymbol{I}_h + \boldsymbol{I}_g \otimes \boldsymbol{H}$, a square matrix of order gh. The transpose notation is denoted by $'$.

2. Model description

We analyze a queueing-inventory system with negative customers and catastrophes in the warehouse as demonstrated in Figure 1.

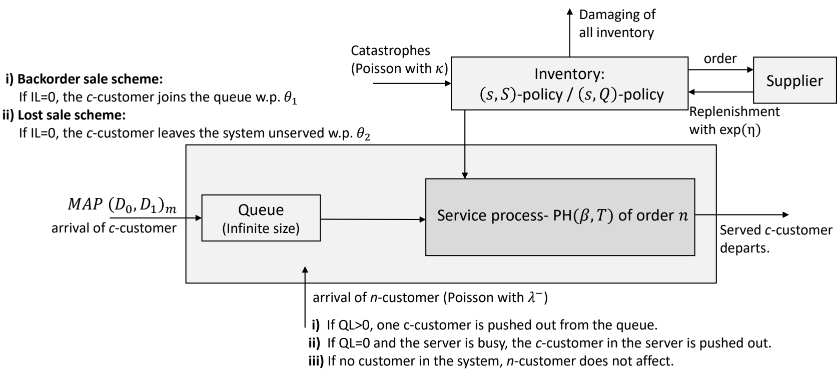

Figure 1. Block diagram of the QIS with negative customer and catastrophe in warehouse.

- The c-customers (consumer customers) arrive in the system according to Markovian arrival process (MAP) with representation $(\boldsymbol{D}_0, \boldsymbol{D}_1)_m$. The underlying Markov chain of the MAP is governed by the matrix $\boldsymbol{D}$ $(= \boldsymbol{D}_0 + \boldsymbol{D}_1)$. Such that, the matrix $\boldsymbol{D}_0$ denotes the transition rates without arrival while the matrix $\boldsymbol{D}_1$ denotes the transition rates with arrival. So, the arrival rate of c-customers is given by $\lambda^+ = \boldsymbol{\delta}\boldsymbol{D}_1\boldsymbol{e}$ where $\boldsymbol{\delta}$ is the stationary probability vector of the generator matrix $\boldsymbol{D}$ and it is satisfied

$$\boldsymbol{\delta}\boldsymbol{D} = \boldsymbol{0}, \ \boldsymbol{\delta}\boldsymbol{e} = 1. \tag{1}$$

For further details on MAP and their usefulness in QIS modelling, the reader may refer to [35–40].

- The service times of the c-customers follow phase-type distribution with representation $(\boldsymbol{\beta}, T)_n$ where $\boldsymbol{\beta}$ is the initial probability vector, $\boldsymbol{\beta e} = 1$, $\boldsymbol{T}$ is an infinitesimal generator matrix holding the transition rates among the n transient states, and $\boldsymbol{T}^0$ is a column vector contains the absorption rates into state 0 from the transient states. It is clear that $\boldsymbol{Te} + \boldsymbol{T}^0 = \boldsymbol{0}$. The phase-type distribution has the service rate $\mu = 1/[\boldsymbol{\beta}(-\boldsymbol{T})^{-1}\boldsymbol{e}]$.
- The system also receives n-customers (negative customers) that the arrivals occur according to Poisson process with rate λ^-. When a n-customer arrives in the system, there are three possible cases; (i) if there is least one c-customer in the queue $(QL > 0)$ at the time an n-customer arrives, then only the c-customer is pushed out from the queue (i.e., the servicing of the c-customer in the server continues), (ii) if the queue has no c-customer $(QL = 0)$ and the server is busy with a c-customer, then the c-customer in the server is forced out of the system. However in this case, the inventory level does not change, since it is assumed that stocks are released after the completion of servicing a c-customer and (iii) the received n-customer does not affect the operation of the system if there are no c-customers in the system (in the queue and in the server).
- Hybrid sales scheme is used in the system. When a c-customer arrives in the system, if the inventory level is zero $(IL = 0)$, then the c-customer either joins the queue of infinite capacity with probability θ_1 (called *backorder sale scheme*), or leaves the system unserved with probability θ_2 (called *lost sale scheme*). Note that $\theta_1 + \theta_2 = 1$. If th inventory level occurs to be zero with completion servicing of a c-customer, the c-customer in the queue (if any) waits for a replenishment.
- In the warehouse part of the system, catastrophic events can occur according to Poisson process with parameter κ. At the moment of arrival of such an event, all the items in the system are instantly destroyed. As a result of the catastrophes, even the item, which is at the status of release to the c-customer, is destroyed. The c-customer whose service was interrupted due to a catastrophe is returned to the queue. We can say that the catastrophe only destroys the items of the system and does not force c-customers out of the system. If the inventory level is zero, then the disaster does not affect the operation of the system warehouse.
- Two inventory replenishment policies are considered in this study. That is, as (s, S)-type policy for the Model-1 and an (s, Q)-type policy for the Model-2. The lead time of an order follows exponential distribution with parameter η for both replenishment policies. In a (s, S)-type policy (sometimes this policy is called "Up to S"), when the inventory level drops to the reorder point s, $0 \leq s < S$, an order is placed for replenishment and upon replenishment the inventory level becomes S. This policy states that the replenishment quantity varies in order to fill the maximum capacity of the inventory when the reorder is placed. In a (s, Q)-type policy, when the inventory level drops to the reorder point s, $s < \frac{S}{2}$, an order quantity of a $Q = S - s$ is placed for replenishment and upon replenishment the inventory level becomes sum of the current items in the inventory and order quantity. This policy states that the replenishment quantity is always fixed.

3. The steady-state analysis

In this section, the steady-state analysis of the queueing-inventory model described in Section 2 is performed. That is, we discuss Model-1 with (s, S)-type replenishmet policy in Subsection 3.1 and Model-2 with (s, Q)-type replenishmet policy in Subsection 3.2.

Let $K(t)$, $I(t)$, $J_1(t)$ and $J_2(t)$ denote, respectively, the number of c-customers in the system, the inventory level, the phase of the service and the phase of the arrival, at time t. The process $\{(K(t), I(t), J_1(t), J_2(t)), t \geq 0\}$ is a continuous-time Markov chain (CTMC) and the state space in the lexicographical ordering is given by

$$\begin{aligned}\Omega = \;& \{(0, i, j_2) : 0 \leq i \leq S,\ j_2 = 1, ..., m\} \bigcup \\ & \{(k, i, j_1, j_2) : k > 0,\ 0 \leq i \leq S,\ j_1 = 1, ..., n,\ j_2 = 1, ..., m\}.\end{aligned}$$

The level $\{(0,i,j_2) : 0 \le i \le S,\ j_2 = 1,...,m\}$ of dimension $m(S+1)$ corresponds to the case when there are no c-customers in the system and the inventory level is i. The arrival process is in one of m phases. The level $\{(k,i,j_1,j_2) : k > 0,\ 0 \le i \le S,\ j_1 = 1,...,n,\ j_2 = 1,...,m\}$ of dimension $mn(S+1)$ corresponds to the case when there are k c-customers in the system and the inventory level is i. The service process and the arrival process are in one of n phases and in one of m phases, respectively.

3.1. Model-1 with (s,S)-type replenishment policy

The infinitesimal generator matrix of the Markov chain governing the queueing-inventory system under (s,S)-type policy has a block-tridiagonal matrix structure and is given by

$$\boldsymbol{G} = \begin{pmatrix} \boldsymbol{B}_0 & \boldsymbol{A}_0 & & & \\ \boldsymbol{C}_0 & \boldsymbol{B} & \boldsymbol{A} & & \\ & \boldsymbol{C} & \boldsymbol{B} & \boldsymbol{A} & \\ & & \boldsymbol{C} & \boldsymbol{B} & \boldsymbol{A} \\ & & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2}$$

The matrices $\boldsymbol{A}_0$ and $\boldsymbol{A}$ in the upper diagonal of the matrix $\boldsymbol{G}$ have dimensions $m(S+1) \times mn(S+1)$ and $mn(S+1) \times mn(S+1)$, respectively.

$$\boldsymbol{A}_0 = \begin{pmatrix} \boldsymbol{\beta} \otimes \boldsymbol{D_1}\boldsymbol{\theta_1} & & & \\ & \boldsymbol{\beta} \otimes \boldsymbol{D_1} & & \\ & & \ddots & \\ & & & \boldsymbol{\beta} \otimes \boldsymbol{D_1} \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} \boldsymbol{I}_n \otimes \boldsymbol{D}_1\theta_1 & & & \\ & \boldsymbol{I}_n \otimes \boldsymbol{D}_1 & & \\ & & \ddots & \\ & & & \boldsymbol{I}_n \otimes \boldsymbol{D}_1 \end{pmatrix}.$$

The matrices $\boldsymbol{C}_0$ and $\boldsymbol{C}$ in the lower diagonal of the matrix $\boldsymbol{G}$ have dimensions $mn(S+1) \times m(S+1)$ and $mn(S+1) \times mn(S+1)$, respectively.

$$\boldsymbol{C}_0 = \begin{pmatrix} (\boldsymbol{e}_n \otimes \boldsymbol{I}_m)\lambda^- & & & \\ \boldsymbol{T}^0 \otimes \boldsymbol{I}_m & (\boldsymbol{e}_n \otimes \boldsymbol{I}_m)\lambda^- & & \\ & \ddots & \ddots & \\ & & \boldsymbol{T}^0 \otimes \boldsymbol{I}_m & (\boldsymbol{e}_n \otimes \boldsymbol{I}_m)\lambda^- \end{pmatrix},$$

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{I}\lambda^- & & & & \\ \boldsymbol{T^0}\boldsymbol{\beta} \otimes \boldsymbol{I_m} & \boldsymbol{I}\lambda^- & & & \\ & \boldsymbol{T^0}\boldsymbol{\beta} \otimes \boldsymbol{I_m} & \boldsymbol{I}\lambda^- & & \\ & & \ddots & \ddots & \\ & & & \boldsymbol{T^0}\boldsymbol{\beta} \otimes \boldsymbol{I_m} & \boldsymbol{I}\lambda^- \end{pmatrix}.$$

The matrices $\boldsymbol{B}_0$ and $\boldsymbol{B}$ in the main diagonal of the matrix $\boldsymbol{G}$ have dimensions $m(S+1) \times m(S+1)$ and $mn(S+1) \times mn(S+1)$, respectively.

$$\boldsymbol{B}_0 = \begin{pmatrix} \boldsymbol{D}_0\theta_1 - \eta\boldsymbol{I} & & & & & & \eta\boldsymbol{I} \\ \kappa\boldsymbol{I} & \boldsymbol{D}_0 - (\eta+\kappa)\boldsymbol{I} & & & & & \eta\boldsymbol{I} \\ \vdots & & \ddots & & & & \vdots \\ \kappa\boldsymbol{I} & & & \boldsymbol{D}_0 - (\eta+\kappa)\boldsymbol{I} & & & \eta\boldsymbol{I} \\ \kappa\boldsymbol{I} & & & & \boldsymbol{D}_0 - \kappa\boldsymbol{I} & & \\ \vdots & & & & & \ddots & \\ \kappa\boldsymbol{I} & & & & & & \boldsymbol{D}_0 - \kappa\boldsymbol{I} \end{pmatrix},$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 November 2023 doi:10.20944/preprints202311.0961.v1

$$
\boldsymbol{B} = \begin{pmatrix}
b_0 & & & & & & \eta \boldsymbol{I} \\
\kappa \boldsymbol{I} & b_1 & & & & & \eta \boldsymbol{I} \\
\vdots & & \ddots & & & & \vdots \\
\kappa \boldsymbol{I} & & & b_1 & & & \eta \boldsymbol{I} \\
\kappa \boldsymbol{I} & & & & b_2 & & \\
\vdots & & & & & \ddots & \\
\kappa \boldsymbol{I} & & & & & & b_2
\end{pmatrix}
$$

where
$b_0 = \boldsymbol{I}_n \otimes \boldsymbol{D}_0 \theta_1 - (\eta + \lambda^-)\boldsymbol{I}$, $b_1 = (\boldsymbol{T} \oplus \boldsymbol{D}_0) - (\eta + \kappa + \lambda^-)\boldsymbol{I}$ and $b_2 = (\boldsymbol{T} \oplus \boldsymbol{D}_0) - (\kappa + \lambda^-)\boldsymbol{I}$

3.1.1. Stability condition

Let $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \cdots, \boldsymbol{\pi}_S)$ be the steady-state probability vector of the finite generator $\boldsymbol{F} = \boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C}$. The probability vector $\boldsymbol{\pi}_i$ of dimension mn means that the inventory level is i, the service process and the arrival process are in one of n phases and in one of m phases, respectively. That is, $\boldsymbol{\pi}$ satisfies

$$\boldsymbol{\pi}\boldsymbol{F} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{\pi}\boldsymbol{e} = 1. \tag{3}$$

The steady-state equations in (3) can be rewritten as

$$
\begin{aligned}
\boldsymbol{\pi}_0[(\boldsymbol{I}_n \otimes \boldsymbol{D}_1\theta_1) + (\boldsymbol{I}_n \otimes \boldsymbol{D}_0\theta_1) - \eta\boldsymbol{I}] + \boldsymbol{\pi}_1[(\boldsymbol{T}^0\boldsymbol{\beta} \otimes \boldsymbol{I}_m) + \kappa\boldsymbol{I}] + [\boldsymbol{\pi}_2 + \cdots + \boldsymbol{\pi}_S] + \kappa\boldsymbol{I} &= \boldsymbol{0}, & \\
\boldsymbol{\pi}_i[(\boldsymbol{I}_n \otimes \boldsymbol{D}_1) + (\boldsymbol{T} \oplus \boldsymbol{D}_0) - (\kappa + \eta)\boldsymbol{I}] + \boldsymbol{\pi}_{i+1}(\boldsymbol{T}^0\boldsymbol{\beta} \otimes \boldsymbol{I}_m) &= \boldsymbol{0}, & 1 \le i \le s, \\
\boldsymbol{\pi}_i[(\boldsymbol{I}_n \otimes \boldsymbol{D}_1) + (\boldsymbol{T} \oplus \boldsymbol{D}_0) - \kappa\boldsymbol{I}] + \boldsymbol{\pi}_{i+1}(\boldsymbol{T}^0\boldsymbol{\beta} \otimes \boldsymbol{I}_m) &= \boldsymbol{0}, & s+1 \le i \le S-1, \\
[\boldsymbol{\pi}_0 + \cdots + \boldsymbol{\pi}_s]\eta\boldsymbol{I} + \boldsymbol{\pi}_S[(\boldsymbol{I}_n \otimes \boldsymbol{D}_1) + (\boldsymbol{T} \oplus \boldsymbol{D}_0) - \kappa\boldsymbol{I}] &= \boldsymbol{0}, &
\end{aligned} \tag{4}
$$

with the normalizing condition

$$\sum_{i=0}^{S} \boldsymbol{\pi}_i \boldsymbol{e} = 1.$$

Theorem 1. *The defined queuing-inventory system under an (s, S)-policy is stable if and only if the following condition is satisfied:*

$$\rho \;=\; \frac{(1 - \theta_2\boldsymbol{\pi}_0\boldsymbol{e})\lambda^+}{\mu(1 - \boldsymbol{\pi}_0\boldsymbol{e}) + \lambda^-} \;<\; 1. \tag{5}$$

Proof of Theorem 1. The defined queueing-inventory system is a QBD process thus it will be stable *if and only if* $\boldsymbol{\pi}\boldsymbol{A}\boldsymbol{e} < \boldsymbol{\pi}\boldsymbol{C}\boldsymbol{e}$ (See in [35]). That is,

$$\left[\theta_1\boldsymbol{\pi}_0 + \sum_{j=1}^{S}\boldsymbol{\pi}_j\right](\boldsymbol{I}_n \otimes \boldsymbol{D}_1)\boldsymbol{e} \;<\; \lambda^- + \sum_{j=1}^{S}\boldsymbol{\pi}_j(\boldsymbol{T}^0\boldsymbol{\beta} \otimes \boldsymbol{I}_m)\boldsymbol{e}. \tag{6}$$

Adding the equations given in (4), the following equation is obtained

$$\theta_1\boldsymbol{\pi}_0(\boldsymbol{I}_n \otimes \boldsymbol{D}) + \sum_{j=1}^{S}\boldsymbol{\pi}_j\left[(\boldsymbol{T} + \boldsymbol{T}^0\boldsymbol{\beta}) \oplus \boldsymbol{D}\right] = \boldsymbol{0}. \tag{7}$$

Post-multiplying the equation in (7) by $(\boldsymbol{e}_n \otimes \boldsymbol{I}_m)$ and using the arrival rate of the c-customers $\lambda^+ = \boldsymbol{\delta}\boldsymbol{D}_1\boldsymbol{e}$ and the normalizing condition in (4), the left-side of the inequality in (6) is given

$$\left[\theta_1\boldsymbol{\pi}_0 + \sum_{j=1}^{S}\boldsymbol{\pi}_j\right](\boldsymbol{I}_n \otimes \boldsymbol{D}_1)\boldsymbol{e} = \left[\theta_1\boldsymbol{\pi}_0\boldsymbol{e} + \sum_{j=1}^{S}\boldsymbol{\pi}_j\boldsymbol{e}\right]\lambda^+ = (1 - \theta_2\boldsymbol{\pi}_0\boldsymbol{e})\lambda^+.$$

Post-multiplying the equation in (7) by $(\boldsymbol{I}_n \otimes \boldsymbol{e}_m)$ and using the service rate $\mu = 1/[\boldsymbol{\beta}(-\boldsymbol{T})^{-1}\boldsymbol{e}]$ and the normalizing condition in (4), we get

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 November 2023 doi:10.20944/preprints202311.0961.v1

$$\sum_{j=1}^{S} \boldsymbol{\pi}_j (\boldsymbol{T}^0 \boldsymbol{\beta} \otimes \boldsymbol{I}_m) \boldsymbol{e} = \mu(1 - \boldsymbol{\pi}_0 \boldsymbol{e}).$$

The right-side of the inequality in (6) is obtained. So, the proof of Theorem is completed. □

The probability vector $\boldsymbol{\pi}_0$ in (5) can be calculated by solving the equations given in (4).

Not: In the paper [34], the authors studied the queueing-inventory system in which we have discussed in here by considering Poisson arrival and exponentially distributed service times. They obtained the closed-form solution of the probabilities for the special case. We suggest the paper in [34] to see the stability condition of the system under Poisson arrival and exponential service.

3.1.2. The steady-state probability vector of the matrix $\boldsymbol{G}$

Let $\boldsymbol{x} = (\boldsymbol{x}(0),\ \boldsymbol{x}(1),\ \boldsymbol{x}(2),\ \cdots)$ denote the steady-state probability vector of the generator matrix $\boldsymbol{G}$ in (2). That is, $\boldsymbol{x}$ satisfies

$$\boldsymbol{x}\,\boldsymbol{G} = \boldsymbol{0} \ \text{ and } \ \boldsymbol{x}\,\boldsymbol{e} = 1. \tag{8}$$

$m(S+1)$ dimensional row vector $\boldsymbol{x}(0)$ is further partitioned into vectors represented as $\boldsymbol{x}(0) = [\boldsymbol{x}(0,0), \boldsymbol{x}(0,1), \cdots, \boldsymbol{x}(0,S)]$ and the dimension of the each vector is m. The vector $\boldsymbol{x}(0,i)$ gives the steady-state probability that there are no c-customers in the system, the inventory level is $i,\ 0 \leq i \leq S$, and the arrival process is in one of m phases.

$mn(S+1)$ dimensional row vector $\boldsymbol{x}(k),\ k \geq 1$, is further partitioned into vectors represented as $\boldsymbol{x}(k) = [\boldsymbol{x}(k,0), \boldsymbol{x}(k,1), \cdots, \boldsymbol{x}(k,S)]$ and the dimension of the each vector is mn. The vector $\boldsymbol{x}(k,i)$ gives the steady-state probability that there are k c-customers in the system, the inventory level is $i,\ 0 \leq i \leq S$, and the service process and the arrival process are in one of n phases and m phases, respectively.

Under the stability condition given in (5) the steady-state probability vector $\boldsymbol{x}$ is obtained (See [35]) as

$$\boldsymbol{x}(k) = \boldsymbol{x}(1)\boldsymbol{R}^{k-1},\ \ k > 1, \tag{9}$$

where the matrix R is the minimal nonnegative solution to the following matrix quadratic equation

$$\boldsymbol{R}^2\boldsymbol{C} + \boldsymbol{R}\boldsymbol{B} + \boldsymbol{A} = \boldsymbol{0}, \tag{10}$$

and the vector $\boldsymbol{x}(0)$ and $\boldsymbol{x}(1)$ are obtained by solving

$$\begin{aligned} \boldsymbol{x}(0)\boldsymbol{B}_0 + \boldsymbol{x}(1)\boldsymbol{C}_0 &= \boldsymbol{0}, \\ \boldsymbol{x}(0)\boldsymbol{A}_0 + \boldsymbol{x}(1)[\boldsymbol{B} + \boldsymbol{R}\boldsymbol{C}] &= \boldsymbol{0}, \end{aligned} \tag{11}$$

subject to the normalizing condition

$$\boldsymbol{x}(0)\boldsymbol{e} + \boldsymbol{x}(1)(\boldsymbol{I} - \boldsymbol{R})^{-1}\boldsymbol{e} = 1. \tag{12}$$

3.2. Model-2 with (s, Q)-type replenishment policy

The infinitesimal generator matrix of the Markov chain governing the queueing-inventory system under (s, Q)-type policy has a block-tridiagonal matrix structure and is given by

$$\tilde{G} = \begin{pmatrix} \tilde{B}_0 & A_0 & & & \\ C_0 & \tilde{B} & A & & \\ & C & \tilde{B} & A & \\ & & C & \tilde{B} & A \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{13}$$

The matrices A_0, A, C_0 and C are the same in the both generator matrices in (2) and (13). Considering a different replenishment policy only the modification occurs in the main diagonal. The matrices $\tilde{B}_0$ and $\tilde{B}$ in the main diagonal of the matrix $\tilde{G}$ are given by

$$\tilde{B}_0 = \begin{pmatrix} D_0\theta_1 - \eta I & & & & & & \eta I & & & \\ \kappa I & D_0 - (\eta+\kappa)I & & & & & & \eta I & & \\ \vdots & & \ddots & & & & & & \ddots & \\ \kappa I & & & D_0 - (\eta+\kappa)I & & & & & & \eta I \\ \kappa I & & & & D_0 - \kappa I & & & & & \\ \vdots & & & & & \ddots & & & & \\ \kappa I & & & & & & D_0 - \kappa I & & & \end{pmatrix},$$

$$\tilde{B} = \begin{pmatrix} b_0 & & & & & & \eta I & & & \\ \kappa I & b_1 & & & & & & \eta I & & \\ \vdots & & \ddots & & & & & & \ddots & \\ \kappa I & & & b_1 & & & & & & \eta I \\ \kappa I & & & & b_2 & & & & & \\ \vdots & & & & & \ddots & & & & \\ \kappa I & & & & & & b_2 & & & \end{pmatrix}$$

where
$b_0 = I_n \otimes D_0\theta_1 - (\eta+\lambda^-)I,\ b_1 = (T \oplus D_0) - (\eta+\kappa+\lambda^-)I$ and $b_2 = (T \oplus D_0) - (\kappa+\lambda^-)I$

3.2.1. Stability condition

Let $\tilde{\pi} = (\tilde{\pi}_0, \tilde{\pi}_1, \tilde{\pi}_2, \cdots, \tilde{\pi}_S)$ be the steady-state probability vector of the finite generator $\tilde{F} = A + \tilde{B} + C$. The probability vector $\tilde{\pi}_i$ of dimension mn means that the inventory level is i, the service process and the arrival process are in one of n phases and in one of m phases, respectively. That is, π satisfies

$$\tilde{\pi}\tilde{F} = \mathbf{0} \quad \text{and} \quad \tilde{\pi}e = 1. \tag{14}$$

The steady-state equations in (14) can be rewritten as

$$\begin{aligned} \tilde{\pi}_0[(I_n \otimes D_1\theta_1) + (I_n \otimes D_0\theta_1) - \eta I] + \tilde{\pi}_1[(T^0\beta \otimes I_m) + \kappa I] + [\tilde{\pi}_2 + \cdots + \tilde{\pi}_S] + \kappa I &= 0, & \\ \tilde{\pi}_i[(I_n \otimes D_1) + (T \oplus D_0) - (\kappa+\eta)I] + \tilde{\pi}_{i+1}(T^0\beta \otimes I_m) &= 0, & 1 \le i \le s, \\ \tilde{\pi}_i[(I_n \otimes D_1) + (T \oplus D_0) - \kappa I] + \tilde{\pi}_{i+1}(T^0\beta \otimes I_m) &= 0, & s+1 \le i \le Q-1, \\ \tilde{\pi}_{i-Q}\eta I + \tilde{\pi}_i[(I_n \otimes D_1) + (T \oplus D_0) - \kappa I] + \tilde{\pi}_{i+1}(T^0\beta \otimes I_m) &= 0, & Q \le i \le S-1, \\ \tilde{\pi}_s\eta I + \tilde{\pi}_S[(I_n \otimes D_1) + (T \oplus D_0) - \kappa I] &= 0, & \end{aligned} \tag{15}$$

with the normalizing condition

$$\sum_{i=0}^{S} \tilde{\pi}_i e = 1.$$

The system is a QBD process thus it will be stable *if and only if* $\tilde{\pi}Ae < \tilde{\pi}Ce$. The stability condition is given in the equation (16). The proof of Theorem 2 can be performed similar to Theorem 1 in the equation (5).

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 November 2023 doi:10.20944/preprints202311.0961.v1

Theorem 2. *The defined queuing-inventory system under an (s, Q)-policy is stable if and only if the following condition is satisfied:*

$$\tilde{\rho} \;=\; \frac{(1-\theta_2\tilde{\boldsymbol{\pi}}_0\boldsymbol{e})\lambda^+}{\mu(1-\tilde{\boldsymbol{\pi}}_0\boldsymbol{e})+\lambda^-} \;<\; 1. \tag{16}$$

The probability vector $\tilde{\boldsymbol{\pi}}_0$ can be calculated by solving the equations given in (15).

3.2.2. The steady-state probability vector of the matrix $\tilde{\boldsymbol{G}}$

Let $\tilde{\boldsymbol{x}} = (\tilde{\boldsymbol{x}}(0),\ \tilde{\boldsymbol{x}}(1),\ \tilde{\boldsymbol{x}}(2),\ \cdots)$ denote the steady-state probability vector of the generator matrix $\tilde{\boldsymbol{G}}$ in (13). That is, $\tilde{\boldsymbol{x}}$ satisfies

$$\tilde{\boldsymbol{x}}\,\tilde{\boldsymbol{G}} = \boldsymbol{0} \;\text{ and }\; \tilde{\boldsymbol{x}}\,\boldsymbol{e} = 1. \tag{17}$$

$m(S+1)$ dimensional row vector $\tilde{\boldsymbol{x}}(0)$ is further partitioned into vectors represented as $\tilde{\boldsymbol{x}}(0) = [\tilde{\boldsymbol{x}}(0,0), \tilde{\boldsymbol{x}}(0,1), \cdots, \tilde{\boldsymbol{x}}(0,S)]$ and the dimension of the each vector is m. The vector $\tilde{\boldsymbol{x}}(0,i)$ gives the steady-state probability that there are no c-customers in the system, the inventory level is $i,\ 0 \le i \le S$, and the arrival process is in one of m phases.

$mn(S+1)$ dimensional row vector $\tilde{\boldsymbol{x}}(k),\ k \ge 1$, is further partitioned into vectors represented as $\tilde{\boldsymbol{x}}(k) = [\tilde{\boldsymbol{x}}(k,0), \tilde{\boldsymbol{x}}(k,1), \cdots, \tilde{\boldsymbol{x}}(k,S)]$ and the dimension of the each vector is mn. The vector $\tilde{\boldsymbol{x}}(k,i)$ gives the steady-state probability that there are k c-customers in the system, the inventory level is $i,\ 0 \le i \le S$, and the service process and the arrival process are in one of n phases and m phases, respectively.

The steady-state probability vector $\tilde{\boldsymbol{x}}$ is obtained by using the matrix-geometric solution given in (9)-(12). Recall that the matrices $\tilde{\boldsymbol{B}}_0$ and $\tilde{\boldsymbol{B}}$ are used for this solution.

4. Performance measures of Model-1 and Model-2

In this section, some performance measures of the queueing-inventory system under (s, S)-type and (s, Q)-type policies are listed. The following first seven items are valid for the both models. But, we recall that one should use the probabilities $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$ for the (s, S)-type policy (Model-1) and for the (s, Q)-type policy (Model-2), respectively. On the other hand, the last item (item 8) includes different formula for each model.

1. *The probability that there is no c-customer in the system*

$$P_{idle} = \boldsymbol{x}(0)\boldsymbol{e}.$$

2. *The mean number of c-customers in the system*

$$E(N) = \sum_{k=1}^{\infty} k\,\boldsymbol{x}(k)\boldsymbol{e} = \boldsymbol{x}(1)(\boldsymbol{I}-\boldsymbol{R})^{-2}\boldsymbol{e}.$$

3. *The mean loss rate of c-customers because of no inventory*

$$E_I(LR) = \lambda^+\theta_2\Big[\boldsymbol{x}(0,0)\boldsymbol{e}_m + \sum_{k=1}^{\infty}\boldsymbol{x}(k,0)\boldsymbol{e}_{mn}\Big].$$

4. *The mean loss rate of c-customers because of n-customer*

$$E_N(LR) = \lambda^-\Big[1-\boldsymbol{x}(0)\boldsymbol{e}\Big].$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 November 2023 doi:10.20944/preprints202311.0961.v1

5. *The mean loss rate of c-customers*

$$E(LR) = E_I(LR) + E_N(LR).$$

6. *The mean number of items in the inventory*

$$E(I) = \sum_{i=1}^{S} i\, x(0,i)\boldsymbol{e}_m + \sum_{k=1}^{\infty}\sum_{i=1}^{S} i\, x(k,i)\boldsymbol{e}_{mn}.$$

7. *The mean reorder rate*

$$E(RR) = \mu \sum_{k=1}^{\infty} i\, x(k,s+1)\boldsymbol{e}_{mn} + \kappa\Big[\sum_{i=1}^{S} x(0,i)\boldsymbol{e}_m + \sum_{k=1}^{\infty}\sum_{i=1}^{S} x(k,i)\boldsymbol{e}_{mn}\Big].$$

8. *The mean order size*

$$E_1(OS) = \sum_{i=S-s}^{S} i\, x(0,S-i)\boldsymbol{e}_m + \sum_{k=1}^{\infty}\sum_{i=S-s}^{S} i\, x(k,S-i)\boldsymbol{e}_{mn}.$$

$$E_2(OS) = Q\Big[\sum_{i=0}^{s} \tilde{x}(0,i)\boldsymbol{e}_m + \sum_{k=1}^{\infty}\sum_{i=0}^{s} \tilde{x}(k,i)\boldsymbol{e}_{mn}\Big].$$

5. Numerical study

For the arrival process, the following five sets of values for $\boldsymbol{D}_0$ and $\boldsymbol{D}_1$ are considered. The arrival processes have the same mean of 1 but each one of them is qualitatively different. The values of the standard deviation of the inter-arrival times of the arrival processes with respect to ERLA are, respectively, 1, 1.41421, 3.17451, 1.99336, and 1.99336. The MAP processes are normalized to have a specific arrival rate λ^+ as given in [41]. The arrival processes labeled MNCA and MPCA have negative and positive correlation for two successive inter-arrival times with values -0.4889 and 0.4889, respectively, whereas the first three arrival processes have zero correlation for two successive inter-arrival times.

Erlang distribution (**ERLA**):

$$\boldsymbol{D}_0 = \begin{pmatrix} -2 & 2 \\ 0 & -2 \end{pmatrix}, \; \boldsymbol{D}_1 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.$$

Exponential distribution (**EXPA**):

$$\boldsymbol{D}_0 = \begin{pmatrix} -1 \end{pmatrix}, \; \boldsymbol{D}_1 = \begin{pmatrix} 1 \end{pmatrix}.$$

Hyperexponential distribution (**HEXA**):

$$\boldsymbol{D}_0 = \begin{pmatrix} -1.9 & 0 \\ 0 & -0.19 \end{pmatrix}, \; \boldsymbol{D}_1 = \begin{pmatrix} 1.71 & 0.19 \\ 0.171 & 0.019 \end{pmatrix}.$$

MAP with negative correlation (**MNCA**):

$$\boldsymbol{D}_0 = \begin{pmatrix} -1.00222 & 1.00222 & 0 \\ 0 & -1.00222 & 0 \\ 0 & 0 & -225.75 \end{pmatrix}, \; \boldsymbol{D}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0.01002 & 0 & 0.9922 \\ 223.4925 & 0 & 2.2575 \end{pmatrix}.$$

MAP with positive correlation (**MPCA**):

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 November 2023 doi:10.20944/preprints202311.0961.v1

$$D_0 = \begin{pmatrix} -1.00222 & 1.00222 & 0 \\ 0 & -1.00222 & 0 \\ 0 & 0 & -225.75 \end{pmatrix}, \; D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0.9922 & 0 & 0.01002 \\ 2.2575 & 0 & 223.4925 \end{pmatrix}.$$

For the service times, we consider three phase-type distributions with parameter $(\boldsymbol{\beta}, \boldsymbol{T})$. The phase-type distributions have the same mean of 1 but each one of them is qualitatively different. The values of the standard deviation of the distributions are, respectively, 0.70711, 1, and 2.24472. The distributions are normalized at a specific value for the service rate μ.

Erlang distribution (**ERLS**):

$$\boldsymbol{\beta} = \begin{pmatrix} 1, & 0 \end{pmatrix}, \; \boldsymbol{T} = \begin{pmatrix} -2 & 2 \\ 0 & -2 \end{pmatrix}.$$

Exponential distribution (**EXPS**):

$$\boldsymbol{\beta} = \begin{pmatrix} 1 \end{pmatrix}, \; \boldsymbol{T} = \begin{pmatrix} -1 \end{pmatrix}.$$

Hyperexponential distribution (**HEXS**):

$$\boldsymbol{\beta} = \begin{pmatrix} 0.9, & 0.1 \end{pmatrix}, \; \boldsymbol{T} = \begin{pmatrix} -1.9 & 0 \\ 0 & -0.19 \end{pmatrix}.$$

5.1. The Effect of parameters on performance measures

We discuss the behavior of the performance measures under various the service time distributions and the arrival processes for the Model-1 with (s, S)-policy and Model-2 with (s, Q)-policy in Tables 2–13. Towards this end, the reorder point is fixed by $s = 3$ and the maximum inventory level is fixed by $S = 10$. The values of the other parameters can be seen in Table 1.

Table 1. The values of the parameters in Tables 2–13

As it is varied	It is fixed
the arrival rate of c-customers: λ^+	$\lambda^- = 1,\ \mu = 8,\ \eta = 1,\ \kappa = 1,\ \theta_1 = 0.6$
the arrival rate of n-customers: λ^-	$\lambda^+ = 5,\ \mu = 8,\ \eta = 1,\ \kappa = 1,\ \theta_1 = 0.6$
the service rate of c-customers: μ	$\lambda^+ = 5,\ \lambda^- = 1,\ \eta = 1,\ \kappa = 1,\ \theta_1 = 0.6$
the rate of the catastrophic events: κ	$\lambda^+ = 5,\ \lambda^- = 1,\ \mu = 8,\ \eta = 1,\ \theta_1 = 0.6$
the probability that c-customer joins the queue when the inventory level is zero: θ_1	$\lambda^+ = 5,\ \lambda^- = 1,\ \mu = 8,\ \eta = 1,\ \kappa = 1$

Firstly, we investigate the effects of the rates λ^+, λ^-, μ and κ on the mean number of c-customers in the system $E(N)$ under the various scenarios in Table 2 for Model-1 with (s, S)-policy and in Table 3 for Model-2 with (s, Q)-policy.

As expected, the mean number of c-customers in the system increases with increasing values of λ^+ in Table 2. When looking only at ERLA arrivals, it is seen that the variability in PH-distribution is important. Especially in high traffic intensity situations. For example, at $\lambda^+ = 5$ (high intensity), the values of $E(N)$ are 7.559, 8.458 and 16.444 for ERLS, EXPS, and HEXS, respectively, and at $\lambda^+ = 4.2$ (low intensity), the values occur 3.239, 3.490 and 5.611 for ERLS, EXPS, and HEXS, respectively. Similar comment can be made when HEXA arrivals occur. On the other hand, variability in MAP affects the values of $E(N)$ more compared to the variability in PH-distribution. Let's look ERLS services. The values of $E(N)$ are 3.239 for ERLA and 7.730 for HEXA at $\lambda^+ = 4.2$; are 7.559 for ERLA and 20.759 for

 doi:10.20944/preprints202311.0961.v1

HEXA at $\lambda^+ = 5$. Also, we can say that the values of $E(N)$ dramatically increases in the case of HEXS (service with high variability) compared to the other PH-distributions.

As values of κ increase, the values of $E(N)$ increase in Table 2. Comments similar to those above can be made regarding the effect of variability in MAP process and PH-distribution.

In Table 2, the mean number of c-customers in the system decreases with increasing the arrival rate of n-customers λ^- or the service rate of c-customers μ as expected. The effect of variability in MAP process and PH-distribution on the values of $E(N)$ is seen as μ (or λ^-) increases. Again, variability in the MAP process (variability in the inter-arrival times in other words) appears to be more significant compared to variability in PH-distribution, especially when the system has high traffic intensity (i.e., see the cases of $\mu = 7.6$ or $\lambda^- = 1$).

All comments made for Table 2 can also be made for Table 3. Compared to the values in Table 2, it can be seen that the values of $E(N)$ in Table 3 are higher, especially at high traffic intensity. In addition, we can say that the variability in MAP process or PH-distribution is more effective when the inventory policy is (s, Q). That is, as the system becomes denser, the increment or decrement becomes faster.

Table 2. $E(N)$ under (s, S)-policy

		ERLA			HEXA		
Values of the parameters		**ERLS**	**EXPS**	**HEXS**	**ERLS**	**EXPS**	**HEXS**
	4.2	3.239	3.490	5.611	7.730	8.133	10.894
	4.4	3.848	4.179	6.994	9.530	10.046	13.654
λ^+	4.6	4.663	5.106	8.925	11.967	12.646	17.501
	4.8	5.811	6.426	11.789	15.438	16.373	23.198
	5	7.559	8.458	16.444	20.759	22.140	32.449
	0.4	3.401	3.707	6.344	9.298	9.772	13.120
	0.6	4.384	4.808	8.496	11.889	12.534	17.199
κ	0.8	5.686	6.291	11.589	15.463	16.380	23.117
	1	7.559	8.458	16.444	20.759	22.140	32.449
	1.2	10.577	12.023	25.194	29.468	31.767	49.303
	7.6	9.620	10.940	22.927	27.554	29.633	45.447
	8	7.559	8.458	16.444	20.759	22.140	32.449
μ	8.4	6.323	6.989	12.837	16.701	17.717	25.201
	8.8	5.499	6.018	10.549	14.009	14.802	20.592
	9.2	4.909	5.329	8.975	12.095	12.741	17.411
	1	7.559	8.458	16.444	20.759	22.140	32.449
	1.4	4.317	4.701	7.931	11.502	12.095	16.254
λ^-	1.8	2.957	3.159	4.778	7.644	7.979	10.175
	2.2	2.216	2.331	3.200	5.555	5.767	7.059
	2.6	1.753	1.822	2.296	4.262	4.405	5.205

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 November 2023 doi:10.20944/preprints202311.0961.v1

Table 3. $E(N)$ under (s, Q)-policy

		ERLA			HEXA		
Values of the parameters		**ERLS**	**EXPS**	**HEXS**	**ERLS**	**EXPS**	**HEXS**
	4.2	3.701	4.001	6.579	9.563	10.081	13.596
	4.4	4.560	4.976	8.584	12.213	12.924	17.831
λ^+	4.6	5.811	6.412	11.701	16.100	17.133	24.402
	4.8	7.803	8.737	17.165	22.329	23.979	35.903
	5	11.486	13.156	29.116	33.888	37.021	61.022
	0.4	4.462	4.861	8.427	13.026	13.702	18.572
	0.6	5.900	6.499	11.895	17.145	18.173	25.651
κ	0.8	7.997	8.947	17.641	23.348	25.032	37.437
	1	11.486	13.156	29.116	33.888	37.021	61.022
	1.2	18.705	22.381	63.549	55.978	63.556	131.820
	7.6	16.591	19.688	52.949	50.813	57.091	111.116
	8	11.486	13.156	29.116	33.888	37.021	61.022
μ	8.4	8.971	10.066	20.110	25.573	27.542	42.060
	8.8	7.472	8.265	15.396	20.636	22.028	32.114
	9.2	6.477	7.086	12.507	17.370	18.426	26.003
	1	11.486	13.156	29.116	33.888	37.021	61.022
	1.4	5.187	5.675	9.862	14.842	15.683	21.456
λ^-	1.8	3.270	3.498	5.346	9.048	9.451	12.058
	2.2	2.354	2.476	3.412	6.281	6.516	7.939
	2.6	1.822	1.892	2.386	4.682	4.833	5.677

Secondly, we discuss the effects of the rates λ^+, λ^-, κ and the probability θ_1 on the mean number of items in the inventory $E(I)$ under the various scenarios in Table 4 for Model-1 with (s, S)-policy and in Table 5 for Model-2 with (s, Q)-policy.

Table 4. $E(I)$ under (s, S)-policy

		ERLA		HEXA		MPCA	
Values of the parameters		**ERLS**	**HEXS**	**ERLS**	**HEXS**	**ERLS**	**HEXS**
	4	3.266	3.324	3.345	3.408	3.334	3.397
	4.2	3.209	3.275	3.280	3.350	3.268	3.338
λ^+	4.4	3.154	3.228	3.217	3.294	3.204	3.281
	4.6	3.099	3.182	3.154	3.238	3.141	3.226
	4.8	3.046	3.138	3.092	3.184	3.080	3.172
	0.2	4.000	4.088	4.140	4.227	4.054	4.147
	0.4	3.696	3.797	3.807	3.907	3.747	3.851
κ	0.6	3.431	3.537	3.513	3.616	3.475	3.582
	0.8	3.199	3.303	3.255	3.358	3.234	3.339
	1	2.994	3.094	3.030	3.130	3.020	3.120
	0.1	3.655	3.665	3.774	3.795	3.767	3.796
	0.3	3.500	3.526	3.606	3.643	3.598	3.639
θ_1	0.5	3.343	3.390	3.432	3.487	3.422	3.478
	0.7	3.191	3.259	3.256	3.328	3.245	3.316
	0.9	3.039	3.127	3.077	3.165	3.068	3.155
	1	2.994	3.094	3.030	3.130	3.020	3.120
	1.4	3.108	3.184	3.159	3.242	3.150	3.231
λ^-	1.8	3.212	3.260	3.270	3.336	3.266	3.328
	2.2	3.306	3.325	3.368	3.416	3.368	3.412
	2.6	3.391	3.380	3.453	3.483	3.459	3.486

 doi:10.20944/preprints202311.0961.v1

Table 5. $E(I)$ under (s, Q)-policy

		ERLA		HEXA		MPCA	
Values of the parameters		**ERLS**	**HEXS**	**ERLS**	**HEXS**	**ERLS**	**HEXS**
	4	2.266	2.289	2.275	2.303	2.250	2.277
	4.2	2.214	2.240	2.221	2.252	2.200	2.231
λ^+	4.4	2.162	2.192	2.167	2.201	2.150	2.184
	4.6	2.109	2.143	2.113	2.150	2.101	2.138
	4.8	2.057	2.095	2.060	2.100	2.051	2.091
	0.2	2.949	2.984	2.976	3.015	2.960	3.000
	0.4	2.634	2.671	2.648	2.689	2.633	2.675
κ	0.6	2.382	2.421	2.390	2.432	2.377	2.420
	0.8	2.176	2.217	2.180	2.223	2.171	2.215
	1	2.005	2.047	2.007	2.050	2.001	2.045
	0.1	2.559	2.563	2.624	2.635	2.581	2.594
	0.3	2.456	2.467	2.496	2.515	2.454	2.473
θ_1	0.5	2.335	2.354	2.351	2.377	2.320	2.345
	0.7	2.193	2.219	2.195	2.225	2.177	2.207
	0.9	2.030	2.059	2.027	2.059	2.020	2.053
	1	2.005	2.047	2.007	2.050	2.001	2.045
	1.4	2.121	2.152	2.124	2.161	2.112	2.148
λ^-	1.8	2.218	2.236	2.222	2.252	2.205	2.233
	2.2	2.301	2.303	2.306	2.327	2.285	2.303
	2.6	2.371	2.355	2.378	2.389	2.353	2.362

As the number of c-customers (by λ^+ or θ_1) or catastrophic events (by κ) in the system increase, the mean inventory level in the system decreases. As expected, the values of $E(I)$ increase with the increment of the n-customer in the system (λ^-). On the other hand, the values of $E(I)$ increase with increasing variability (from ERLS to HEXS for PH-distribution or from ERLA to HEXA for MAP process). Also, it is seen that when the system is dense, the effect of variation in arrival process is greater than the effect of variation in service times in Table 4 and Table 5. We note the values in Table 5 (at (s, Q)-policy) are slightly lower.

Thirdly, we examine the effects of the rates λ^+, λ^-, κ and the probability θ_1 on the mean reorder rate in Tables 6–7 and the mean order size in Tables 8–9 under the various scenarios.

As seen in Tables 4–5, the decrease in the mean number of items in the inventory occurs with the increase in the number of customers in the system (by increasing the λ^+ and θ_1 rates) or with the increase of catastrophes events (by increasing the κ rate). The more customers there are, the more item in the inventory is needed. Therefore, it is seen that by increasing the values of λ^+ (by increasing the values of κ or θ_1), the values of the mean reorder rate increase in Tables 6–7 and the values of the mean order size in Tables 8–9. On the other hand, it is obvious that as n-customers come more frequently, the number of c-customers in the system will decrease (i.e., less item in the inventory will be needed). For the system under (s, S)-policy, it is seen that the values of $E(RR)$ and $E_1(OS)$ decrease with increasing λ^- in Table 6 and Table 8, respectively. Similarly, the values of $E(RR)$ and $E_2(OS)$ decrease with increasing λ^- in Table 7 and Table 9, respectively, for the system under (s, Q)-policy.

In all four parts (parts related to λ^+, κ, θ_1, λ^-) of Table 6 or Table 7, the values of the mean reorder rate decrease with increasing the variability in PH-distribution (ERLS and HEXS). On the other hand, with increasing the variability in MAP (ERLA and HEXA), the values of the mean reorder rate decrease in some parts (i.e., part κ in Table 6) and first increase and then decrease in some parts (i.e., part λ^- in Table 6). Similarly, when looking at the four parts of Table 8 or Table 9, it is seen that with the increase in the variability of PH-distribution, the values of the mean order size increase in some parts (i.e., part θ_1 in Table 8), decrease in some parts (i.e., part κ in Table 9), and first increase and then decrease in some parts (i.e., part λ^+ in Table 9). That is, we cannot talk about a specific behavior

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 November 2023 doi:10.20944/preprints202311.0961.v1

regarding the effect of variation. Tables 8–9 also shows an irregular behavior with increasing variation in MAP.

Table 6. $E(RR)$ under (s, S)-policy

		ERLA		HEXA		MPCA	
Values of the parameters		**ERLS**	**HEXS**	**ERLS**	**HEXS**	**ERLS**	**HEXS**
λ^+	4	0.642	0.607	0.646	0.609	0.633	0.598
	4.2	0.653	0.615	0.655	0.615	0.643	0.605
	4.4	0.663	0.621	0.663	0.620	0.653	0.612
	4.6	0.673	0.628	0.672	0.626	0.663	0.619
	4.8	0.682	0.634	0.680	0.632	0.673	0.626
κ	0.2	0.511	0.472	0.496	0.466	0.496	0.464
	0.4	0.572	0.526	0.561	0.521	0.558	0.516
	0.6	0.620	0.570	0.613	0.566	0.607	0.561
	0.8	0.659	0.608	0.655	0.605	0.649	0.600
	1	0.691	0.639	0.689	0.637	0.683	0.633
θ_1	0.1	0.587	0.566	0.594	0.571	0.581	0.559
	0.3	0.604	0.580	0.613	0.585	0.599	0.573
	0.5	0.629	0.598	0.634	0.601	0.621	0.589
	0.7	0.656	0.617	0.658	0.617	0.646	0.607
	0.9	0.682	0.635	0.682	0.634	0.675	0.628
λ^-	1	0.691	0.639	0.689	0.637	0.683	0.633
	1.4	0.672	0.627	0.671	0.625	0.663	0.618
	1.8	0.656	0.614	0.656	0.615	0.646	0.606
	2.2	0.640	0.603	0.644	0.606	0.632	0.596
	2.6	0.627	0.593	0.632	0.598	0.620	0.587

Table 7. $E(RR)$ under (s, Q)-policy

		ERLA		HEXA		MPCA	
Values of the parameters		**ERLS**	**HEXS**	**ERLS**	**HEXS**	**ERLS**	**HEXS**
λ^+	4	0.777	0.699	0.762	0.687	0.752	0.678
	4.2	0.788	0.705	0.774	0.694	0.766	0.687
	4.4	0.798	0.711	0.785	0.701	0.779	0.695
	4.6	0.807	0.716	0.796	0.708	0.792	0.704
	4.8	0.816	0.721	0.807	0.714	0.804	0.711
κ	0.2	0.623	0.576	0.608	0.569	0.610	0.568
	0.4	0.692	0.627	0.679	0.619	0.679	0.618
	0.6	0.747	0.667	0.735	0.660	0.734	0.658
	0.8	0.790	0.699	0.780	0.693	0.779	0.691
	1	0.825	0.725	0.817	0.720	0.816	0.719
θ_1	0.1	0.697	0.646	0.684	0.636	0.666	0.619
	0.3	0.729	0.668	0.714	0.656	0.698	0.640
	0.5	0.762	0.690	0.746	0.677	0.733	0.665
	0.7	0.792	0.708	0.779	0.698	0.771	0.691
	0.9	0.820	0.723	0.814	0.719	0.811	0.717
λ^-	1	0.825	0.725	0.817	0.720	0.816	0.719
	1.4	0.806	0.715	0.794	0.706	0.789	0.701
	1.8	0.789	0.704	0.774	0.693	0.766	0.686
	2.2	0.773	0.693	0.757	0.682	0.746	0.672
	2.6	0.758	0.682	0.742	0.672	0.728	0.659

 doi:10.20944/preprints202311.0961.v1

Table 8. $E_1(OS)$ under (s,S)-policy

		ERLA		HEXA		MPCA	
Values of the parameters		**ERLS**	**HEXS**	**ERLS**	**HEXS**	**ERLS**	**HEXS**
λ^+	4	5.891	5.928	5.953	5.983	5.896	5.927
	4.2	5.960	5.998	6.012	6.043	5.960	5.993
	4.4	6.028	6.066	6.071	6.103	6.025	6.058
	4.6	6.095	6.133	6.130	6.163	6.090	6.125
	4.8	6.161	6.200	6.188	6.222	6.155	6.191
κ	0.2	4.852	4.828	4.797	4.772	4.795	4.767
	0.4	5.267	5.257	5.247	5.234	5.222	5.210
	0.6	5.629	5.636	5.632	5.635	5.599	5.604
	0.8	5.947	5.970	5.962	5.982	5.930	5.952
	1	6.227	6.265	6.247	6.281	6.220	6.257
θ_1	0.1	5.545	5.567	5.607	5.625	5.547	5.562
	0.3	5.654	5.682	5.732	5.754	5.671	5.691
	0.5	5.806	5.840	5.876	5.903	5.816	5.843
	0.7	5.980	6.020	6.032	6.066	5.982	6.017
	0.9	6.167	6.215	6.199	6.241	6.166	6.212
λ^-	1	6.227	6.265	6.247	6.281	6.220	6.257
	1.4	6.087	6.127	6.125	6.158	6.085	6.118
	1.8	5.966	6.010	6.021	6.055	5.973	6.005
	2.2	5.861	5.912	5.931	5.969	5.880	5.912
	2.6	5.770	5.831	5.853	5.896	5.802	5.835

Table 9. $E_2(OS)$ under (s,Q)-policy

		ERLA		HEXA		MPCA	
Values of the parameters		**ERLS**	**HEXS**	**ERLS**	**HEXS**	**ERLS**	**HEXS**
λ^+	4	4.605	4.611	4.613	4.614	4.573	4.574
	4.2	4.666	4.669	4.671	4.670	4.637	4.637
	4.4	4.725	4.726	4.728	4.724	4.700	4.698
	4.6	4.784	4.781	4.784	4.778	4.763	4.759
	4.8	4.841	4.835	4.840	4.832	4.825	4.818
κ	0.2	4.036	4.006	3.993	3.959	3.994	3.959
	0.4	4.319	4.293	4.294	4.265	4.288	4.259
	0.6	4.548	4.527	4.535	4.511	4.524	4.501
	0.8	4.738	4.722	4.732	4.714	4.720	4.704
	1	4.897	4.888	4.896	4.885	4.886	4.876
θ_1	0.1	4.236	4.248	4.241	4.247	4.181	4.182
	0.3	4.365	4.375	4.379	4.382	4.322	4.323
	0.5	4.521	4.529	4.532	4.534	4.485	4.486
	0.7	4.691	4.696	4.698	4.698	4.665	4.668
	0.9	4.872	4.875	4.875	4.876	4.861	4.865
λ^-	1	4.897	4.888	4.896	4.885	4.886	4.876
	1.4	4.771	4.770	4.773	4.766	4.750	4.745
	1.8	4.659	4.671	4.668	4.668	4.634	4.634
	2.2	4.562	4.589	4.579	4.586	4.536	4.542
	2.6	4.476	4.519	4.501	4.518	4.452	4.464

The results in Tables 6–9 are for specific values of the parameters. The increases or decreases seen with increasing of variability depend on the values of the parameters. So, what we can clearly say is that the values of the mean order rate and the mean order size will definitely be affected by variability (instead of increase or decrease with variability).

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 November 2023 doi:10.20944/preprints202311.0961.v1

When Table 6 and Table 7 are compared (when Table 8 and Table 9 are compared), it is seen that the results in the system under (s, Q)-policy are larger (smaller) than the results in the system under (s, S)-policy. Additionally, as the values of the performance measures faster increase (or decrease) with the increase of the values of the parameters in the system under (s, Q)-policy.

Finally, we examine the effects of system parameters on the mean lost rate of c-customers in the system. Let's recall, c-customers can lost in the system studied in two cases; If there is no inventory at the time the c-customer comes to the system, he does not enter the system with probability θ_2 (he is said to be lost)- this case is indicated by $E_I(LR)$ in Tables 10–11, and the arrival of n-customers to the system causes the loss of one c-customer- this case is denoted by $E_N(LR)$ in Tables 12–13.

As the value of λ^+ or κ increases, the probability that the inventory is stock-out increases. This increases the rate at which c-customers are lost due to lack of item in the inventory. On the other hand, as λ^- increases, the probability of the inventory falling to zero decreases (as it reduces the number of c-customers in the system), which causes the values of $E_I(LR)$ to decrease. As an interesting result, it is seen that as θ_1 probability increases, the values of $E_I(LR)$ decrease even though the number of c-customers in the system increases. All results can be seen in Table 10 for the system under (s, S)-policy and Table 11 for the system under (s, Q)-policy.

As expected, as long as there are c-customers in the system, c-customers will disappear as n-customers arrive. Therefore, it can be seen in Table 12 and Table 13 that $E_N(LR)$ values increase as the values of all parameters increase.

Table 10. $E_I(LR)$ under (s, S)-policy

		ERLA		HEXA		MPCA	
Values of the parameters		**ERLS**	**HEXS**	**ERLS**	**HEXS**	**ERLS**	**HEXS**
λ^+	4	0.838	0.850	0.857	0.870	0.860	0.875
	4.2	0.887	0.901	0.906	0.921	0.908	0.925
	4.4	0.937	0.954	0.956	0.973	0.957	0.976
	4.6	0.989	1.008	1.006	1.026	1.007	1.028
	4.8	1.041	1.064	1.057	1.080	1.057	1.081
κ	0.2	0.645	0.670	0.679	0.702	0.673	0.698
	0.4	0.790	0.815	0.819	0.843	0.816	0.841
	0.6	0.910	0.936	0.934	0.959	0.932	0.958
	0.8	1.010	1.037	1.029	1.054	1.028	1.055
	1	1.095	1.122	1.109	1.134	1.108	1.135
θ_1	0.1	1.838	1.845	1.867	1.878	1.877	1.894
	0.3	1.437	1.447	1.468	1.480	1.476	1.493
	0.5	1.039	1.050	1.063	1.076	1.068	1.084
	0.7	0.635	0.646	0.649	0.660	0.650	0.663
	0.9	0.217	0.222	0.220	0.226	0.221	0.226
λ^-	1	1.095	1.122	1.109	1.134	1.108	1.135
	1.4	1.074	1.094	1.093	1.114	1.094	1.117
	1.8	1.058	1.073	1.080	1.098	1.083	1.102
	2.2	1.046	1.058	1.069	1.085	1.074	1.091
	2.6	1.037	1.047	1.060	1.074	1.067	1.082

 doi:10.20944/preprints202311.0961.v1

Table 11. $E_I(LR)$ under (s, Q)-policy

		ERLA		HEXA		MPCA	
Values of the parameters		**ERLS**	**HEXS**	**ERLS**	**HEXS**	**ERLS**	**HEXS**
λ^+	4	0.883	0.902	0.907	0.926	0.905	0.926
	4.2	0.939	0.961	0.961	0.984	0.958	0.982
	4.4	0.996	1.022	1.017	1.042	1.013	1.040
	4.6	1.055	1.085	1.073	1.102	1.070	1.100
	4.8	1.115	1.149	1.130	1.163	1.127	1.161
κ	0.2	0.772	0.808	0.809	0.843	0.799	0.836
	0.4	0.906	0.943	0.936	0.971	0.928	0.965
	0.6	1.014	1.052	1.037	1.073	1.031	1.069
	0.8	1.103	1.141	1.119	1.156	1.116	1.153
	1	1.177	1.216	1.188	1.225	1.186	1.223
θ_1	0.1	1.887	1.901	1.931	1.948	1.930	1.954
	0.3	1.486	1.503	1.530	1.550	1.529	1.553
	0.5	1.087	1.106	1.119	1.139	1.117	1.139
	0.7	0.674	0.691	0.690	0.707	0.688	0.706
	0.9	0.234	0.242	0.237	0.245	0.236	0.244
λ^-	1	1.177	1.216	1.188	1.225	1.186	1.223
	1.4	1.144	1.174	1.164	1.194	1.161	1.192
	1.8	1.117	1.141	1.144	1.170	1.141	1.168
	2.2	1.097	1.117	1.127	1.150	1.125	1.149
	2.6	1.081	1.098	1.113	1.134	1.113	1.134

Table 12. $E_N(LR)$ under (s, S)-policy

		ERLA		HEXA		MPCA	
Values of the parameters		**ERLS**	**HEXS**	**ERLS**	**HEXS**	**ERLS**	**HEXS**
λ^+	4	0.759	0.752	0.724	0.727	0.745	0.745
	4.2	0.787	0.782	0.756	0.761	0.776	0.777
	4.4	0.815	0.813	0.788	0.794	0.806	0.809
	4.6	0.843	0.843	0.819	0.827	0.836	0.840
	4.8	0.871	0.873	0.851	0.860	0.865	0.871
κ	0.2	0.743	0.736	0.747	0.748	0.756	0.755
	0.4	0.790	0.787	0.783	0.788	0.796	0.798
	0.6	0.830	0.830	0.818	0.825	0.831	0.835
	0.8	0.866	0.869	0.851	0.860	0.864	0.870
	1	0.898	0.903	0.882	0.893	0.894	0.902
θ_1	0.1	0.438	0.417	0.446	0.434	0.459	0.446
	0.3	0.601	0.583	0.572	0.567	0.586	0.578
	0.5	0.713	0.702	0.675	0.676	0.696	0.693
	0.7	0.802	0.799	0.771	0.778	0.791	0.795
	0.9	0.882	0.889	0.864	0.878	0.878	0.889
λ^-	1	0.898	0.903	0.882	0.893	0.894	0.902
	1.4	1.173	1.166	1.142	1.148	1.161	1.163
	1.8	1.410	1.384	1.363	1.359	1.387	1.379
	2.2	1.615	1.562	1.554	1.536	1.579	1.559
	2.6	1.790	1.707	1.720	1.685	1.744	1.710

Table 13. $E_N(LR)$ under (s, Q)-policy

		ERLA		HEXA		MPCA	
Values of the parameters		ERLS	HEXS	ERLS	HEXS	ERLS	HEXS
λ^+	4	0.778	0.776	0.755	0.762	0.772	0.777
	4.2	0.809	0.810	0.788	0.798	0.805	0.812
	4.4	0.840	0.844	0.822	0.834	0.836	0.845
	4.6	0.870	0.877	0.856	0.870	0.868	0.879
	4.8	0.900	0.910	0.889	0.905	0.899	0.912
κ	0.2	0.783	0.783	0.794	0.800	0.800	0.805
	0.4	0.828	0.832	0.829	0.839	0.837	0.846
	0.6	0.867	0.874	0.862	0.875	0.871	0.882
	0.8	0.901	0.911	0.893	0.909	0.901	0.915
	1	0.930	0.944	0.923	0.940	0.929	0.945
θ_1	0.1	0.435	0.415	0.452	0.439	0.469	0.459
	0.3	0.604	0.589	0.587	0.583	0.602	0.597
	0.5	0.726	0.719	0.700	0.704	0.719	0.720
	0.7	0.828	0.831	0.808	0.821	0.824	0.833
	0.9	0.924	0.942	0.915	0.938	0.923	0.943
λ^-	1	0.930	0.944	0.923	0.940	0.929	0.945
	1.4	1.206	1.208	1.187	1.200	1.201	1.212
	1.8	1.441	1.424	1.410	1.414	1.430	1.431
	2.2	1.642	1.597	1.600	1.590	1.624	1.612
	2.6	1.813	1.738	1.764	1.738	1.789	1.764

5.2. *Optimization*

For the described two models, the function of the expected total cost, ETC, is constructed and an optimization discussion about inventory policies is provided for some specific parameters. In the equation (18), we note that $E_i(OR)$ is the mean order size of the system with (s, S)-policy for $i = 1$ and of the system with (s, Q)-policy for $i = 2$.

$$ETC = \left[c_k + c_r E_i(OS)\right] E(RR) + c_h E(I) + c_{ps}\kappa E(I) + c_l E(LR) + c_w E(N) \tag{18}$$

where
c_k : the fixed cost of one order,
c_r : the unit cost of the order size,
c_h : the holding cost per item in the inventory per unit of time,
c_{ps} : the damaging cost per item in the inventory,
c_l : the cost incured due to the loss of a c-customer,
c_w : the waiting cost of a c-customer in the system.

Towards finding the optimum values of the inventory level (that minimize ETC) for the both model, we fix $\lambda^+ = 4$, $\lambda^- = 1$, $\mu = 8$, $\eta = 1$, $\kappa = 1$ and $\theta_1 = 0.6$ and vary the reorder points $s = 3, 5, 7$. Also, we fix the unit values of the defined above costs by $c_k = 10$, $c_r = 15$, $c_h = 10$, $c_{ps} = 15$, $c_l = 350$ and $c_w = 300$. Under various distributions of the service times and arrival processes, we give the optimum values of ETC and S in Table 14 for the system under (s, S)-policy and in Table 15 for the system under (s, Q)-policy.

Let's look at the cases of ERLA, EXPA and HEXA in Table 14. As the variability in arrival processes increases (respectively, ERLA, EXA and HEXA), the optimum value of S also increases. For both ERLS and EXPS services, the optimum S is generally the same, while the optimum cost varies slightly. In all cases, HEXS services with high variability require more inventory in the system. When the reorder point s is increased, the values of S generally do not change except for HEXA arrivals. However, in the case of HEXA, the optimum S is seen to decrease as s increases.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 November 2023 doi:10.20944/preprints202311.0961.v1

Table 14. Optimum values of ETC^* and S^* for the system under (s, S)-policy

		$s=3$		$s=5$		$s=7$	
MAP	**PH**	S^*	ETC^*	S^*	ETC^*	S^*	ETC^*
ERLA	ERLS	12	1523.049	12	1526.263	12	1538.455
	EXPS	12	1577.435	12	1579.782	12	1590.842
	HEXS	14	2027.068	14	2025.895	14	2030.171
EXPA	ERLS	13	1657.027	13	1657.273	13	1665.452
	EXPS	13	1714.634	13	1714.218	13	1721.526
	HEXS	15	2169.740	15	2167.181	14	2169.473
HEXA	ERLS	18	2413.463	17	2402.938	16	2398.154
	EXPS	18	2496.819	17	2486.839	17	2482.051
	HEXS	19	3043.694	19	3034.463	18	3028.903
MNCA	ERLS	13	1706.068	13	1706.237	13	1714.395
	EXPS	13	1760.549	13	1760.072	13	1767.381
	HEXS	15	2209.347	15	2206.767	15	2209.113
MPCA	ERLS	39	28273.270	38	28245.217	36	28217.794
	EXPS	40	28343.298	39	28316.825	37	28290.718
	HEXS	45	28862.495	43	28840.115	42	28818.031

In Table 14 let's look at the MNCA and MPCA cases where there is correlation. In negatively correlated arrivals (MNCA), the results in the HEXS service are significantly different from the others and the increase in the values of s is of no significance. On the other hand, in positively correlated arrivals (MPCA), the increase in the values of s and the increase in the variability in service times are separately very important. That is, as the variability in PH-distribution increases, the values of S increase, and as the reorder point increases, the values of S decrease.

First, it is noticeable that the optimum values of S in Table 15 are larger than the values in Table 14, while there is not much difference between the optimum cost values. In other words, in the (s, Q)-policy, there is a need to keep more inventory in the system. Although more inventory is carried, the total cost is almost the same as under the (s, S)-policy.

Table 15. Optimum values of ETC^* and S^* for the system under (s, Q)-policy

		$s=3$		$s=5$		$s=7$	
MAP	**PH**	S^*	ETC^*	S^*	ETC^*	S^*	ETC^*
ERLA	ERLS	15	1522.919	17	1529.208	19	1547.543
	EXPS	15	1577.272	17	1582.646	19	1599.781
	HEXS	17	2026.762	19	2027.477	21	2035.853
EXPA	ERLS	16	1656.737	18	1659.345	20	1672.515
	EXPS	16	1714.313	18	1716.220	20	1728.459
	HEXS	18	2169.372	20	2168.303	22	2173.935
HEXA	ERLS	21	2412.971	22	2403.011	23	2399.884
	EXPS	21	2496.307	22	2486.853	24	2483.935
	HEXS	22	3043.138	24	3034.403	25	3029.949
MNCA	ERLS	16	1705.783	18	1708.339	20	1721.544
	EXPS	16	1760.236	18	1762.111	20	1774.414
	HEXS	18	2208.991	20	2207.933	21	2213.632
MPCA	ERLS	42	28273.128	43	28244.999	43	28217.403
	EXPS	43	28343.163	44	28316.618	44	28290.344
	HEXS	48	28862.372	48	28840.132	49	28817.682

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 November 2023 doi:10.20944/preprints202311.0961.v1

The comments made for Table 14 regarding the variability of service times or arrival process can also be said for Table 15. As variation increases, more inventory is needed. Also, positive correlation is important for the system under (s, Q)-policy similar to the system under (s, S)-policy.

Finally, the important difference between the two tables is the effect of the reorder point s. As the values of s increases, the values of S remain the same or decrease in Table 14 (as we mentioned above). In Table 15, as the values of s increases, the values of S remain the same or increase.

6. Discussion

We study two queueing-inventory systems with catastrophes in the warehouse. Upon arrival of a catastrophe all inventory in the system is instantly destroyed. The arrivals of the c-customers follow a Markovian Arrival Process (MAP) and they can be queued in an infinite buffer. Service time of a c-customer follows a phase-type distribution. The system receives n-customers to service facility and upon arrival of a n-customer one c-customer is pushed out from the system, if any. One of two replenishment policies can be used in the system: either (s, S) or (s, Q). If upon arrival of the c-customer, the inventory level is zero, then according to the Bernoulli scheme, this customer is either lost (lost sale scheme) or join the queue (backorder sale scheme).

The system is formulated by a four-dimensional continuous-time Markov chain. Steady state distribution is obtained using the matrix-geometric method. A comprehensive numerical study is performed on the performance measures and an optimization under various the service time distributions and the arrival processes. As a result of numerical studies, it is seen that the variability in service distribution, the variability in the arrival process and the arrivals with positive correlation have an impact on both the performance measures of the system and the optimum inventory policy. Also, it has been observed that the effect of variability is more specifically in the system with (s, Q)-policy than the system with (s, S)-policy.

For the future work, ones can improve the studied system by considering the batch service and/or batch arrival.

Author Contributions: Conceptualization, A.M. and S.O.; methodology, A.M. and S.O.; software, S.O.; investigation, A.M., S.O. and J.S.; writing—review and editing, A.M., S.O. and J.S.; supervision and project administration, A.M.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

QS	Queueing System
QIS	Queueing Inventory System
ICS	Inventory Control System
MAP	Markovian Arrival Process
PH	Phase-type distribution
IL	Inventory Level
QL	Queue Length
CTMC	Continuous Time Markov Chain
QBD	Quasi-birth-and-death process
ETC	Expected Total Cost

 doi:10.20944/preprints202311.0961.v1

References

1. Schwarz, M.; Daduna, H. Queuing Systems with Inventory Management with Random Lead Times and with Backordering. *Math. Methods Oper. Res.* **2006**, *64*, 383–414.
2. Schwarz, M.; Sauer, C.; Daduna, H.; Kulik, R.; Szekli, R. $M/M/1$ Queuing Systems with Inventory. *Queuing Syst. Theory Appl.* **2006**, *54*, 55–78.
3. Melikov, A.; Molchanov, A. Stock Optimization in Transport/Storage Systems. *Cybernetics* **1992**, *28*, 484–487.
4. Sigman, K.; Simchi-Levi, D. Light Traffic Heuristic for an $M/G/1$ Queue with Limited Inventory. *Ann. Oper. Res.* **1992**, *40*, 371-380.
5. Krishnamoorthy, A.; Shajin, D.; Narayanan, W. Inventory with Positive Service Time: A Survey, Advanced Trends in Queueing Theory; Series of Books "Mathematics and Statistics" Sciences. V. 2; Anisimov, V., Limnios, N., Eds.; ISTE & Wiley: London, UK, 2021; pp. 201-238.
6. Krishnamoorthy, A.; Lakshmy B.; Manikandan R. A survey on inventory models with positive service time. *OPSEARCH* **2011**, *48*(2), 153–169.
7. Karthikeyan, K.; Sudhesh R. Recent review article on queueing inventory systems. *Research J. Pharm. and Tech.* **2016**, *9*(11), 1451–1461.
8. Ko, S.S. A Nonhomogeneous Quasi-Birth Process Approach for an (s, S) Policy for a Perishable Inventory System with Retrial Demands. *J. Ind. Manag. Opt.* **2020**, *16*, 1415-1433.
9. Melikov, A.; Krishnamoorthy, A.; Shahmaliyev, M.O. Numerical Analysis and Long Run Total Cost Optimization of Perishable Queuing Inventory Systems with Delayed Feedback. *Queuing Model. Serv. Manag.* **2019**, *2*, 83-111.
10. Jenifer J.S.A.; Sangeetha N.; Sivakumar B. Optimal Control of Service Parameter for a Perishable Inventory System with Service Facility, Postponed Demands and Finite Waiting Hall. *Int. J. Inf.* & *Manag Sci.* **2014**, *25*, 349-370.
11. Reshmi, P.S.; Jose, K. A Perishable (s, S) Inventory System with an Infinite Orbit and Retrials. *Math. Sci. Int. Res. J.* **2022**, *7*(1), 121-126.
12. Melikov, A.; Shahmaliyev, M.; Nair, S.S. Matrix-Geometric Method to Study Queuing System with Perishable Inventory. *Autom. Remote Control* **2021**, *82*, 2168-2181.
13. Goyal S., GiriB. Recent trends in modeling of deteriorating inventory *Eur. J. Oper. Res.* **2001**, *134*(1), 1-16.
14. Jeganathan, K.; Selvakumar, S.; Saravanan, S.; Anbazhagan, N.; Amutha, S.; Cho, W.; Joshi, G. P.; Ryoo, J. (2022). Performance of stochastic inventory system with a fresh item, returned item, refurbished item, and multi-class customers. Mathematics 2022, 10(7), 1137.
15. Karaesmen I., Scheller-Wolf A., Deniz B. Managing perishable and aging inventories: Review and future research directions- Planning production and inventories in the extended enterprise. A state-of-the-art handbook. (Eds. Kempf K., Keskinocak P, Uzsoy P.). Vol. 1. Springer, 2011, 393-438.
16. Nahmias S. Perishable inventory theory. Heidelberg: Springer. 2011.
17. Melikov, A.; Aliyeva, S.; Nair, S.; Krishna Kumar, B. Retrial queuing-inventory systems with delayed feedback and instantaneous damaging of items. *Axioms* **2022**, *11*, 241.
18. Melikov, A.; Mirzayev, R.R.; Nair, S.S. Numerical investigation of double source queuing-inventory systems with destructive customers. *J. Comput. Syst. Sci. Int.* **2022**, *61*, 581–598.
19. Melikov, A.; Mirzayev, R.R.; Nair, S.S. Double Sources Queuing-Inventory System with Hybrid Replenishment Policy. *Mathematics* **2022**, *10*, 2423.
20. Melikov, A.; Mirzayev, R.R.; Sztrik, J. Double Sources QIS with finite waiting room and destructible stocks. *Mathematics* **2023**, *11*, 226.
21. Lian, Z.; Liu, L.; Neuts, F. A Discrete-Time Model for Common Lifetime Inventory Systems. *Math. Oper. Res.* **2005**, *30*, 718–732.
22. Chakravarthy, S.R. An inventory system with Markovian demands, and phase-type distributions for perishability and replenishment. *OPSEARCH* **2010**, *47*, 266–283.
23. Krishnamoorthy, A.; Shajin, D.; Lakshmy, B. On a queueing-inventory with reservation, cancellation, common life time and retrial. *Ann. Oper. Res.* **2016**, *247*, 365–389.
24. Shajin, D.; Krishnamoorthy, A.; Manikandan, R. On a Queueing-Inventory System with Common Life Time and Markovian Lead Time Process. *Oper. Res.* **2022**, *22*, 651–684.

25. Demircioglu, M.; Bruneel, H.; Wittevrongel, S. Analysis of a Discrete-Time Queueing Model with Disasters. *Mathematics* **2021**, *9*, 3283.
26. Krishna Kumar, B.; Arivudainambi, D. Transient Solution of an $M/M/1$ Queue with Catastrophes. *Comput. Math. Appl.* **2000**, *40*, 1233–1240.
27. Vinodhini, G.A.F.; Vidhya, V. Computational Analysis of Queues with Catastrophes in a Multiphase Random Environment. *Math. Probl. Eng.* **2016**, 2917917.
28. Ye, J.; Liu, L.; Jiang, T. Analysis of a Single-Sever Queue with Disasters and Repairs under Bernoulli Vacation Schedule. *J. Syst. Sci. Inf.* **2016**, *4*, 547-559.
29. Chakravarthy, S. R. A catastrophic queueing model with delayed action. *Appl. Math. Model.* **2017**, *46*, 631-649.
30. Chakravarthy, S. R.; Dudin, A. N.; Klimenok, V. I. A retrial queueing model with MAP arrivals, catastrophic failures with repairs, and customer impatience. *Asia-Pacific J. Oper. Res.* **2010**, *27*, 727-752.
31. Raj, R.; Jain, V. (2023) Resource and traffic control optimization in $MMAP[c]/PH[c]/S$ queueing system with PH retrial times and catastrophe phenomenon. *Telecommun Syst.* **2023**, *84*, 341-362.
32. Sivakumar, B.; Arivarignan, G. A Perishable Inventory System with Service Facilities and Negative Customers. *Adv. Model. Optim.* **2005**, *7*, 193–210.
33. Soujanya, M.L.; Laxmi, P.V. Analysis on Dual Supply Inventory Model Having Negative Arrivals and Finite Lifetime Inventory. *Reliab. Theory Appl.* **2021**, *16*, 295–301.
34. Melikov A.; Poladova L.; Edayapurath S.; Sztrik J. Single-server queuing-inventory systems with negative customers and catastrophes in the warehouse. *Mathematics* **2023**, *11*, 2380.
35. Neuts, M.F. (1981) *Matrix-geometric solutions in stochastic models: An algorithmic approach*; The Johns Hopkins University Press: Baltimore, MD, USA, 1981.
36. Chakravarthy, S.R. Introduction to Matrix-Analytic Methods in Queues; John Wiley & Sons, Inc.: London, UK, 2022; Volume 1.
37. Chakravarthy, S.R. Introduction to Matrix-Analytic Methods in Queues; John Wiley & Sons, Inc.: London, UK, 2022; Volume 2.
38. Dudin, A.N.; Klimenok, V.I.; Vishnevsky, V.M. The Theory of Queueing Systems with Correlated Flows; Springer Nature Switzerland AG: Basel, Switzerland, 2020.
39. He, Q.-M. Fundamentals of Matrix-Analytic Methods; Springer: New York, NY, USA, 2014.
40. Latouche, G.; Ramaswami, V. Introduction to Matrix Analytic Methods in Stochastic Modeling; SIAM: Philadelphia, PA, USA, 1999.
41. Chakravarthy, S. R. Markovian arrival processes. *Wiley Encyclopedia of Operations Research and Management Science* **2010** https://doi.org/10.1002/9780470400531.eorms0499