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Article

Exergy as Lyapunov Function for Studying the Dynamic Stability of a Flow, Reacting to Self-Oscillations Excitation

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Abstract: This article introduces a new physically based Lyapunov function definition, which can be used as acoustic energy for finite amplitude pressure oscillations in reacting systems especially in combustion chambers. Reacting flow is seen as an open, non-equilibrium (parameters are distributed unevenly locally) thermodynamic system. This Lyapunov function is defined as the maximum mechanical work, which could be extracted by a heat engine from studied system, if this system would be disconnected from inlet and outlet and from any other surrounding environment, and the engine could transfer to the surrounding environment only mechanical work.

Keywords: reacting flow stability; self-sustained oscillations; combustion instability; finite amplitude acoustic

1. Introduction

Combustion instability is one of the most significant problems for low emission combustion chambers and high energy density rocket engines currently under development. Combustion instability has a significant meaning in land-based gas turbine installations due to the widespread method of burning lean, preliminarily blended mixtures. This approach allows significantly reducing the most harmful nitrous oxides (NO_x), while sacrificing combustion stability [1].

Under these operating conditions, burners are severely damaged by self-exciting oscillations, caused by combustion. This effect has been investigated with different types of combustion chamber in the following works [2–4]. Particularly the instability problem emerges in the combustion of hydrogen, which is a perspective type of fuel, for both aviation engines and land-based gas turbine installations. Combustion instability, a coupling between resonant combustor acoustics and heat release rate fluctuations, is one of the leading challenges in developing and operating both aircraft and power-generation gas turbines [5]. Combustion stability and velocity of heat dissipation, depending also on acoustic oscillations, impact both on the engine characteristics and on its reliability. It is also known that the method of vortex burning, which is commonly employed because of its numerous advantages, also has a tendency to instability onset, which is documented in the following sources [6–10]. Therefore, as for producing a complete mathematical model of self-oscillating processes in the combustion chamber, it is necessary to consider the total energy, including acoustic component.

In some devices such as thermoacoustic engine and pulsejets this instability may be used to produce work. In system where combustion instability is inhibited such phenomena produces work, but structurally destructive. Pressure oscillations, which occur in combustion instability process, have high availability to produce work. Moreover, a system that uses pressure oscillations can be added to any internal combustion engine to create additional work.

First study for singing flames was produced by Reyleigh [11]. In his work the criteria for maintaining the oscillations was elaborated: "...it follows that at the phase of greatest condensation heat is received by the air, and at the phase of greatest rarefaction heat is given up from it, and thus there is a tendency to maintain the vibrations." Analyzing combustion stability has been carried out in two ways. First way was started by work [12] using Kovaznay ideas. The purpose of this work was a physical based explanation of Rayleigh criteria by dividing the volume of studied system into small parts and studying the interaction between them. It is assumed that these parts work as cylinder and piston of reciprocating engine and move with main flow. These parts interact with acoustic waves, if parts produce work, then this work is used to increase energy of acoustic waves, if these parts consume work, then it decreases energy of acoustic waves.

The second way was started from Putnam interpretation [13] of Rayleigh criteria, in this work mean flow was omitted from study, and from linear acoustic analysis for pressure wave equation it was derived that the mean power added to acoustic wave for unit of volume is equal to N_A :

$$N_A = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^{\tau} \frac{(k-1)}{kP_0} P'Q' dt. \quad (1)$$

Polifke in his work [14] divided acoustic emission from combustor to combustion noise and self-excited fluctuation. The combustion noise is not having a feedback from the sound, emitted behind the combustion region, which does not influence the combustion process and species concentration. The self-excited fluctuations are produced by right phase correlation between combustion and reflected sound waves. In his explanation of thermodynamic cycle which increases internal energy of sound waves he also neglected mean flow effect, so he gets as a result Putnam interpretation of Rayleigh's criterion Eq. (1). In following chapters of his work, author studied moving systems in which the produced work can be compared with Eq. (1):

$$N_A = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^{\tau} \frac{(k-1)}{kP_0} P'Q' \left(1 - \frac{P'Q}{P_0 Q'} \right) dt. \quad (2)$$

If Rayleigh's criterion (in Putnam interpretation [3]) is analyzed from thermodynamically point of view it can be shown what it is another form of Carnot formula:

$$\begin{aligned} N_A &= \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^{\tau} \frac{(k-1)}{kP_0} P'Q' dt \approx \\ &\approx f \oint \frac{T'}{T_0} dQ' \approx f \oint \left(1 - \frac{T_0}{T} \right) dQ'. \end{aligned} \quad (3)$$

In another words, Rayleigh's criteria only means: the system is unstable if more heat has been added when system has higher temperature and less heat has been added when system has lower temperature.

This analogy between Rayleigh criteria and Carnot formula clearly shows that, in the same way as efficiency of real internal combustion engines is less than efficiency of Carnot cycle, the real combustion instability efficiency in transforming heat to acoustic energy is less than in the model used to formulate Rayleigh's criteria. And Eq. (2) from work [14] can be seen as correction to Brayton cycle efficiency in comparison with Carnot cycle.

The definitions of stability criteria are usually produced from some adaptation of Lyapunov function, in the first work it was acoustic energy, later it was extended to a function, which takes into account entropy pulsations. But Lyapunov function was defined using algebraic manipulations only, without physical based explanation.

This method to derive equation for acoustical energy was described in Mayers' work [15]. In works [16–21] this method to derive acoustic energy was extended for non-isentropic flow and to analyze combustion systems, in all this works acoustic energy is studied as second order polynomial by disturbances. In work [20] authors studied which type of acoustic energy quadratic form does not support spurious solution growth. In work [21] authors extend Mayers' method to derive equation

for third order polynomial by disturbances which is used to study nonlinear interaction between acoustic waves having different frequencies but physical meaning of this quantity was not explained. In this study the origin of acoustical energy is analyzed in the base of thermodynamics of open systems, and found the definition of acoustic energy as the system availability to produce work (or, in another words, the exergy of a system). Previously this method was used in works [22,23], but definition of exergy used in that works was only a first order polynomial by disturbances, and proposed approximation of acoustic energy has zero mean value and due to this, it could not be used to study growth and attenuation of combustion instability.

2. Simple model

Thermodynamic formulation of Lyapunov function for reacting flow is performed according to Bystrai [24]. The studied system is seen as having internal and external parameters. If the system is steady, it means it reaches equilibrium between external thermodynamic forces and dissipation, internal parameters are fully defined and additional work extraction from this system is impossible. Thus, the simplest case is when there are no external thermodynamic forces at all, we just have a non-equilibrium closed system.

To study influence of pressure, velocity and entropy fluctuations on the availability to produce work, we start our analysis from closed volume which contains ideal gas having uneven pressure, entropy and velocity fields. Suppose these fluctuations are small enough compared to mean internal energy in the volume, where ε is a small parameter:

$$p = p_0 + \varepsilon p'(x); \quad s = \varepsilon s'(x); \quad \vec{v} = \varepsilon \vec{v}'(x) \quad (4)$$

Mean parameters were taken in such manner that these conditions are met, where $\langle \rangle$ is an operator to find mean value in volume:

$$\langle \vec{v}' \rangle = \frac{1}{V} \int_V \vec{v}' dV = \vec{0}; \quad (5a)$$

$$\langle p' \rangle = \frac{1}{V} \int_V p' dV = 0; \quad (5b)$$

$$\langle s' \rangle = \frac{1}{V} \int_V s' dV = 0. \quad (5c)$$

For ideal gas we will use equation of state represented in variables p and s , at mean pressure and mean entropy density is equal to ρ_0 :

$$\rho = \rho_0 \left(\frac{p}{p_0} \right)^{1/k} \exp \left(-\frac{s}{C_p} \right). \quad (6)$$

It is supposed that no dissipative processes are present in this volume, it is supposed also that we have some engine which could produce work from fluctuation in this volume, but without working gas' mass loss. Moreover, this engine has no connection with surrounding environment except by mechanical work's transfer. Initial distribution of parameters in this system will be named "state 1". Final distribution of parameters will be named "state 2". When this final distribution will be achieved, then this engine will have extracted all available work. In "state 2" distribution of parameters should be uniform.

To find maximum work that can be produced from such system we will use statement from Landau and Lifshits [25]: «System does maximum work when its entropy remains constant, i.e. when the process of reaching equilibrium is reversible». The same method to derive availability of work is used in [26] for optics field. This maximal work will be named internal exergy in this text.

According to this statement maximum work Ex will be:

$$Ex = E_1 - E_2, \quad (7)$$

where E_1 is this system's energy in the "state 1", E_2 is this system's energy in the "state 2". The systems in "state 1" and in "state 2" should have the same mass (M), same entropy (S) and the same volume (V) but system in "state 2" should have uniform parameters. According to this we have:

$$\begin{aligned} M &= \rho_2 V = \int_V \rho_1 dV = \\ &= \int_V \rho_0 \left(\frac{p_0 + \varepsilon p'(x)}{p_0} \right)^{1/k} \exp\left(-\frac{\varepsilon s'(x)}{C_p}\right) dV \end{aligned} \quad (8)$$

$$\begin{aligned} S &= \rho_2 s_2 V = \int_V \rho_1 s_1 dV = \\ &= \int_V \rho_0 \left(\frac{p_0 + \varepsilon p'(x)}{p_0} \right)^{1/k} \exp\left(-\frac{\varepsilon s'(x)}{C_p}\right) \varepsilon s'(x) dV. \end{aligned} \quad (9)$$

Full derivation of energies in "state 1" and "state 2" is presented in appendix A. The final result is:

$$\begin{aligned} \frac{Ex}{V\varepsilon^2} = \frac{E_1 - E_2}{V\varepsilon^2} &= \frac{\rho_0 \langle |\bar{v}|^2 \rangle}{2} + \frac{\langle (p')^2 \rangle}{2kp_0} + \frac{p_0 \langle (s')^2 \rangle}{2kRC_p} + \\ &- \frac{\varepsilon(2k-1) \langle (p')^3 \rangle}{6k^2 (p_0)^2} - \frac{\varepsilon p_0 \langle (s')^3 \rangle}{3R(C_p)^2} + \\ &+ \frac{\varepsilon \rho_0 \langle p' |\bar{v}|^2 \rangle}{2kp_0} - \frac{\varepsilon \rho_0 \langle s' |\bar{v}|^2 \rangle}{2C_p} + O(\varepsilon^2). \end{aligned} \quad (10)$$

If we take only quadratic part of this expression, it is the same as in works [17,18,21]. From Strumpe and Furletov work [17] it is:

$$E_{III} = \frac{\rho(v')^2}{2} + \frac{(p')^2}{2\rho c^2} + \frac{\rho T(s')^2}{2C_p}. \quad (11)$$

From Thierry Poinot and Denis Veynante work [8] it is:

$$e_{tot} = \rho_0 \frac{(\bar{v}')^2}{2} + \frac{1}{\rho_0 c^2} \frac{(p')^2}{2} + \frac{p_0 (s')^2}{2RC_p}. \quad (12)$$

The third power term part of expression (14) can be found only in Jacob's work [27]:

$$\begin{aligned} E_3 &= \frac{1}{2} \rho_0 (\bar{v}')^2 + \frac{1}{\rho_0 c^2} \frac{(p')^2}{2} + \\ &+ \frac{\rho_0 T_0 (s')^2}{2(C_p)^2} + \\ &+ \frac{(1-2k)(p')^3}{6(\rho_0)^2 c^4} + \frac{p'(s')^2}{2kRC_p} - \\ &- \frac{\rho_0 T_0 (s')^3}{3(C_p)^3}. \end{aligned} \quad (13)$$

but it has not all the terms in the Eq. (10) that can be found in this work, mainly because the authors of that works think they are negligible.

It is valuable to note that this equation for internal exergy do not contain mean temperature of volume. This peculiarity can be explained as follows: no heat is added or removed from the system and the entropy is kept the same during this thermodynamic process, so the thermodynamic state is fully described by pressure and entropy.

We can also study such situation as when fluctuations of parameters diminished due to dissipation process while mass in volume and internal energy are kept constant. This will lead to increasing of entropy in volume. This state will be named as "state 3". Using asymptotic expansion by small parameter ε we will get the following result:

$$\frac{P_0}{R} \Delta s_{2 \rightarrow 3} = Ex. \quad (14)$$

Mean thermodynamic temperature may be determined according to Gouy-Stodola theorem [28]:

$$T_0 = \frac{Ex}{\langle \rho \rangle \Delta s_{2 \rightarrow 3}} = \frac{P_0}{R \langle \rho \rangle}, \quad (15)$$

or mean temperature in this case is temperature calculated from mean pressure and density.

This method to find internal exergy of closed volume may be extended to reacting mixture according to internal exergy definition. In this case we can study closed volume having reacting mixture, mass fraction of its components (Y_i) which initially has uneven distribution around the volume. Engine can produce work using such unevenness till concentrations of components became uniform and all chemical reactions will be fully completed to equilibrium state. The work will be maximal if entropy is kept constant by such engine and in "state 2" it will be achieved minimal internal energy in this volume compared to all states, if the constraints are fulfilled.

The defined internal exergy has such valuable properties:

- it is non negative;
- internal exergy of full system is not less than internal exergy of their parts;
- internal exergy in closed system may only decrease;
- internal exergy do not depends on ambient parameters;
- mean temperature could be found from internal exergy and internal entropy production to reach equilibrium state.

3. The simple model's extension to moving system

The case studied in previous section may be extended to the case of closed thermodynamic system interacting with surroundings by work and heat. This interaction with surroundings will lead to change mean entropy and pressure. In this article we study such system as having internal structure, which can be analyzed by internal exergy. To define such system's internal exergy, we will derive evolutionary differential equation, using assumption that irreversible entropy generation is fully defined by dissipative function and by heat exergy definition based on non-isothermal heat transfer. The initial conditions for this equation will be defined according to previous section results.

In this section we will derive equation for second order internal exergy. In this definition it is assumed that volume of averaging is fixed. Defining equation for higher order exergy can be derived using the same method.

We will use the identity:

$$\frac{d}{dt} \left\langle \frac{(\varphi - \langle \varphi \rangle)^2}{2} \right\rangle = \left\langle (\varphi - \langle \varphi \rangle) \frac{d\varphi}{dt} \right\rangle, \quad (16)$$

with the following second order expression of internal exergy:

$$Ex_2 = \frac{\langle \rho \rangle \langle |\bar{v}|^2 \rangle}{2} + \frac{\langle (p')^2 \rangle}{2k \langle p \rangle} + \frac{\langle p \rangle \langle (s')^2 \rangle}{2kRC_p}, \quad (17)$$

and equations of gas dynamics for ideal gas in form:

$$Ex_2 = \frac{\langle \rho \rangle \langle |\bar{v}|^2 \rangle}{2} + \frac{\langle (p')^2 \rangle}{2k \langle p \rangle} + \frac{\langle p \rangle \langle (s')^2 \rangle}{2kRC_p}, \quad (17)$$

$$\rho \frac{du}{dt} + \frac{\partial p}{\partial x} = \nabla(\sigma_x), \quad (18a)$$

$$\rho \frac{dv}{dt} + \frac{\partial p}{\partial y} = \nabla(\sigma_y), \quad (18b)$$

$$\rho \frac{dw}{dt} + \frac{\partial p}{\partial z} = \nabla(\sigma_z), \quad (18c)$$

$$\frac{dp}{dt} + kp(\nabla \cdot \vec{v}) = (k-1)(Q + \Phi), \quad (18d)$$

$$\frac{p}{R} \frac{ds}{dt} = (Q + \Phi). \quad (18e)$$

After representing all quadratic parts in Eq. (17) by Eq. (16) and inserting Eqs. (18a)-(18e) into Eq. (17) we will get such result:

$$\begin{aligned} \frac{dEx_2}{dt} V = & - \int_{\Gamma} p' \vec{v}' \cdot d\Gamma + \\ & + \int_V \left(\left(1 - \frac{T_0}{T_0 + T'} \right) - 1 \right) (Q' + \Phi') - \Phi' \Big) dV + \\ & + \left(Ex_2 + \frac{(k-1) \langle (p')^2 \rangle}{2k \langle p \rangle} \right) \int_{\Gamma} \vec{v}' \cdot d\Gamma + \\ & + \left(\frac{\langle p \rangle \langle (s')^2 \rangle}{2kRC_p} - \frac{\langle (p')^2 \rangle}{2k \langle p \rangle} \right) \frac{(k-1)}{\langle p \rangle} \langle Q \rangle V. \end{aligned} \quad (19)$$

Where the first term is the power at which work is transferred to the studied system, the second term is exergy of heat transferred to system and loss of exergy by dissipation, the third term is additional exergy transfer which is usually neglected due to small velocity in studied system (and this is the effect of flowing), the last term is Brayton cycle correction.

4. Variational principle for study combustion instability.

The analysis in section 1 clearly shows that for computational fluid dynamic methods based on finite volume methods, the exergy loss will occur on each time step because of averaging values of pressure, density, and temperature and species concentration in control volumes. Calculation by first order methods fully follows procedure described in section 1. In second order methods the loss of exergy will be less but even this method could not prevent exergy loss because values reconstruction in control surfaces between volumes achieved on the base of averaged values. In special cases of isentropic flow and in the case of turbulence flow we have examples in which there are introduced equations for parameters that could not be resolved during calculations with energy conservation. Eq. (19) derived in section 3 could be used the same way.

The nonlinear isentropic flow has Hamiltonian structure and may be formulated as minimal action principle in which Hamiltonian has direct relation to exergy [29]. The variational principle for exergy was formulated for incompressible flow in work [30]. Variational principle for exergy may be derived from the principle of least dissipation of energy described in [31]: the difference between the production of irreversible entropy in the system under study, represented by thermodynamic coordinates and forces and dissipative function in real process, is extremum.

$$\delta \int_V \left(\rho T \frac{ds^i}{dt} - \Phi \right) dV = 0. \quad (20)$$

Production of irreversible entropy could be found from exergy such way:

$$\rho T \frac{ds^i}{dt} = A + Q \left(1 - \frac{T_0}{T} \right) + \Phi \left(1 - \frac{T_0}{T} \right) - \frac{dEx}{dt}, \quad (21)$$

and variational principle will have such form

$$\delta \int_V \left(A + Q \left(1 - \frac{T_0}{T} \right) + \Phi \left(1 - \frac{T_0}{T} \right) - \frac{dEx}{dt} - \Phi \right) dV = 0. \quad (22)$$

This principle is formulated for moving closed volume in ambient field, where the A (power at which work transferred to studied system) should be formulated as having two parts representing the influence of ambient pressure and velocity on studied system.

$$A = -\bar{v} \cdot \nabla p^e - p(\nabla \cdot \bar{v}^e), \quad (23)$$

where p^e and \bar{v}^e represent external influences by pressure and velocity on studied volume.

After variational procedure, these external influences should be set equal to pressure and velocity in the system, and derived equations can be seen as a model of self-consistent field theory.

If we use second order exergy, then this variational procedure allows recover equations of compressible flow which is used in work [32] with additions representing addition heat by viscous dissipation:

$$\begin{aligned} & \delta \int_V \left(A + (Q + \Phi) \left(1 - \frac{T_0}{T(p,s)} \right) - \Phi - \frac{dEx}{dt} \right) dV = \\ & = \int_V \delta u \left(-\frac{\partial p^e}{\partial x} + \nabla(\sigma_x) - \rho \frac{du}{dt} \right) dV + \\ & + \int_V \delta v \left(-\frac{\partial p^e}{\partial y} + \nabla(\sigma_y) - \rho \frac{dv}{dt} \right) dV + \\ & + \int_V \delta w \left(-\frac{\partial p^e}{\partial z} + \nabla(\sigma_z) - \rho \frac{dw}{dt} \right) dV + \\ & \int_V \delta p \left(-\nabla \cdot \bar{v}^e + (Q + \Phi) \frac{k-1}{kp_0} - \frac{1}{kp_0} \frac{dp}{dt} \right) dV + \\ & + \int_V \delta s \left((Q + \Phi) \frac{p_0}{pC_p} - \frac{p_0}{RC_p} \frac{ds}{dt} \right) dV. \end{aligned} \quad (24)$$

Eq. (24) may be seen as Galerkin formulation for flow calculation, which, if $\delta p = p$, $\delta s = s$, $\delta u = u$, $\delta w = w$, it represents internal exergy of flow. This equation could be used in numerical computation for study dynamical flame instability to find upper limit to self-sustained oscillations.

5. Conclusions

The novelty of the presented results resides in physical based interpretation of Lyapunov function usually used in analyzing stability in reacting flows.

To find difference between oscillating flow and steady flow it is usually assumed that steady flow exists and that mean values calculated in oscillating flow is equal to steady one. But this is true only up to and inclusive to first order of pressure oscillations magnitude. In this work the ground state is the uniform state from which mechanical work could not be extracted. The amount of mechanical work which could be extracted may be attributed to pressure, velocity and entropy waves and also to turbulent pulsations of flow. Also this amount of work could be attributed to work which could be applied to destroy the structure and this is the upper limit of this mechanical work.

Results of this work shows that the results of algebraic manipulations described in works [17,18,21] has clear physical meaning: The value for which was derived transport equation is maximal mechanical work which could be extracted by heat engine from studied system if this system would be disconnected from inlet and outlet and from any other surroundings and the engine could transfer to surroundings only mechanical work. This value in this article is named internal exergy.

The integral of transport equation for internal exergy by volume may be seen as variational functional for compressible Navie-Stockes equations which allows to "physically correct" model wave dynamics in numerical calculations because this prevents too high numerical dissipations and at the same time avoid incorrect internal entropy negative production.

The effectiveness increase in determining the characteristics of acoustic oscillations in numerical models, with the help of the aforementioned “physical correction” could allow more efficiently evaluating both peak and integral heat dissipation [33]. Also, the use of the present model allows more efficiently solving during design phase one of the main problems in the development and operation of both aviation and land-based gas turbine installations – with the additional effect of minimizing the destabilization of combustion process through acoustic oscillations.

Nomenclature

Table 1. Table of nomenclature part 1.

N_A	<i>acoustic power</i>	$\frac{W}{m^3}$
k	<i>adiabatic index</i>	-
p, P	<i>absolute pressure</i>	Pa
p'	<i>pulsating pressure</i>	Pa
Q	<i>heat source</i>	$\frac{W}{m^3}$
Q'	<i>pulsating heat</i>	$\frac{W}{m^3}$
f	<i>frequency</i>	$\frac{1}{s}$
p_0	<i>mean by volume pressure</i>	Pa
\bar{v}	<i>velocity of flow</i>	$\frac{m}{s}$
\bar{v}'	<i>pulsating velocity</i>	$\frac{m}{s}$
C_p	<i>heat capacity</i>	$\frac{J}{kg \cdot K}$
s	<i>entropy</i>	$\frac{J}{kg \cdot K}$
s'	<i>pulsating entropy</i>	$\frac{J}{kg \cdot K}$

Table 2. Table of nomenclature part 2.

$\langle \dots \rangle$	<i>averaging by volume</i>	
E	<i>energy</i>	J
M	<i>weight</i>	kg
Ex	<i>exergy</i>	J
ρ	<i>density</i>	$\frac{kg}{m^3}$
S	<i>entropy of system</i>	$\frac{J}{K}$
ϵ	<i>small parameter</i>	-

V	<i>volume of system under consideration</i>	m^3
A	<i>applied to system work</i>	W
u, v, w	<i>components of velocity vector</i>	$\frac{m}{s}$
\bar{v}^e	<i>external velocity</i>	$\frac{m}{s}$
p^e	<i>external pressure</i>	Pa
T	<i>field of temperature</i>	K
T_0	<i>reference temperature</i>	K
$\sigma_x, \sigma_y, \sigma_z$	<i>components of viscous stress tensor</i>	$\frac{kg}{m \cdot s^2}$

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Appendix A

In this appendix calculations of equilibrium “state 2” is described.

To found ρ_2 and s_2 values, asymptotic expansion method with respect to small parameter ε could be used:

$$\begin{aligned}
 \rho_2 = \rho_0 + & \frac{\rho_0 \varepsilon^2 \langle (s')^2 \rangle}{2(C_p)^2} - \frac{\rho_0 \varepsilon^2 (k-1) \langle (p')^2 \rangle}{2k^2 (p_0)^2} - \\
 & - \frac{2\rho_0 \varepsilon^2 \langle s' p' \rangle}{k C_p p_0} - \frac{\rho_0 \varepsilon^3 \langle (s')^3 \rangle}{6(C_p)^3} + \\
 & + \frac{\rho_0 \varepsilon^3 (k-1)(2k-1) \langle (p')^3 \rangle}{6k^3 (p_0)^3} + \\
 & + \frac{\rho_0 \varepsilon^3 \langle (s')^2 p' \rangle}{2k(C_p)^2 p_0} + \frac{\rho_0 \varepsilon^3 \langle s' (p')^2 \rangle}{2k^2 C_p (p_0)^2} + \\
 & + O(\varepsilon^4),
 \end{aligned} \tag{25}$$

$$\begin{aligned} \rho_2 s_2 &= \rho_0 \varepsilon^2 \left(\frac{\langle s' p' \rangle}{k p_0} - \frac{\langle (s')^2 \rangle}{C_p} \right) + \\ &+ \frac{\rho_0 \varepsilon^3}{2} \left(\frac{\langle (s')^3 \rangle}{(C_p)^2} - \frac{(k-1) \langle (p')^2 s' \rangle}{k^2 (p_0)^2} - \frac{2 \langle (s')^2 p' \rangle}{k C_p p_0} \right) + \\ &O(\varepsilon^4). \end{aligned} \quad (26)$$

First order terms are eliminated according to definitions of mean values Eq.(4).

We find p_2 using relation: $p_2 = p_0 \left(\frac{\rho_2}{\rho_0} \right)^k \exp \left(\frac{(\rho_2 s_2) k}{\rho_2 C_p} \right)$.

$$p_2 = p_0 \left(\frac{\rho_2}{\rho_0} \right)^k \exp \left(\frac{(\rho_2 s_2) k}{\rho_2 C_p} \right). \quad (27)$$

After substitution ρ_2 and $(\rho_2 s_2)$ into this relation one can get:

$$\begin{aligned} p_2 &= p_0 - \frac{p_0 \varepsilon^2 k \langle (s')^2 \rangle}{2 (C_p)^2} - \frac{p_0 \varepsilon^2 (k-1) \langle (p')^2 \rangle}{2k (p_0)^2} + \\ &+ \frac{p_0 \varepsilon^3 (k-1) (2k-1) \langle (p')^3 \rangle}{6k^2 (p_0)^3} + \frac{p_0 \varepsilon^3 k \langle (s')^3 \rangle}{3 (C_p)^3} - \\ &- \frac{p_0 \varepsilon^3 \langle (s')^2 p' \rangle}{2 (C_p)^2 p_0} + O(\varepsilon^4). \end{aligned} \quad (28)$$

Having pressure and entropy we can find energy of system in "state 2":

$$\frac{E_2}{V} = \frac{p_2}{(k-1)}. \quad (29)$$

And the energy in "state 1" can be also represented as asymptotic expansion:

$$\begin{aligned} \frac{E_1}{V} &= \frac{p_0}{(k-1)} + \frac{\rho_0 \varepsilon^2 \langle (\bar{v}')^2 \rangle}{2} + \\ &+ \frac{\rho_0 \varepsilon^3}{2} \left(\frac{\langle (\bar{v}')^2 p' \rangle}{k p_0} - \frac{\langle (\bar{v}')^2 s' \rangle}{C_p} \right) + O(\varepsilon^4). \end{aligned} \quad (30)$$

According to these relations the exergy of ideal gas volume having fluctuation in entropy, density and velocity may be calculated by Eq. (7). Final result presented in Eq. (10).

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