Technical Note

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Technical Note

On the Equality of 'Stopping Times' of Hailstone Numbers

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Abstract: This short technical note proves some observed novel corollaries of the Collatz Conjecture. For this, a unique 'Collatz Representation' of Hailstone numbers has been employed. After that, an extension of this corollary is explained using the fact that only certain differences in the last digit of the 'Collatz Representation' gives a whole number difference between numbers. A more general result is obtained. The final proof is for a corollary that some numbers the same number of steps to reach 1 if the Collatz Conjecture is true and all variables are whole numbers.

Keywords: Collatz Conjecture; Proof; Induction

Theorem 1. *Any natural number can be represented as follows, for* $x, y, w, r_i \in N$ *and* $x > y > r_i > \cdots$

$n = \frac{2^x - 2^y - \sum_{i=1}^{w} 2^{r_i} 3^i}{3^{w+1}}$ iff the Collatz Conjecture is true.

Proof. The proof is trivial and can be left to the reader.

Definition 1. x, y, w, r_i *will be defined as the digits of a number in its collatz representation. We will define D(n) as a function returning the collatz representation of n. w is equal to the number of digits - 2. EXAMPLE.* $44 = \frac{2^{12} - 2^8 - 2^5 * 3 - 2^3 * 9 - 2^2 * 27}{3^4}$. *Therefore, D(44) = 12, 8, 5, 3, 2*

NOTE: Collatz representation is one in which a number can be represented with the smallest number of digits possible. A test for this: The difference between first two digits should not be 2. If it is, remove the first digits until difference becomes different. The proof for this can be considered unrelated and trivial.

Theorem 2. *Adding one to all digits in collatz representation causes the number to be multiplied by 2.*

Proof. Again, the proof is trivial because adding one to all digits in collatz representation is equivalent to multiplying the numerator in $\frac{2^x - 2^y - \sum_{i=1}^{w} 2^{r_i} 3^i}{3^{w+1}}$ by 2.

Corollary 2.1. *A number with n in its last digit in collatz representation is divisible by* 2^n *but not* 2^{n+1}.

Proof. It is easy to see this given that the number of multiplications by 2 is equal to the last digit.

Theorem 3. *Removing a 0 as the last digit in collatz representation causes the number to be multiplied by 3 and added to 1. The least number of digits has to be 2.*

Proof. $n = \frac{2^x - 2^y - \sum_{i=1}^{w} 2^{r_i} 3^i}{3^{w+1}}$

r_w is the last digit and is 0.
Case 1: If there are more than 2 digits,

$$n_{new} = \frac{2^x - 2^y - \sum_{i=1}^{w_{new}} 2^{r_i} 3^i}{3^{w_{new}+1}}$$

$$w_{new} = w - 1$$

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Therefore, $n_{new} = \frac{2^x - 2^y - \sum_{i=1}^{w-1} 2^{r_i} 3^i}{3^w}$

$$3n + 1 = \frac{2^x - 2^y - \sum_{i=1}^{w} 2^{r_i} 3^i}{3^w} + 1$$

$$= \frac{2^x - 2^y - \sum_{i=1}^{w-1} 2^{r_i} 3^i - 3^w + 3^w}{3^w}$$

$$= \frac{2^x - 2^y - \sum_{i=1}^{w-1} 2^{r_i} 3^i}{3^w} = n_{new}$$

Case 2: If there are 2 digits,

$$n = \frac{2^x - 2^y}{3}$$

is the last digit and is 0.

$$n = \frac{2^x - 1}{3}$$

$$n_{new} = 2^x = 3n + 1$$

Case 3: For 1 digit, removing it means that the number is removed so it does not apply.
Therefore, proved.

Theorem 4. *Number of steps to reach 1 (exclusive) is equal to the sum of number of digits and first digit - 1.*

Proof. We know from THEOREM 2. Division by 2 is removing 1 from all digits. We also know that digits are in descending order. Therefore, there are as many divisions by 2 as first digit. We also know that removal of 0 as the last digit gives 3x+1. Therefore, there will be as many such removals as the number of digits. Therefore, the sum of these – 1 give requisite number of steps.

Corollary 4.1. *Any digit changing except the first one does not change the number of steps.*

Proof. Trivial to prove from THEOREM 4.

Theorem 5. *n is a natural number for* $2^b - 2^a = 3n$ *only if* $b = a + 2c$.

Proof. This could be a two-part proof. Proving that this always creates a number divisible, and another combination creates a number that is not.
For divisibility:
To prove, $2^b - 2^a = 2^{a+2c} - 2^a = (4^c - 1)2^a$
Now we have to prove: $(4^c - 1)$ is divisible.
Base case for c=1: 4-1=3 (true)
Assume true for c=k.
Therefore, let $\left(4^k - 1\right) = 3i$
For c=k+1,

$$\left(4^{k+1} - 1\right) = 4 * \left(4^k\right) - 1 = 4\left(4^k - 1\right) + 3 = 3i + 3$$

$$= 3(i + 1)$$

Therefore, true.
Hence, proven by mathematical induction that $2^{a+2c} - 2^a$ can be divided by 3.
For non-divisibility:
To prove, $2^b - 2^a = 2^{a+2c+1} - 2^a = (2^{2c+1} - 1)2^a$
Now we have to prove: $(2^{2c+1} - 1)$ is not divisible.
Base case for c=1: 8-1=7 (true)
Assume true for c=k.
Therefore, let $\left(2^{2k+1} - 1\right) = 3i + 1$

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For c=k+1

$$\left(2^{2k+3}-1\right)=4*\left(4^{k}\right)-1=4\left(2^{2k+1}-1\right)+3$$
$$=12i+7=3(4i+2)+1$$

Therefore, true.
Hence, proven by mathematical induction that $2^{a+2c+1}-2^{a}$ cannot be divided by 3.
Therefore, we proved only a difference of a multiple of two in the powers of 2 being subtracted from each other can make a product of 3 and another number.

Theorem 6. *For numbers only differing in their last digits in their collatz representation, only a difference of multiple of two in their last digits can create a whole number difference.*

Proof. If n_a-n_b is difference between numbers only differing in their last digits.

$$n_a-n_b=$$

$$\frac{2^{x}-2^{y}-\sum_{i=1}^{w-1}2^{r_i}3^{i}-2^{x}+2^{y}+\sum_{i=1}^{w-1}2^{r_i}3^{i}+3^{w}\left(-2^{a}+2^{b}\right)}{3^{w+1}}$$

$$=\frac{-2^{a}+2^{b}}{3}$$

If n_a-n_b is a whole number $-2^{a}+2^{b}$ is divided by 3.
From THEOREM 5., b=a+2c. Hence, proved.

Theorem 7. $(2i+1)*2^{a+2c}$ *and* $(2i+1)*2^{a+2c}+\frac{(4^{c}-1)*2^{a}}{3}$ *have the same number of steps to reach 1 if the Collatz Conjecture is true and all variables are Whole Numbers.*

Proof. $(2i+1)*2^{a+2c}$ ends with b=a+2c from COROLLARY 2.1. If n_a ends with a and has same digits, $n_a-n_b=\frac{-2^{a}+2^{b}}{3}=\frac{(4^{c}-1)*2^{a}}{3}$.
Verification for Example. Let n = $(2x+1)*2^{0+2}$ = 8x+4

$$n=\frac{2^{x}-2^{y}-\sum_{i=1}^{w}2^{r_i}3^{i}}{3^{w+1}} \tag{1}$$

We know the last digit is 2, implying:

$$n=\frac{2^{x}-2^{y}-\sum_{i=1}^{w-1}2^{r_i}3^{i}-4*3^{w}}{3^{w+1}}$$

Now, $n+1=\frac{2^{x}-2^{y}-\sum_{i=1}^{w-1}2^{r_i}3^{i}-4*3^{w}}{3^{w+1}}+1$

$$=\frac{2^{x}-2^{y}-\sum_{i=1}^{w-1}2^{r_i}3^{i}-4*3^{w}+3*3^{w}}{3^{w+1}}$$

$$=\frac{2^{x}-2^{y}-\sum_{i=1}^{w-1}2^{r_i}3^{i}-1*3^{w}}{3^{w+1}}$$

$$=\frac{2^{x}-2^{y}-\sum_{i=1}^{w-1}2^{r_i}3^{i}-2^{0}*3^{w}}{3^{w+1}}$$

It can clearly be seen that for n+1 and n, the collatz representation is the same but the last digit is 2 for n and 0 for n+1. Therefore, the number of digits and the first digit are equal, making the number of steps equal.
Therefore, proved 8x+4 and 8x+5 have same number of steps. Therefore, $(2x+1)*2^{0+2}$ and $(2x+1)*2^{0+2}+\frac{(4-1)*2^{0}}{3}$ have same number of steps.

Theorem 8. *For only a difference in the n+1th from last digit whole number differences can be produced only from differences of multiple of 2*3n.*

Proof. To prove: $2^{2*3^n k}-1$ is divisible by 3^{n+1} and $2^{2*3^n k+a}-1$ is not for a not equal to 0.
For divisibility:
Base case: THEOREM 5.

$$2^{2*3^n k}=3^{n+1}N+1$$

Assume true for n = n: $2^{2*3^n k}=3^{n+1}N+1$
For n+1:

$$2^{3*2*3^n k}-1$$

$$=(3^{n+1}N+1)^3-1$$

$$=3^{3n+3}N^3+3^{2n+3}N^2+3^{n+2}N$$

$$=3^{n+2}(3^{2n+1}N^3+3^{n+1}N^2+N)$$

Therefore, proved $2^{2*3^n k}-1$ is divisible by 3^{n+1}.
Adding a as $2^{2*3^n k+a}-1$, for n+1,

$$2^a*(3^{3n+3}N^3+3^{2n+3}N^2+3^{n+2}N+1)-1$$

For divisibility, a=0. Therefore, $2^{2*3^n k+a}-1$ is not divided for a not equal to 0
We can prove 8 in a way similar to THEOREM 7 and 8:
Final Corollary to the 3x+1 Problem:
For $r_i>r_{i-1}$, n - number of digits < -1, digit (n+1) from last = a, and all variables being whole numbers:

$$\left[\frac{(2\mathrm{N}+1)*2^{\mathrm{a}}-\sum_{i=1}^{n}3^i\,2^{r_i}}{3^n}\right]$$

And

$$\left[\frac{(2\mathrm{N}+1)*2^{\mathrm{a}}-\sum_{i=1}^{n}3^i\,2^{r_i}}{3^n}\right]+\frac{(2^{2*3^n k}-1)2^a}{3^{n+1}}$$

Have equal stopping times.

Conclusion

In conclusion, we found some very interesting results. More importantly, we learnt things that would be extremely difficult to prove and observe if using the normal form of number notation. However, with the Collatz representation, we were able to prove some very interesting and novel observations, that, if not helpful in proving the Collatz Conjecture right, could reduce search time for counterexamples.

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