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Lax Pairs for the Modified KdV Equation

Georgy I. Burde

Swiss Institute for Dryland Environmental and Energy Research, Jacob Blaustein Institutes for Desert Research, Ben-Gurion University, Sede-Boker Campus, 84990 Israel; georg@bgu.ac.il

Abstract: Multi-parameter families of Lax pairs for the modified Korteweg-de Vries (mKdV) equation are defined by applying a direct method developed in the present study. The gauge transformations converting the defined Lax pairs to some simpler forms are found. The direct method and its possible applications to other types of evolution equations are discussed.

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1. Introduction

Integrable systems of partial differential equations (PDEs) have been actively studied in the context of soliton theory in recent years [1–5]. In the literature, there are several alternative definitions of integrability of nonlinear PDEs: existence of the Lax pair, existence of infinitely many generalized symmetries (or existence of recursion operators), soliton solutions, Bi-hamiltonian structure (infinitely many conserved quantities), Bäcklund transformation, Painleve property, etc. (see, e.g., [5,6] for reviews). Different concepts of integrability are interwoven quite closely. In particular, the concept of Lax pair, a possibility for an equation to be realized as the compatibility condition of two linear eigenvalue equations, the Lax pair [7], appears in several approaches. The notion of Lax pair has played a key role in the development of soliton theory, and in many cases the identification of a corresponding Lax pair has been the first step to recognize the integrable character of important nonlinear PDEs.

However, this feature, the Lax pair existence, is not easy to determine a priori from the equation itself. Obtaining the Lax pair is a highly nontrivial operation and no systematic, general approach exists for this. The methods developed for construction of a Lax pair (some of them are reviewed in [8]) involve nontrivial mathematical problems and so do not allow to algorithmically determine whether a given equation admits the Lax pair, or for which parameter values a class of equations admits the Lax pair. Direct methods, which are based straight on the Lax pair definition, may provide more perspective. A direct method for identifying equations possessing Lax pairs has been developed and applied to some types of equations in [8]. In particular, the method has been applied to the modified Korteweg-de Vries equation (mKdV) which is one of the prototypical examples of integrable equations. The equation has the form

$$u_t + r_1 u^2 u_x + + u_{3x} = 0 \tag{1}$$

where r_1 is an arbitrary parameter that can be changed by scaling and subscripts of the form "nx" denote derivatives of the order n with respect to x. Applying the method of [8] to equation (1) yields two different branches of the Lax pairs one of which is a single Lax pair and the second represents a one-parameter (ϵ) family of Lax pairs. A particular case of the second branch corresponding to $\epsilon = 0$ coincides with the known Lax pair for the mKdV equation while other Lax pairs, the first branch and those of the second branch corresponding to $\epsilon \neq 0$, seemed to be new. Those results imply that a variety of the Lax pairs for the mKdV equation exists.

This paper reports the following results. First, by applying a direct method, that discards the requirement of scaling invariance imbedded into the framework of the method of [8] (see more details in Section 2), it is shown that the variety of the Lax pairs for the mKdV equation can be even more extended. Instead of a single Lax pair of the first branch of [8], the method of the present study yields a three-parameter family of Lax pairs while the Lax pair of [8] becomes a degenerated case of that family.

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The reason for the extension is that the method, which is free of the restriction of scaling invariance, allows terms of different scaling weights to appear in the Lax pair.

Second, it is found by applying proper gauge transformations that only one Lax pair (coinciding with the known one) from a variety of the defined Lax pairs can be useful for finding solutions of equation (1). The three-parameter family of the Lax pairs of the first branch (including the first branch of [8] as a particular case) can be converted by a gauge transformations into the form which, upon using equation (1) and its differential consequences, does not include u. Such a Lax pair is useless for finding solutions by the IST and/or constructing conservation laws and so should be classified as "fake" or "weak". [1] Further, it is found that, for the Lax pairs of the one-parameter (ϵ) family of the second branch (and so for the second branch of [8] coinciding with it), there exists a gauge transformation eliminating the parameter ϵ which reduces it to the known Lax pair for the mKdV equation.

This paper is organized, as follows. In Section 2 following the Introduction, the direct method is outlined and discussed. The Lax pairs for the mKdV equation obtained by applying the method are listed in Section 3. The issue of gauge equivalence and the gauge transformations converting the defined Lax pairs to simpler forms are discussed in Section 4. Some remarks on the results and possible applications of the method are furnished in Section 5.

2. Direct method

The Lax pair in an operator form

$$\mathbf{L}\psi + \lambda\psi = 0; \quad \frac{\partial\psi}{\partial t} + \mathbf{A}\psi = 0 \tag{2}$$

admitted by a given PDE is determined from the condition of compatibility of (2) with the PDE. The condition can be written, in the form convenient for calculations, as

$$\left(\mathbf{L}f(x)\right)_t = \mathbf{L}\left(\mathbf{A}f(x)\right) - \mathbf{A}\left(\mathbf{L}f(x)\right) \tag{3}$$

where $f(x)$ is an auxiliary function. It should only hold on solutions of the original PDE.

In the procedure of the method, the operators $\mathbf{L}$ and $\mathbf{A}$ are sought as linear differential operators expressed in powers of $\mathbf{D}_x$, as follows

$$\mathbf{L} = \mathbf{D}_x^m + U^{(1)}(u, u_x, u_{2x}, \ldots)\mathbf{D}_x^{m-1} + \cdots + U^{(m)}(u, u_x, u_{2x}, \ldots)\mathbf{I}, \tag{4}$$

$$\mathbf{A} = Q^{(0)}\mathbf{D}_x^n + Q^{(1)}(u, u_x, u_{2x}, \ldots)\mathbf{D}_x^{n-1} + \cdots + Q^{(n)}(u, u_x, u_{2x}, \ldots)\mathbf{I} \tag{5}$$

where $\mathbf{D}_x$ is the total derivative operator for the space variable x, $\mathbf{I}$ is the identity operator, n is the order of the original PDE, $Q^{(0)}$ is a constant and it is assumed that the functions $U^{(i)}(u, u_x, u_{2x}, \ldots)$ and $Q^{(i)}(u, u_x, u_{2x}, \ldots)$ depend only on finitely many derivatives. The initial assumptions include choosing the order m of the differential operator (4) and assigning the functional forms of $U^{(i)}(u, u_x, u_{2x}, \ldots)$ (making "ansatz") as differential polynomials in u and its derivatives with the coefficients to be determined.

After that the "ansatz" is specified, the procedure of the method becomes completely algorithmic. The functions $Q^{(i)}$ and the unknown coefficients of the differential polynomials $U^{(i)}$ are determined

[1] At the early stages of the development of soliton theory, the opinion was widespread that the discovery of a Lax pair associated with a nonlinear evolution PDE implied that that PDE was integrable. However, after the observation of the fake (weak) Lax pairs phenomenon [9], the fact that a Lax pair can be associated to a nonlinear PDE cannot be considered as a proof of its integrability. There is no formal definition of what Lax pairs should be classified as weak but it is impled that weak Lax pairs are useless for applications. Essentially, a Lax pair is useless if it is gauge equivalent to a matrix without a spectral parameter since the presence of the spectral parameter is crucial for finding solutions by the inverse scattering transform (IST). The present (more rare) case is when a spectral parameter is non-removable but there exist gauge transformations allowing to remove all dependent variables of the associated nonlinear system from the Lax pair.

from the relation obtained by introducing (4) and (5) into the Lax equation (3) expressing the condition of compatibility of the system (2) with the original PDE. The resulting expression is a linear differential polynomial in $f(x)$ and its derivatives and, in view of arbitrariness of $f(x)$, its coefficients dependent on $U^{(i)}$, $Q^{(i)}$ and their derivatives should vanish. This provides $n+1$ relations that are considered as differential equations for the functions $Q^{(i)}$. Some of the equations can be solved even without specifying the forms of the functions $U^{(i)}$. The remaining (usually two) equations contain time derivatives of $U^{(i)}$ so that one needs to assign the forms of the differential polynomials for $U^{(i)}$ and use the original PDE and its differential consequences for eliminating terms with time derivatives of u. As the result, one has two equations for one function $Q^{(n)}$ which are compatible if the coefficients of a differential polynomial in u and its derivatives obtained by eliminating $Q^{(n)}$ vanish. After solving the algebraic equations expressing this condition a single differential equation for the last unknown function $Q^{(n)}$ remains. Solving this equation completes the derivation but for having the result in quadratures some additional constraints on the coefficients are to be imposed.

A direct method for the Lax pairs calculations has been developed in [8]. The method of [8] differs from the present one in two aspects. First, the scaling symmetry condition is imposed which implies that all the terms in the operators **L** and **A** have uniform weight. (In the case of equation (1), if the weight of the x–derivative is assumed to be equal to one, $W(\partial_x) = 1$, then $W(u) = 1$, $W(u_x) = 2$ and so on, and, correspondingly, $W(u_t) = 4$.) The second difference, partially related to the scaling symmetry requirement, is that not only the forms of the functions $U^{(i)}$ but also the forms of $Q^{(i)}$ are assigned. Using the scaling invariance condition simplifies the equations that determine the Lax pair and allows to formalize the procedure: the functional forms of both $U^{(i)}(u, u_x, u_{2x}, \ldots)$ and $Q^{(i)}(u, u_x, u_{2x}, \ldots)$ can be algorithmically assigned and a finite system of algebraic equations for the unknown coefficients arising at the final stage of the procedure (instead of a system of differential equations for $Q^{(i)}$ of the present method) can be algorithmically solved by the Gröbner basis methods. However, the scaling invariance condition reduces the generality of the method: the Lax pairs that include terms of different scaling weights are out of its scope. In particular, this prevents application of the method of [8] to nonhomogeneous, mixed scaling weight, equations, since the Lax pairs with the terms of lower and higher weights originating respectively from the lower and higher scaling weight parts of the equation inevitably arise for such equations. In general, the Lax pairs for both homogeneous and non-homogeneous equations may include terms of different scaling weights as a practice of applying the method of the present study shows. In most cases, the method yields not single Lax pairs but multi-parameter families of Lax pairs consisting of terms of different scaling weights. An example of such a multi-parameter family of Lax pairs yielded by applying the method to the mKdV equation (1) is given in Section 3. Although, as it is shown in Section 4, the Lax pairs of that family should be treated as weak, the appearance of such terms is a sign that one should keep the functional forms for the operators **L** and **A** as general as possible not to miss some variants. In this respect, the procedure of the present method, in which the ansatzes for $U^{(i)}$ include terms of different scaling weights and the forms of $Q^{(i)}$ are not assigned but determined in the course of calculations, is preferable.

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3. Lax pairs for the mKdV equation

For the mKdV equation (1), two different families of the Lax pairs, with the operator **L** of order 2, are available using the method of the present study. The first family is given by

$$\mathbf{L} = \mathbf{D}_x^2 + (\mu_2 + 2\mu_1 u)\,\mathbf{D}_x + \left(\mu_1\mu_2 u + \mu_1^2 u^2 + \mu_1 u_x\right)\mathbf{I}, \tag{6}$$

$$\begin{aligned}\mathbf{A} &= q_0\mathbf{D}_x^3 + 3q_0\left(\mu_1 u + \frac{\mu_2}{2}\right)\mathbf{D}_x^2 + 3q_0\left(\mu_1^2u^2 + \mu_1\mu_2 u + \frac{\mu_2^2}{8} + \mu_1 u_x\right)\mathbf{D}_x\\ &+\mu_1\Bigg(\left(q_0\mu_1^2 - \frac{r_1}{3}\right)u^3 + \frac{3}{2}q_0\mu_1\mu_2 u^2 + \frac{3}{8}q_0\mu_2^2 u\\ &+\left(\frac{3}{2}q_0\mu_2 + 3q_0\mu_1 u\right)u_x + (q_0-1)\,u_{2x}\Bigg)\mathbf{I}\end{aligned} \tag{7}$$

where μ_1, μ_2 and q_0 are arbitrary constants ($\mu_1 \neq 0$). Note that the scaling weight of the terms multiplied by μ_2 is different from that of other terms.

The second family is defined by

$$\mathbf{L} = \mathbf{D}_x^2 + 2\epsilon u\mathbf{D}_x + \frac{1}{6}\left(\left(r_1 + 6\epsilon^2\right)u^2 + \left(6\epsilon \pm \sqrt{-6r_1}\right)u_x\right)\mathbf{I}, \tag{8}$$

$$\begin{aligned}\mathbf{A} &= 4\mathbf{D}_x^3 + 12\epsilon u\mathbf{D}_x^2 + \left(\left(r_1 + 12\epsilon^2\right)u^2 + \left(12\epsilon \pm \sqrt{-6r_1}\right)u_x\right)\mathbf{D}_x\\ &+\left(\epsilon\left(\frac{2r_1}{3} + 4\epsilon^2\right)u^3 + \left(r_1 \pm \sqrt{-6r_1}\epsilon + 12\epsilon^2\right)uu_x + \left(3\epsilon \pm \frac{1}{2}\sqrt{-6r_1}\right)u_{2x}\right)\mathbf{I}\end{aligned} \tag{9}$$

where ϵ is an arbitrary constant.

Two branches of Lax pairs have been identified for the mKdV equation in [8]. Their 'first branch' is a degenerate case ($q_0 = 0$, $\mu_2 = 0$, $\mu_1 = 1$) of the family defined by ((6), (7)) and their 'second branch' is ((8), (9)) (the notation coincides except for that their α should be replaced by r_1 and the sign of the operator **A** should be changed).

4. Gauge equivalence

For Lax pairs in matrix form (zero-curvature representation [2–4,10,11]), scalar equations (2) are replaced by the matrix equations

$$\mathbf{D}_x\mathbf{\Psi} = \mathbf{X}\mathbf{\Psi}, \quad \mathbf{D}_t\mathbf{\Psi} = \mathbf{T}\mathbf{\Psi} \tag{10}$$

where $\mathbf{\Psi}$ is a vector function on the jet space of u and $\mathbf{X}(\lambda, u, u_x, \ldots)$ and $\mathbf{T}(\lambda, u, u_x, \ldots)$ are, in general, $(n \times n)$ matrices dependent on the spectral parameter λ. The Lax equation (3) is replaced by the equation expressing the compatibility condition for (10), as follows

$$\left(\mathbf{D_t X} - \mathbf{D_x T} + [\mathbf{X}, \mathbf{T}]\right)\mathbf{\Psi} = \mathbf{0} \tag{11}$$

where $[\mathbf{X}, \mathbf{T}] = \mathbf{XT} - \mathbf{TX}$ is the matrix commutator. Equation (11) is called the matrix form Lax equation or zero-curvature condition.

For any pair of matrices $(\mathbf{X}, \mathbf{T})$ which satisfy (11), an infinite number of equivalent pairs

$$\mathbf{D}_x\tilde{\mathbf{\Psi}} = \tilde{\mathbf{X}}\tilde{\mathbf{\Psi}}, \quad \mathbf{D}_t\tilde{\mathbf{\Psi}} = \tilde{\mathbf{T}}\tilde{\mathbf{\Psi}} \tag{12}$$

may be found through a gauge transformation of the form

$$\tilde{\mathbf{\Psi}} = \mathbf{G}\mathbf{\Psi}, \quad \tilde{\mathbf{X}} = \mathbf{G}\mathbf{X}\mathbf{G}^{-1} + \mathbf{D}_x(\mathbf{G})\mathbf{G}^{-1}, \quad \tilde{\mathbf{T}} = \mathbf{G}\mathbf{T}\mathbf{G}^{-1} + \mathbf{D}_t(\mathbf{G})\mathbf{G}^{-1} \tag{13}$$

where **G** is a nonsingular matrix.

The gauge transformations that convert the families of Lax pairs ((6), (7)) and ((8), (9)) into some equivalent forms can be found. Let us consider first the one-parameter (ϵ) family ((8), (9)). In what follows, it is set $r_1 = -6$ to simplify the formulas. A family of matrices $(\mathbf{X}, \mathbf{T})$ corresponding to the operators ((8), (9)) is given by

$$\mathbf{X} = \begin{pmatrix} 0 & 1 \\ -\lambda + \left(1-\epsilon^2\right) u^2 - (1+\epsilon)\, u_x & -2\epsilon u \end{pmatrix}$$

$$\mathbf{T} = \begin{pmatrix} 4\epsilon\lambda u - 2\,(1+\epsilon)\, u u_x + (1+\epsilon)\, u_{2x} & 4\lambda + 2u^2 - 2u_x \\ \begin{array}{c} -4\lambda^2 + \left(2-4\epsilon^2\right)\lambda u^2 + \left(2-2\epsilon^2\right) u^4 - 2\lambda u_x \\ -2\left(2+\epsilon-\epsilon^2\right) u^2 u_x - 2\,(1+\epsilon)\, u u_{2x} + (1+\epsilon)\, u_{3x} \end{array} & \begin{array}{c} -4\epsilon\lambda u - 4\epsilon u^3 \\ +2\,(1+\epsilon)\, u u_x - (1-\epsilon)\, u_{2x} \end{array} \end{pmatrix}$$

Applying the gauge transformation (13) with the matrix

$$\mathbf{G} = \begin{pmatrix} e^{\epsilon \int u(x,t)dx} & 0 \\ \epsilon u(x,t)\, e^{\epsilon \int u(x,t)dx} & e^{\epsilon \int u(x,t)dx} \end{pmatrix} \tag{14}$$

to the family of matrices (14) and (14) yields the matrices $(\tilde{\mathbf{X}}, \tilde{\mathbf{T}})$ not containing the parameter ϵ, as follows

$$\mathbf{X}_1 = \begin{pmatrix} 0 & 1 \\ -\lambda + u^2 - u_x & 0 \end{pmatrix} \tag{15}$$

$$\mathbf{T}_1 = \begin{pmatrix} -2uu_x + u_{2x} & 4\lambda + 2u^2 - 2u_x \\ -4\lambda^2 + 2\lambda u^2 + 2u^4 - 2\lambda u_x - 4u^2 u_x - 2uu_{2x} + u_{3x} & 2uu_x - u_{2x} \end{pmatrix} \tag{16}$$

The corresponding Lax pairs in an operator form are ((8), (9)) taken for $\epsilon = 0$. The Lax pair ((15), (16)) is the known Lax pair for the mKdV equation which can be obtained from the Lax pair

$$\mathbf{X}_{KdV} = \begin{pmatrix} 0 & 1 \\ -\lambda - v & 0 \end{pmatrix} \tag{17}$$

$$\mathbf{T}_{KdV} = \begin{pmatrix} v_x & 4\lambda - 2v \\ -4\lambda^2 - 2\lambda v + 2v^2 + v_{2x} & -v_x \end{pmatrix} \tag{18}$$

for the KdV equation

$$v_t + 6vv_x + +v_{3x} = 0 \tag{19}$$

through Miura's transformation

$$v = u_x - u^2 \tag{20}$$

The Lax pair ((15), (16)) is equivalent to another known Lax pair for the mKdV equation [3,12]

$$\mathbf{X}_2 = \begin{pmatrix} -ik & -u \\ -u & ik \end{pmatrix} \tag{21}$$

$$\mathbf{T}_2 = \begin{pmatrix} -4ik^3 - 2iku^2 & -4k^2u - 2u^3 - 2iku_x + u_{2x} \\ -4k^2u - 2u^3 + 2iku_x + u_{2x} & 4ik^3 + 2iku^2 \end{pmatrix} \tag{22}$$

to which it is related by the gauge transformation

$$\mathbf{G} = \begin{pmatrix} 6(ik-u) & -6 \\ 6(ik+u) & 6 \end{pmatrix} \tag{23}$$

Analyzing the matrix forms of the first family of Lax pairs ((6), (7)) reveals that it can be converted by a gauge transformations into the form which, upon using equation (1) and its differential consequences, does not include u. In order not to overload the presentation, the matrix form of the Lax pair ((6), (7)) is not shown, only the transformation matrix is given below. The Lax pairs ((6), (7)), being written in matrix forms, can be reduced by the gauge transformation (13) with the matrix

$$\mathbf{G} = \begin{pmatrix} e^{\mu_1 \int u(x,t)dx} & 0 \\ \mu_1 u(x,t)\, e^{\mu_1 \int u(x,t)dx} & e^{\mu_1 \int u(x,t)dx} \end{pmatrix} \tag{24}$$

to the constant matrices

$$\tilde{\mathbf{X}} = \begin{pmatrix} 0 & 1 \\ -\lambda & -\mu_2 \end{pmatrix}, \quad \tilde{\mathbf{T}} = \begin{pmatrix} \frac{1}{2}\mu_2 q_0 \lambda & \frac{1}{8} q_0 \left(8\lambda + \mu_2^2\right) \\ -\frac{1}{8} q_0 \lambda \left(8\lambda + \mu_2^2\right) & -\frac{1}{8} q_0 \mu_2 \left(4\lambda + \mu_2^2\right) \end{pmatrix} \tag{25}$$

Such a Lax pair not containing the dependent variable u is useless for finding solutions by the IST and/or constructing conservation laws. Thus, the Lax pairs ((6), (7)) should be treated as weak.

5. Concluding comments

In the present paper, the results of application of the direct method to the mKdV equation are considered. It is demonstrated that the method can produce multi-parameter families of the Lax pairs. At the same time, by applying proper gauge transformations it is found that, (at least) in the case of that specific equation, the defined Lax pairs can be either reduced to a single Lax pair, which is known, or converted into the form not containing the dependent variable which is useless for applications. Thus, only the known Lax pair is what remains from the variety of Lax pairs defined (and so from its particular case found in [8]).

As a matter of fact, the direct method should be most useful for the problem of classification of integrable equations of some specific type but not as applied to a single equation. Being applied to an equation or system with parameters, the method yields conditions on the parameters for the Lax pair existence. Since the fake Lax pairs cannot be avoided a positive result of testing a PDE with respect to the Lax pair existence does not give right to place the equation on the list of integrable equations. At the same time, a reliable *negative* result of testing may be considered as a strong argument *against* its integrability since it is commonly believed that a completely integrable nonlinear PDE can be associated with a Lax pair. In the context of the problem of classification of integrable equations, detecting equations that cannot be integrable should be as important as finding candidates for integrable equations. It allows to substantially reduce the list of candidates for integrable equations of that specific type. From a somewhat different perspective, it may allow to complete classification if applying the method yields only equations that have been proved to be integrable.

The method developed in the present study provides a reliable testing for the Lax pairs of a quite general form and, practically, the freedoms in the choice of the "ansatz" for the differential operator **L** do not result in a loss of generality. With the computer algebra capabilities, there are no principal obstacles to implementing calculations for any reasonable candidate. Note in this connection that the Lax pairs presented in this paper have been separated from substantially more complicated initial forms. With such general ansatzes, a negative result of testing a PDE with respect to the Lax pairs existence is as close to a proof as is possible using direct methods. Due to the fact that the requirement of scaling invariance is not imposed the method can be effective in application to both homogeneous and mixed scaling weight equations.

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