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Fermatean Fuzzy Versions of Tanimoto Similarity Measure and Distance Measure and Their Application to Decision-Making Processes

Hongpeng Wang [1], Caikuan Tuo [2], Guoye Feng [3] and Chenglong Li [1,*]

[1] School of Computer Science and Technology, Shandong Jianzhu University, Jinan 250000, China; 202011110145@stu.sdjzu.edu.cn
[2] School of Business Administration, Chongqing Technology and Business University, Chongqing 400000, China; tuocaikuan@ctbu.edu.cn
[3] School of Accounting, Chongqing Technology and Business University, Chongqing 400000, China; fengguoye@ctbu.edu.cn
* Correspondence: lichenglong18@sdjzu.edu.cn

Abstract: Fermatean fuzzy sets (FFSs) are a versatile tool for handling uncertain problems and have shown great effectiveness in practical applications. However, many existing Fermatean fuzzy similarity measures exhibit counter-intuitive situations, making it challenging to accurately measure the similarity or difference between FFSs. To address this issue, several similarity and distance measures for FFSs are proposed, inspired by the Tanimoto similarity measure. The properties of the proposed measures are also explored, along with several comparative examples with existing measures for FFSs, which illustrate their superior performance in processing fuzzy information from FFSs. The proposed measures have effectively overcome the counter-intuitive challenges posed by many existing measures and significantly outperforms existing measures in differentiating different FFSs. Furthermore, we demonstrate the practical applicability of the proposed measures in solving pattern recognition, medical diagnosis and multiple-attribute decision-making problems.

Keywords: fermatean fuzzy sets; similarity measures; distance measures; pattern recognition; medical diagnosis; multiple-attribute decision-making

1. Introduction

Uncertainty has become more widespread in various fields due to the complexity of objective phenomena and the inherent limitations of human knowledge [1–3]. This indistinct feature often occurs in a random and indeterminate fashion, thereby making accurate descriptions difficult. Consequently, multiple new theories and methods have emerged, allowing for effective representation of uncertain information in practical problems [4–8]. One prominent theory is fuzzy sets, which has gained significant attention since its introduction by Zadeh in 1965 [4]. Fuzzy sets theory extends classical set theory to address situations where the boundaries between different categories are not clearly defined. By assigning memberships with degrees to each element in a set, fuzzy sets offer a natural and more accurate depiction of vague concepts. Fuzzy sets theory has overcome limitations of traditional decision-making methods by enabling us to reason about uncertain information and make decisions based on incomplete or ambiguous data. This novel theory provides a new method to modeling and describing fuzzy and uncertain information, and has been widely applied in control systems, pattern recognition, decision-making and artificial intelligence [9–15].

In increasingly complex decision-making problems, traditional fuzzy sets theory has limitations in accurately representing uncertain information. To address this issue, scholars have proposed several extensions to traditional fuzzy sets, such as Intuitionistic fuzzy sets (IFSs) [16], hesitant fuzzy sets [6], evidence theory [17] and rough sets [18] . Among these, IFSs have attracted considerable attention due to their ability to represent fuzzy and uncertain information with the inclusion of both membership and non-membership degrees of elements. This salient feature has made IFSs

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a valuable tool in several fields for addressing uncertainties [19–21]. As an extension of IFSs, in 2013, Yager first introduced Pythagorean fuzzy sets (PFSs) [22]. This model introduces the notion of Pythagorean membership function, which extends the concept of membership and non-membership degrees to a triplet of parameters, including membership, non-membership, and hesitancy degrees. PFSs impose the restriction that a membership degree plus a non-membership degree must not exceed one, enabling them to more effectively capture and represent uncertain information [23–25]. Recently, Yager and Filev [26] extended PFSs and introduced a novel type of fuzzy sets called Fermatean fuzzy sets (FFSs) in 2019. FFSs is founded on the notion of Fermatean distance, which mandates that the sum of the degrees of membership and non-membership is not greater than 1, and encompasses hesitation in capturing and representing uncertain information. This attribute enables it to encompass a greater amount of information, making it more powerful than PFSs and IFSs. Currently, FFSs has evoked considerable interest among researchers. Ghorabaee [27] proposed a novel decision-making approach based on FFSs. Garg [28] demonstrated the application of Fermatean fuzzy aggregation functions in COVID-19 testing facilities. Aydemir and Yilmaz [29] introduced the TOPSIS method, the technique for order of preference by similarity to ideal solution, for FFSs. Shahzadi and Akram [30] introduced the concept of fermatean fuzzy soft sets (FFSSs) and demonstrated their applicability in selecting an antivirus mask. Gul [31] showed the applicability of FFSs for occupational risk assessment in manufacturing. Sergi and Sari [32] proposed some Fermatean fuzzy capital budgeting techniques. Ali and Ansari [33] introduced the concept of fermatean fuzzy bipolar soft sets and demonstrated their utility in multiple-criteria decision-making (MCDM). Rani and Mishra [34] proposed a novel divergence measure and multi-objective optimization based on ratio analysis with the full multiplicative form method in the Fermatean fuzzy environment. In addition, FFSs have also been applied in various fields such as data mining, image processing, and clustering [27,35–37].

The concepts of similarity measure and distance measure are two other important concepts in the theory of fuzzy sets. They respectively refer to a mathematical function that evaluates the similarity and distance between two objects based on their respective attributes. In classical set theory, similarity is usually quantified using set-theoretic metrics, such as Jaccard similarity or cosine similarity [38,39]. However, these metrics perform poorly for fuzzy sets because they ignore the membership degrees of each element in the set. The concept of similarity measure is used to determine the similarity between individuals, while the concept of distance measure is used to quantify the degree of difference between individuals. Similarity measures include broader metrics than distance measures, which specifically calculate differences in Cartesian space. These terms are often used interchangeably, with distance often serving as the reciprocal of similarity and vice versa. In terms of evaluation, for distance measure, the shortest distance is observed between the closest points, while for similarity measure, the highest level of similarity is observed between the closest points. Recent research has introduced innovative similarity measures and distance measures:

- Garg [40] has proposed a similarity measure that utilizes transformed right-angled triangles. Li and Zeng investigated the normalized Hamming distance and the normalized Euclidean distance, seeking to improve the accuracy and efficiency of fuzzy set distance measures[41]. Duan [42] has presented a novel intuitionistic fuzzy similarity measure, while Olgun [43] has designed an intuitionistic fuzzy set cosine similarity measure based on Choquet integral. Huang [44] has proposed a similarity measure for intuitionistic fuzzy sets that incorporates the transformation of an isosceles right triangle's area. Kumar [45] has also introduced a new intuitionistic fuzzy set similarity measure, which has been applied to clustering problems. Garg [46] devised a correlation measure grounded in PFSs to address the issue of multiple-attribute decision-making(MADM). Li [47] introduced a fresh similarity assessment for PFSs, which is founded on the concept of spherical arc distance from a geometric standpoint. Additionally, a MADM approach was established in a Pythagorean fuzzy setting. Hussian and Yang [48] put forward brand-new similarity measures for PFSs based on Hausdorff measures. Moreover, many other similarity measures have been proposed in recent study [49–55].

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- Atanassov has defined four 2D distance measures based on Hamming and Euclidean distances [16]. The Hausdorff distance measures were proposed by Glazoczewski[56] to distinguish differences between intuitionistic fuzzy sets. More recently, Mahanta and Panda proposed a nonlinear distance measure that accounts for differences between intuitionistic fuzzy sets with high hesitation degrees [57]. Gohain introduced a measure based on the difference between the minimum and maximum cross-evaluation factor [58]. Xiao suggested a new Jensen-Shannon divergence-based distance measure of intuitionistic fuzzy sets [59], while Li and Zeng investigated the normalized Hamming and Euclidean distances [60]. Li and Lu [61] proposed a novel distance measure for PFSs, and Xu [62] proposed a Hamming distance measure. Furthermore, Ren, Xu, and Gou [63] presented a novel distance measure that builds upon the Euclidean distance model for PFSs. Moreover, many other distance measures have been proposed in recent study [64–69].

At present, the similarity and distance measures for IFSs and PFSs are relatively complete.

Currently, several similarity and distance measures have been put forward for FFSs, including the new distance using Hellinger distance and triangular divergence [70], the cosine similarity measure by Kirişci [71], the cosine similarity by Sahoo [72] and the similarity measure based on linguistic scale function [73]. These measures are designed to confront challenges such as asymmetry, differences in intersection size, and complexity. Despite this, there is still limited research on the measures of similarity and distance in FFSs, and the existing measures may produce counter-intuitive phenomena under certain circumstances and are not applicable under many conditions. Therefore, the study of similarity and distance measures of FFSs has become an important research field with significant academic significance.

This paper introduces a set of novel similarity and distance measures for FFSs. The properties of these measures are thoroughly analyzed through abundant examples. Additionally, we propose two models that employ these measures for tasks such as pattern recognition, medical diagnosis and MADM problems in Fermatean fuzzy environments. We present a series of experiments comparing our measures to existing ones. Results demonstrate that our proposed measures not only overcome numerous counter-intuitive situations, but also provide more reliable decision-making capabilities when discerning dissimilarities between FFSs. These qualities exemplify the superior nature of our proposed measures.

The main contributions of this paper are as follows:

(1) We introduce novel similarity and distance measures for FFSs utilizing the Tanimoto similarity measure, and provide proofs of their properties.

(2) Two models employing these measures are proposed for pattern recognition, medical diagnosis, and MADM problems, demonstrating their effectiveness.

(3) Through comparative analysis with existing measures for FFSs, our proposed measures exhibit superior performance, with improved sensitivity to discriminating dissimilarities between FFSs and a capacity to circumvent counter-intuitive limitations of existing measures. Our measures offer greater reliability and superiority in distinguishing FFSs.

The following study is presented below. Specifically, in Section 2, we briefly review the fundamental concepts of fuzzy sets theory. In Section 3, we propose several novel similarity and distance measures for FFSs and establish their properties. Meanwhile, a large number of experiments have been conducted to demonstrate the superiority of the proposed measures in overcoming counter-intuitive situations, distinguishing FFSs, and making more reliable decisions. In Section 4, based on proposed measures, two models are introduced to address pattern recognition, medical diagnosis and MADM problems. Finally, in Section 5, we draw conclusions and provide future research directions.

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2. Preliminaries

In this section, some basic concepts related to fuzzy sets, Tanimoto similarity measure and several existing measures for FFSs will be given.

2.1. Intuitionistic Fuzzy Sets

Definition 1 ([16]). *We utilize the symbol Z to denote a finite set. An Intuitionistic fuzzy set I is given by:*

$$I = \{\langle z, \rho_I(z), \sigma_I(z)\rangle : z \in Z\} \tag{1}$$

where $\rho_I(z) : Z \rightarrow [0,1]$ signifies the membership degree of z, and $\sigma_I(z) : Z \rightarrow [0,1]$ expresses the nonmembership degree of z. $\forall z$, $\rho_I(z)$ and $\sigma_I(z)$ satisfy:

$$0 \leq \rho_I(z) + \sigma_I(z) \leq 1 \tag{2}$$

$\forall z$, the indeterminacy degree of the element z is:

$$\theta_I(z) = 1 - \rho_I(z) - \sigma_I(z) \tag{3}$$

2.2. Pythagorean Fuzzy Sets

Definition 2 ([22]). *The Pythagorean fuzzy set P is defined as:*

$$P = \{\langle z, \rho_P(z), \sigma_P(z)\rangle : z \in Z\} \tag{4}$$

where $\rho_P(z) : Z \rightarrow [0,1]$ and $\sigma_P(z) : Z \rightarrow [0,1]$ denote, respectively, the membership degree and nonmembership degree of z. $\forall z$, $\rho_P(z)$ and $\sigma_P(z)$ satisfy:

$$0 \leq \rho_P^2(z) + \sigma_P^2(z) \leq 1 \tag{5}$$

$\forall z$, the indeterminacy degree of the element z is:

$$\theta_P(z) = \sqrt{1 - \rho_P^2(z) - \sigma_P^2(z)} \tag{6}$$

2.3. Fermatean Fuzzy Sets

Definition 3 ([26]). *The Fermatean fuzzy set F is defined as:*

$$F = \{\langle z, \rho_F(z), \sigma_F(z)\rangle : z \in Z\} \tag{7}$$

where $\rho_F(z) : Z \rightarrow [0,1]$ and $\sigma_F(z) : Z \rightarrow [0,1]$ denote, respectively, the membership degree and nonmembership degree of z. $\forall z$, $\rho_F(z)$ and $\sigma_F(z)$ satisfy:

$$0 \leq \rho_F^3(z) + \sigma_F^3(z) \leq 1 \tag{8}$$

For any $z \in Z$, the indeterminacy degree of the element z is:

$$\theta_F(z) = \sqrt[3]{1 - \rho_F^3(z) - \sigma_F^3(z)} \tag{9}$$

Definition 4. *Suppose that $Z = \{z_1, z_2, \ldots, z_n\}$, $F = \{(z, [\rho_F(z_i,), \sigma_F(z_i)]) : z_i \in Z\}$ and $G = \{(z, [\rho_G(z_i,), \sigma_G(z_i)]) : z_i \in Z\}$ are two FFSs. The cosine similarity measure between F and G is defined as:*

$$C_{FFS}(F,G) = \frac{1}{n}\sum_{i=1}^{n} \frac{\rho_F^3(z_i)\rho_G^3(z_i) + \sigma_F^3(z_i)\sigma_G^3(z_i) + \theta_F^3(z_i)\theta_G^3(z_i)}{\sqrt{\rho_F^6(z_i) + \sigma_F^6(z_i) + \theta_F^6(z_i)}\sqrt{\rho_G^6(z_i) + \sigma_G^6(z_i) + \theta_G^6(z_i)}} \tag{10}$$

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Definition 5. *Suppose that* $Z = \{z_1, z_2, \ldots, z_n\}$, $F = \{(z, [\rho_F(z_i,), \sigma_F(z_i)]) : z_i \in Z\}$ *and* $G = \{(z, [\rho_G(z_i,), \sigma_G(z_i)]) : z_i \in Z\}$ *are two FFSs. The Euclidean distance is defined as:*

$$D_{FFS}(F,G) = \frac{1}{2n}\sum_{z_i \in Z}(|\rho_F^3(z_i) - \rho_G^3(z_i)|^2 + |\sigma_F^3(z_i) - \sigma_G^3(z_i)|^2 + |\theta_F^3(z_i) - \theta_G^3(z_i)|^2)^{\frac{1}{2}} \tag{11}$$

Definition 6 ([71])**.** *Suppose that* $Z = \{z_1, z_2, \ldots, z_n\}$, $F = \{(z, [\rho_F(z_i,), \sigma_F(z_i)]) : z_i \in Z\}$ *and* $G = \{(z, [\rho_G(z_i,), \sigma_G(z_i)]) : z_i \in Z\}$ *are two FFSs. The new cosine similarity proposed by Kirişci is defined as:*

$$S^1_{FFS}(F,G) = \frac{C_{FFS}(F,G) + 1 - D_{FFS}(F,G)}{2} \tag{12}$$

Definition 7 ([72])**.** *Suppose that* $Z = \{z_1, z_2, \ldots, z_n\}$, $F = \{(z, [\rho_F(z_i,), \sigma_F(z_i)]) : z_i \in Z\}$ *and* $G = \{(z, [\rho_G(z_i,), \sigma_G(z_i)]) : z_i \in Z\}$ *are two FFSs. Several similarities proposed by Sahoo are defined as:*

$$S^2_{FFS}(F,G) = 1 - \left(\frac{1}{2n}\sum_{i=1}^{n}\left(|\rho_F(z_i) - \rho_G(z_i)|^3 + |\sigma_F(z_i) - \sigma_G(z_i)|^3 + |\theta_F(z_i) - \theta_G(z_i)|^3\right)\right)^{\frac{1}{3}} \tag{13}$$

$$S^3_{FFS}(F,G) = 1 - \left(\frac{1}{2n}\sum_{i=1}^{n}(|\rho_F(z_i) - \rho_G(z_i)| + |\sigma_F(z_i) - \sigma_G(z_i)| + |\theta_F(z_i) - \theta_G(z_i)|)\right) \tag{14}$$

$$S^4_{FFS}(F,G) = 1 - \left(\frac{1}{2n}\sum_{i=1}^{n}\left(\left|\rho_F^3(z_i) - \rho_G^3(z_i)\right| + \left|\sigma_F^3(z_i) - \sigma_F^3(z_i)\right| + \left|\theta_F^3(z_i) - \rho_F^3(z_i)\right|\right)\right) \tag{15}$$

$$S^5_{FFS}(F,G) = 1 - \left(\frac{1}{2n}\sum_{i=1}^{n}\left(|\psi(F_i) - \psi(G_i)|^3\right)\right)^{\frac{1}{3}} \tag{16}$$

$$S^6_{FFS}(F,G) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\psi(F_i)\,\psi(G_i) + \theta_F^3(z_i)\,\theta_G^3(z_i)}{\sqrt{\psi^2(F_i) + \theta_F^3(z_i)}\sqrt{\psi^2(G_i) + \theta_G^3(z_i)}}\right) \tag{17}$$

The score function is defined as:

$$\psi(z) = \rho^3 - \sigma^3 \tag{18}$$

Definition 8 ([70])**.** *Suppose that* $Z = \{z_1, z_2, \ldots, z_n\}$, $F = \{(z, [\rho_F(z_i,), \sigma_F(z_i)]) : z_i \in Z\}$ *and* $G = \{(z, [\rho_G(z_i,), \sigma_G(z_i)]) : z_i \in Z\}$ *are two FFSs. The Hellinger distance is defined as:*

$$d_H(F,G) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left(\left(\sqrt{\rho_F^3(z_i)} - \sqrt{\rho_G^3(z_i)}\right)^2 + \left(\sqrt{\sigma_F^3(z_i)} - \sqrt{\sigma_G^3(z_i)}\right)^2\right)} \tag{19}$$

Definition 9 ([70])**.** *Suppose that* $Z = \{z_1, z_1, \ldots, z_n\}$, $F = \{(z, [\rho_F(z_i,), \sigma_F(z_i)]) : z_i \in Z\}$ *and* $G = \{(z, [\rho_G(z_i,), \sigma_G(z_i)]) : z_i \in Z\}$ *are two FFSs. The triangular divergence distance is defined as:*

$$d_T(F,G) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{(\rho_F^3(z_i) - \rho_G^3(z_i))^2}{\rho_F^3(z_i) + \rho_G^3(z_i)} + \frac{(\sigma_F^3(z_i) - \sigma_G^3(z_i))^2}{\sigma_F^3(z_i) + \sigma_G^3(z_i)}\right)} \tag{20}$$

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2.4. Tanimoto Similarity Measure

Definition 10 ([74]). *Suppose* $A = \{a_1, a_2, \ldots, a_n\}$ *and* $B = \{b_1, b_2, \ldots, b_n\}$ *are two probability distributions. The definition of the Tanimoto measure between A and B is depicted as:*

$$T(A,B) = \frac{\sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 - \sum_{i=1}^{n} a_i b_i} \tag{21}$$

3. Some Novel Tanimoto Similarity Measures and Distance Measures for FFSs

In this section, we present Tanimoto similarity measures and weighted Tanimoto similarity measures for Fermatean fuzzy sets (FFSs) using the concept of Tanimoto similarity measure. To compare with existing distance measures, we propose their distance measure forms. We verify their properties through several numerical experiments. Furthermore, we have demonstrated the proposed measures' ability to overcome the counter-intuitive nature of existing measures and their superiority on distinguishing different FFSs over existing measures through several illustrative examples.

3.1. Tanimoto Similarity Measures for FFSs

Definition 11. *Considering a fixed set* $Z = \{z_1, z_2, \ldots, z_n\}$, $F = \{(z, [\rho_F(z_i,), \sigma_F(z_i)]) : z_i \in Z\}$ *and* $G = \{(z, [\rho_G(z_i,), \sigma_G(z_i)]) : z_i \in Z\}$ *are two FFSs. A Tanimoto similarity measure between F and G is defined as:*

$$T_{FFS}(F,G) = \frac{1}{n}\sum_{i=1}^{n} \frac{\rho_F^3(z_i)\rho_G^3(z_i) + \sigma_F^3(z_i)\sigma_G^3(z_i)}{\rho_F^6(z_i) + \sigma_F^6(z_i) + \rho_G^6(z_i) + \sigma_G^6(z_i) - \rho_F^3(z_i)\rho_G^3(z_i) - \sigma_F^3(z_i)\sigma_G^3(z_i)} \tag{22}$$

Theorem 1. *If F and G are any two FFSs, the* $T_{FFS}(F,G)$ *satisfies the conditions:*

1. $0 \leq T_{FFS}(F,G) \leq 1$;
2. $T_{FFS}(F,G) = T_{FFS}(G,F)$;
3. $T_{FFS}(F,G) = 1$, *if* $F = G, (\rho_F(z_i) = \rho_G(z_i), \sigma_F(z_i) = \sigma_G(z_i))$.

Proof of Theorem 1. 1. Considering the ith item of the summation in Equation 11:

$$T_{FFS}(F,G) = \frac{\rho_F^3(z_i)\rho_G^3(z_i) + \sigma_F^3(z_i)\sigma_G^3(z_i)}{\rho_F^6(z_i) + \sigma_F^6(z_i) + \rho_G^6(z_i) + \sigma_G^6(z_i) - \rho_F^3(z_i)\rho_G^3(z_i) - \sigma_F^3(z_i)\sigma_G^3(z_i)} \tag{23}$$

According to $0 \leq \rho(z_i) \leq 1$ and $0 \leq \sigma(z_i) \leq 1$, we can get $\rho_F^3(z_i)\rho_G^3(z_i) + \sigma_F^3(z_i)\sigma_G^3(z_i) \geq 0$. According to the inequality $a^2 + b^2 \geq 2ab$, $\rho_F^6(z_i) + \sigma_F^6(z_i) + \rho_G^6(z_i) + \sigma_G^6(z_i) - \rho_F^3(z_i)\rho_G^3(z_i) - \sigma_F^3(z_i)\sigma_G^3(z_i) \geq \rho_F^3(z_i)\rho_G^3(z_i) + \sigma_F^3(z_i)\sigma_G^3(z_i)$.Therefore, $0 \leq T_{FFS}(F,G) \leq 1$. From the Equation 11, the sumation of n terms is $0 \leq T_{FFS}(F,G) \leq 1$.

2.

$$\begin{aligned} T_{FFS}(F,G) &= \frac{1}{n}\sum_{i=1}^{n} \frac{\rho_F^3(z_i)\rho_G^3(z_i) + \sigma_F^3(z_i)\sigma_G^3(z_i)}{\rho_F^6(z_i) + \sigma_F^6(z_i) + \rho_G^6(z_i) + \sigma_G^6(z_i) - \rho_F^3(z_i)\rho_G^3(z_i) - \sigma_F^3(z_i)\sigma_G^3(z_i)} \\ &= \frac{1}{n}\sum_{i=1}^{n} \frac{\rho_G^3(z_i)\rho_F^3(z_i) + \sigma_G^3(z_i)\sigma_F^3(z_i)}{\rho_G^6(z_i) + \sigma_G^6(z_i) + \rho_F^6(z_i) + \sigma_F^6(z_i) - \rho_G^3(z_i)\rho_F^3(z_i) - \sigma_G^3(z_i)\sigma_F^3(z_i)} \\ &= T_{FFS}(G,F) \end{aligned}$$

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3. When F = G, there are $\rho_F(z_i) = \rho_G(z_i)$, $(\sigma_F(z_i) = \sigma_G(z_i)$, for $i = 1, 2, \ldots, n$. So, there is

$$\begin{aligned} T_{FFS}(F,G) &= \frac{1}{n}\sum_{i=1}^{n}\frac{\rho_F^3(z_i)\rho_G^3(z_i)+\sigma_F^3(z_i)\sigma_G^3(z_i)}{\rho_F^6(z_i)+\sigma_F^6(z_i)+\rho_G^6(z_i)+\sigma_G^6(z_i)-\rho_F^3(z_i)\rho_G^3(z_i)-\sigma_F^3(z_i)\sigma_G^3(z_i)} \\ &= \frac{1}{n}\sum_{i=1}^{n}\frac{\rho_F^3(z_i)\rho_F^3(z_i)+\sigma_F^3(z_i)\sigma_F^3(z_i)}{\rho_F^6(z_i)+\sigma_F^6(z_i)+\rho_F^6(z_i)+\sigma_F^6(z_i)-\rho_F^3(z_i)\rho_F^3(z_i)-\sigma_F^3(z_i)\sigma_F^3(z_i)} \\ &= \frac{1}{n}\sum_{i=1}^{n}\frac{\rho_F^6(z_i)+\sigma_F^6(z_i)}{\rho_F^6(z_i)+\sigma_F^6(z_i)} \\ &= 1 \end{aligned}$$

Therefore, we have finished the proofs. □

If we consider the weights of z_i, a weighted Tanimoto similarity measure between FFSs F and G is proposed as fllows:

Definition 12. *For $z_i \in Z$, take the weight ω_i. The weighted Tanimoto measure $T^{\omega}_{FFS}(F,G)$ is described as:*

$$T^{\omega}_{FFS}(F,G) = \sum_{i=1}^{n}\frac{\omega_i(\rho_F^3(z_i)\rho_G^3(z_i)+\sigma_F^3(z_i)\sigma_G^3(z_i))}{\rho_F^6(z_i)+\sigma_F^6(z_i)+\rho_G^6(z_i)+\sigma_G^6(z_i)-\rho_F^3(z_i)\rho_G^3(z_i)-\sigma_F^3(z_i)\sigma_G^3(z_i)} \tag{24}$$

Similar to the Proof of Theorem 1, we can get:

Theorem 2. *If F and G are any two FFSs, the $T^{\omega}_{FFS}(F,G)$ satisfies the conditions:*

1. *$0 \le T^{\omega}_{FFS}(F,G) \le 1$;*
2. *$T^{\omega}_{FFS}(F,G) = T^{\omega}_{FFS}(G,F)$;*
3. *$T^{\omega}_{FFS}(F,G) = 1$, if $F = G$, $(\rho_F(z_i) = \rho_G(z_i), \sigma_F(z_i) = \sigma_G(z_i))$.*

When considering the degree of indeterminacy, we can get:

Definition 13. *For $z_i \in Z$, take the degree of indeterminacy θ_i. The Tanimoto measure $T^{\theta}_{FFS}(F,G)$ is described as:*

$$T^{\theta}_{FFS}(F,G) = \frac{1}{n}\sum_{i=1}^{n}\frac{\rho_F^3(z_i)\rho_G^3(z_i)+\sigma_F^3(z_i)\sigma_G^3(z_i)+\theta_F^3(z_i)\theta_G^3(z_i)}{\rho_F^6(z_i)+\sigma_F^6(z_i)+\theta_F^6(z_i)+\rho_G^6(z_i)+\sigma_G^6(z_i)+\theta_G^6(z_i)-\rho_F^3(z_i)\rho_G^3(z_i)-\sigma_F^3(z_i)\sigma_G^3(z_i)-\theta_F^3(z_i)\theta_G^3(z_i)} \tag{25}$$

When considering the degree of indeterminacy and the weights of z_i, we can get:

Definition 14. *For $z_i \in Z$, take the degree of indeterminacy θ_i and the weight ω_i. The Tanimoto measure $T^{\theta\omega}_{FFS}(F,G)$ is described as:*

$$T^{\theta\omega}_{FFS}(F,G) = \sum_{i=1}^{n}\frac{\omega_i(\rho_F^3(z_i)\rho_G^3(z_i)+\sigma_F^3(z_i)\sigma_G^3(z_i)+\theta_F^3(z_i)\theta_G^3(z_i))}{\rho_F^6(z_i)+\sigma_F^6(z_i)+\theta_F^6(z_i)+\rho_G^6(z_i)+\sigma_G^6(z_i)+\theta_G^6(z_i)-\rho_F^3(z_i)\rho_G^3(z_i)-\sigma_F^3(z_i)\sigma_G^3(z_i)-\theta_F^3(z_i)\theta_G^3(z_i)} \tag{26}$$

3.2. Tanimoto Distance Measures for FFSs

To facilitate comparison with existing distance measures, we propose the distance form for Tanimoto similarity.

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Definition 15. *Considering a fixed set* $Z = \{z_1, z_2, \ldots, z_n\}$, $F = \{(z, [\rho_F(z_i,), \sigma_F(z_i)]) : z_i \in Z\}$ *and* $G = \{(z, [\rho_G(z_i,), \sigma_G(z_i)]) : z_i \in Z\}$ *are two FFSs, the Tanimoto distance is described as:*

$$DT_{FFS}(F,G) = 1 - T_{FFS}(F,G) \tag{27}$$

The larger the $DT_{FFS}(F,G)$ is, the greater the difference between two FFSs.

Theorem 3. *If F and G are any two FFSs, the* $DT_{FFS}(F,G)$ *satisfies the conditions:*

1. $0 \le DT_{FFS}(F,G) \le 1$;
2. $DT_{FFS}(F,G) = DT_{FFS}(G,F)$;
3. $DT_{FFS}(F,G) = 0$, *if* $F = G((\rho_F(z_i) = \rho_G(z_i), \sigma_F(z_i) = \sigma_G(z_i), \theta_F(z_i) = \theta_G(z_i))$.

Definition 16. *For* $z_i \in Z$, *take the weight* ω_i. *The weighted Tanimoto distance* $DT^{\omega}_{FFS}(F,G)$ *is described as:*

$$DT^{\omega}_{FFS}(F,G) = 1 - T^{\omega}_{FFS}(F,G) \tag{28}$$

Theorem 4. *If F and G are any two FFSs, the* $DT^{\omega}_{FFS}(F,G)$ *satisfies the conditions:*

1. $0 \le DT^{\omega}_{FFS}(F,G) \le 1$;
2. $DT^{\omega}_{FFS}(F,G) = DT^{\omega}_{FFS}(G,F)$;
3. $DT^{\omega}_{FFS}(F,G) = 0$, *if* $F = G, (\rho_F(z_i) = \rho_G(z_i), \sigma_F(z_i) = \sigma_G(z_i))$.

Definition 17. *For* $z_i \in Z$, *take the degree of indeterminacy* θ_i. *The Tanimoto distance* $DT^{\theta}_{FFS}(F,G)$ *is described as:*

$$DT^{\theta}_{FFS}(F,G) = 1 - T^{\theta}_{FFS}(F,G) \tag{29}$$

Definition 18. *For* $z_i \in Z$, *take the degree of indeterminacy* θ_i *and the weight* ω_i. *The Tanimoto measure* $DT^{\theta\omega}_{FFS}(F,G)$ *is described as:*

$$DT^{\theta\omega}_{FFS}(F,G) = 1 - T^{\theta\omega}_{FFS}(F,G) \tag{30}$$

3.3. Numerical Experiments

Example 1. *There are three FFSs,* F_1, F_2 *and* F_3, *with*

$$F_1 = \{\langle z_1, 0.30, 0.20\rangle, \langle z_2, 0.60, 0.80\rangle\},$$
$$F_2 = \{\langle z_1, 0.30, 0.20\rangle, \langle z_2, 0.60, 0.80\rangle\},$$
$$F_3 = \{\langle z_1, 0.50, 0.70\rangle, \langle z_2, 0.64, 0.77\rangle\}.$$

According to equations in Subsections 3.1 and 3.2, the Tanimoto similarity (sm) and distance (dm) between FFSs are calculated, and the results are depicted as Table 1 and Table 2. Taking the weights $\omega = \{0.4, 0.6\}$, *the weighted Tanimoto measures between FFSs are shown as Table 3 and Table 4.*

Table 1. The results of Tanimoto similarity measures.

Measures	$sm(F_1,F_2)$	$sm(F_2,F_3)$	$sm(F_1,F_3)$	$sm(F_3,F_1)$
T_{FFS}	1.0000	0.5151	0.5151	0.5151
T^{θ}_{FFS}	1.0000	0.8063	0.8063	0.8063

Table 2. The results of Tanimoto distance measures.

Methods	$dm(F_1,F_2)$	$dm(F_2,F_3)$	$dm(F_1,F_3)$	$dm(F_3,F_1)$
DT_{FFS}	0.0000	0.4849	0.4849	0.4849
DT^{θ}_{FFS}	0.0000	0.1937	0.1937	0.1937

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Table 3. The results of weighted Tanimoto similarity measures.

Measures	$sm(F_1, F_2)$	$sm(F_2, F_3)$	$sm(F_1, F_3)$	$sm(F_3, F_1)$
T^{ω}_{FFS}	1.0000	0.6086	0.6086	0.6086
$T^{\theta\omega}_{FFS}$	1.0000	0.8422	0.8422	0.8422

Table 4. The results of weighted Tanimoto distance measures.

Measures	$dm(F_1, F_2)$	$dm(F_2, F_3)$	$dm(F_1, F_3)$	$dm(F_3, F_1)$
DT^{ω}_{FFS}	0.0000	0.3914	0.3914	0.3914
$DT^{\theta\omega}_{FFS}$	0.0000	0.1578	0.1578	0.1578

According to the results above, we can find that when $F_1 = F_2$, the Tanimoto measures between F_1andF_2 , $T_{FFS}(F_1, F_2) = 1$, $DT_{FFS}(F_1, F_2) = 0$, which satisfy the Property (3) in Definition 11 and the Property (3) in Definition 15 .Besides, $T_{FFS}(F_1, F_3) = T_{FFS}(F_3, F_1)$ and $DT_{FFS}(F_1, F_3) = DT_{FFS}(F_3, F_1)$, which satisfy the Property (2) in Definition 11 and the Property (2) in Definition 15.

Example 2. *The FFSs F_1 and F_2 are found in Z, where $F_1 = \{\langle z, \rho, \sigma \rangle\}$, $F_2 = \{\langle z, \sigma, \rho \rangle\}$. In the present example, the parameters ρ and σ are utilized to denote the membership degree and non-membership degree of F_1 and F_2, respectively. The values of ρ and σ fall within the interval $[0, 1]$, and these parameters satisfy condition $0 \leq \rho^3 + \sigma^3 \leq 1$. As per the proposed measure, Figures 1 and 2, as well as Figures 3 and 4, illustrate the trends of similarity and distance between F_1 and F_2 over a range of parameter values for ρ and σ, respectively.*

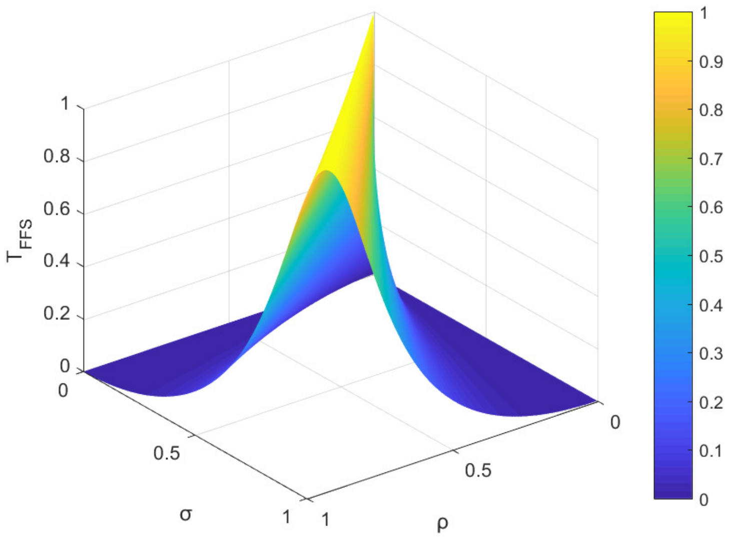

Figure 1. Tanimoto similarity measure between F_1 and F_2.

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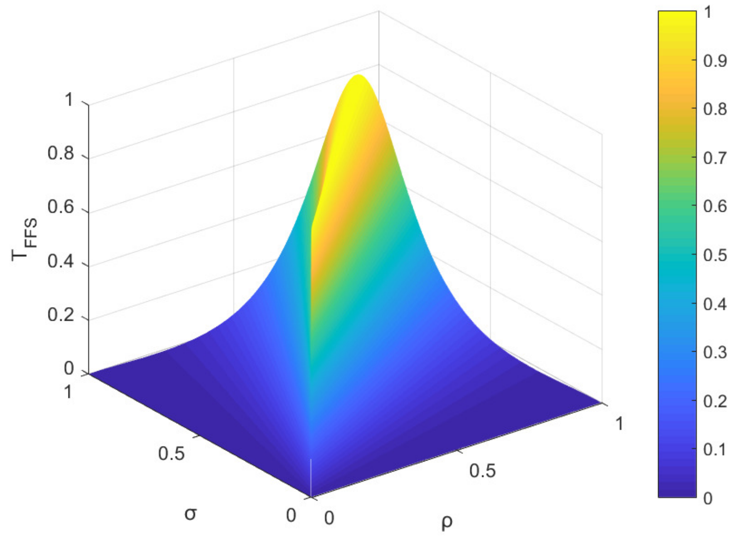

Figure 2. Tanimoto similarity measure between F_1 and F_2.

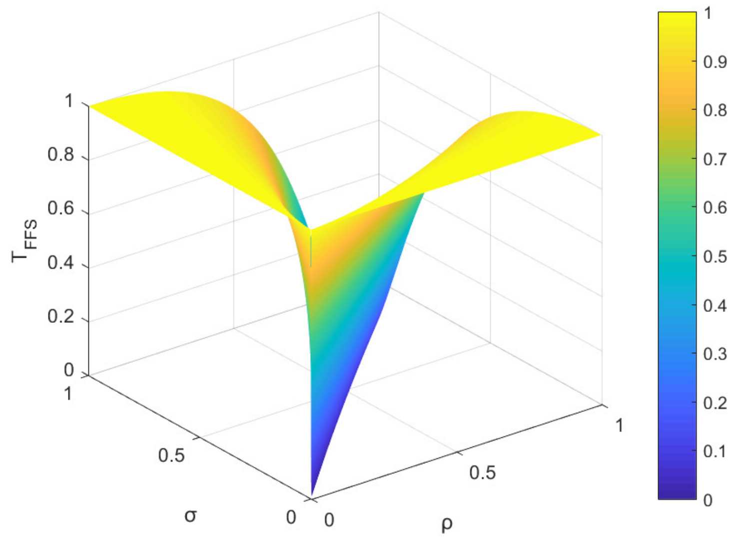

Figure 3. Tanimoto similarity measure between F_1 and F_2.

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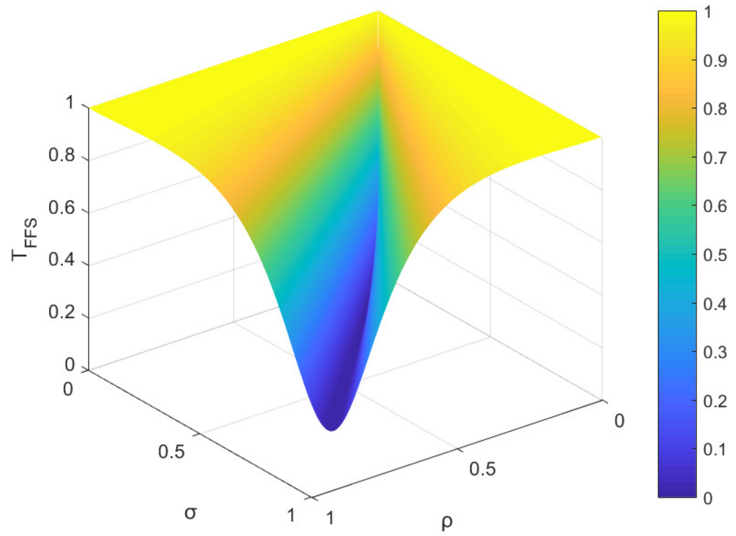

Figure 4. Tanimoto similarity measure between F_1 and F_2.

Based on the observations from Figures 1 and 2, it is evident that the similarity value and the distance value lie within the interval of [0, 1]. Notably, when ρ equals σ, the similarity measure between F_1 and F_2 attains its maximum value of 1, whereas when ρ equals 1 and σ equals 0 (or vice versa), the similarity measure reaches its minimum value of 0. As parameters ρ and σ vary within the range of [0, 1], the similarity measure changes correspondingly within the same range of [0, 1]. The change in distance measure is reversed. These results affirm that the Tanimoto similarity measures and the distance measures satisfy the boundedness Property (1) as defined in Definition 11 and Definition 15, respectively. The overall trend in Figures 1, 2, 3 and 4 aligns with intuitive judgment, indicating that the Tanimoto measures meet the requirements of Property (1) in Definition 11 and Definition 15.

Example 3. *The three FFSs F_1 and F_2 found in Z, denoted as F, G_1 and G_2, and shown in Table 5. It is evident from Table 5 that $G_1 \neq G_2$, implying that the similarity between F and G_1, $sm(F, G_1)$ and the similarity between F and G_2, $sm(F, G_2)$ must be distinct. In Table 6, the outcomes of the Tanimoto similarity measures and those of the cosine similarity measure proposed by Kirişci [71] (indicated as S^1_{FFS}) are compared. Specifically, the Tanimoto similarity measures yield accurate outcomes that align with intuition. Nevertheless, the cosine similarity measure produces counter-intuitive outcomes and may fail to differentiate both FFSs correctly in practice. This highlights the superior precision of our proposed measures and the limitations of the cosine similarity measure. It can be concluded from this example that our proposed measures are more effective and superior to the cosine similarity measure.*

Table 5. FFSs F, G_1 and G_2.

	z_1	z_2
F	$\langle 0.676, 0.446 \rangle$	$\langle 0.428, 0.424 \rangle$
G_1	$\langle 0.985, 0.080 \rangle$	$\langle 0.013, 0.384 \rangle$
G_2	$\langle 0.647, 0.009 \rangle$	$\langle 0.878, 0.336 \rangle$

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Table 6. The results of different similarity measures.

Measures	$sm(F, G_1)$	$sm(F, G_2)$
T_{FFS}	0.404	0.517
T^{θ}_{FFS}	0.642	0.630
S^1_{FFS}	0.655	0.655

Example 4. *There are some FFSs in $Z = \{z_1, z_2\}$, denoted as F^i, G^i_1 and G^i_2 $(i = 1, 2, 3)$, and shown in Tables 7, 8 and 9. It is evident from Tables 7, 8 and 9 that $G^i_1 \neq G^i_2$, implying that $sm(F^i, G^i_1)$ and $sm(F^i, G^i_2)$ must be distinct. In Tables 10, 11 and 12, the outcomes of the Tanimoto similarity measures and those of the similarity measures proposed by Sahoo [72] (indicated as S^2_{FFS}, S^3_{FFS}, S^4_{FFS}, S^5_{FFS} and S^6_{FFS}) are compared. Specifically, the Tanimoto similarity measures yield accurate outcomes that align with intuition. Nevertheless, the similarity measures proposed by Sahoo produce counter-intuitive outcomes and may fail to differentiate both FFSs correctly in practice. This highlights the superior precision of our proposed similarity measures. It can be concluded from this example that our proposed measures are more effective and superior to the measures proposed by Sahoo.*

Table 7. FFSs F^1, G^1_1 and G^1_2.

	z_1	z_2
F^1	$\langle 0.445, 0.579 \rangle$	$\langle 0.310, 0.632 \rangle$
G^1_1	$\langle 0.088, 0.083 \rangle$	$\langle 0.204, 0.059 \rangle$
G^1_2	$\langle 0.572, 0.019 \rangle$	$\langle 0.482, 0.943 \rangle$

Table 8. FFSs F^2, G^2_1 and G^2_2.

	z_1	z_2
F^2	$\langle 0.300, 0.523 \rangle$	$\langle 0.836, 0.094 \rangle$
G^2_1	$\langle 0.742, 0.566 \rangle$	$\langle 0.409, 0.357 \rangle$
G^2_2	$\langle 0.461, 0.752 \rangle$	$\langle 0.140, 0.219 \rangle$

Table 9. FFSs F^3, G^3_1 and G^3_2.

	z_1	z_2
F^3	$\langle 0.486, 0.659 \rangle$	$\langle 0.870, 0.364 \rangle$
G^3_1	$\langle 0.408, 0.806 \rangle$	$\langle 0.567, 0.461 \rangle$
G^3_2	$\langle 0.596, 0.094 \rangle$	$\langle 0.562, 0.355 \rangle$

Table 10. The results of different similarity measures.

Measures	$sm(F^1, G^1_1)$	$sm(F^1, G^1_2)$
T_{FFS}	0.004	0.319
T^{θ}_{FFS}	0.845	0.577
S^2_{FFS}	0.552	0.552
S^3_{FFS}	0.566	0.566

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Table 11. The results of different similarity measures.

Measures	$sm(F^2, G_1^2)$	$sm(F^2, G_2^2)$
T_{FFS}	0.166	0.216
T^{θ}_{FFS}	0.497	0.534
S^4_{FFS}	0.532	0.532
S^5_{FFS}	0.622	0.622

Table 12. The results of different similarity measures.

Measures	$sm(F^3, G_1^3)$	$sm(F^3, G_2^3)$
T_{FFS}	0.541	0.276
T_{FFS}	0.630	0.612
S^6_{FFS}	0.868	0.868

Example 5. *There are three FFSs in $Z = \{z_1, z_2\}$, denoted as F, G_1 and G_2, and shown in Table 13. It is evident from Table 13 that $G_1 \neq G_2$, implying that the distance between F and G_1, $dm(F, G_1)$ and the similarity between F and G_2, $dm(F, G_2)$ must be distinct. In Table 14, the outcomes of the Tanimoto distance measures and those of the Hellinger distance measure and the triangular distance measure proposed by Deng [70] (indicated as d_H and d_T) are compared. Specifically, the Tanimoto distance measures yield accurate outcomes that align with intuition. Nevertheless, the other distance measures produce counter-intuitive outcomes and may fail to differentiate both FFSs correctly in practice. It can be concluded from this example that our proposed measures are more effective and superior to the Hellinger distance measure and the triangular distance measure.*

Table 13. FFSs F, G_1 and G_2.

	z_1	z_2
F	$\langle 0.509, 0.559\rangle$	$\langle 0.541, 0.612\rangle$
G_1	$\langle 0.544, 0.332\rangle$	$\langle 0.748, 0.205\rangle$
G_2	$\langle 0.811, 0.416\rangle$	$\langle 0.569, 0.817\rangle$

Table 14. The results of different distance measures.

Measures	$sm(F, G_1)$	$sm(F, G_2)$
DT_{FFS}	0.525	0.534
DT^{θ}_{FFS}	0.136	0.412
d_H	0.249	0.249
d_T	0.322	0.322

Example 6. *A_i, B_i and C_i are three randomly generated FFSs under case i $(i = 1, 2, \ldots, 50)$. Now we use the proposed similarity measures to measure their similarity and get a difference by subtracting the minimum value from the maximum value of the obtained results, denoted as D_i. We conducted 50 such experiments and averaged the final results to obtain the average difference between the maximum and minimum values of the similarity results in these 50 experiments, denoted as D_{AVG}. Specifically, the D_i and D_{AVG} for FFSs can be calculated as follow:*

$$D_i = \max\left(T_{FFS}\left(A_i, B_i\right), T_{FFS}\left(A_i, C_i\right), T_{FFS}\left(B_i, C_i\right)\right) \quad (31)$$

$$D_{AVG} = \frac{1}{50}\sum_{i=1}^{50} D_i \quad (32)$$

. Similarly, we compare the D_is and D_{AVG}s obtained using other measures. Figure 5 presents the frequency of different similarity measures attaining the maximum D_i in these 50 experiments, while Figure 6 depicts the

D_{AVG}s of different similarity measures across these 50 experiments. We conducted the same experiment using distance measures and the results are presented in Figures 7 and 8.

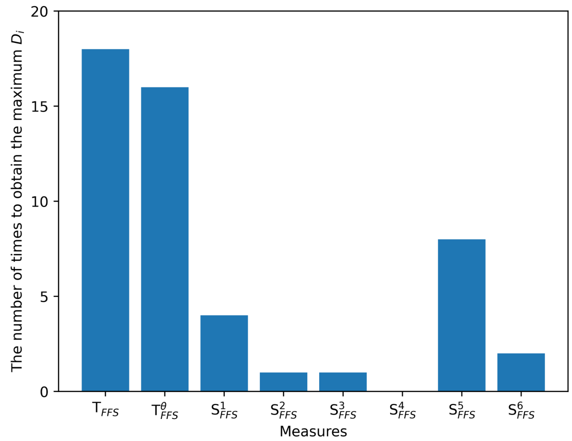

Figure 5. The frequency of different similarity measures attaining the maximum D_i.

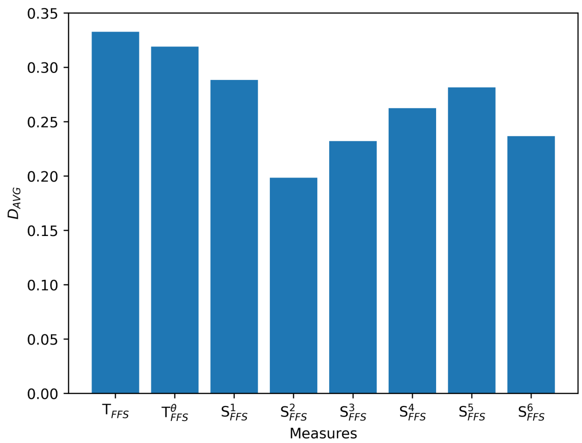

Figure 6. The average difference between the maximum and minimum values of each similarity measure.

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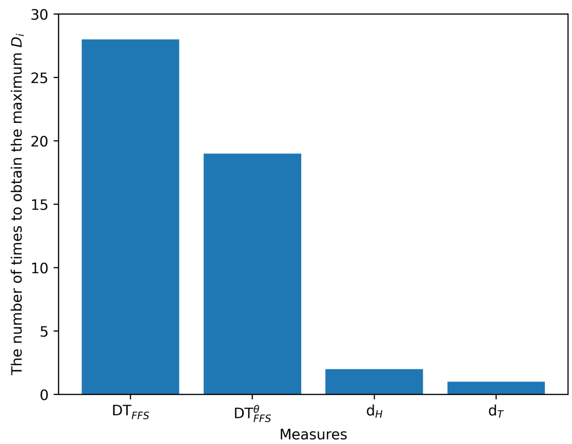

Figure 7. The frequency of different distance measures attaining the maximum D_i.

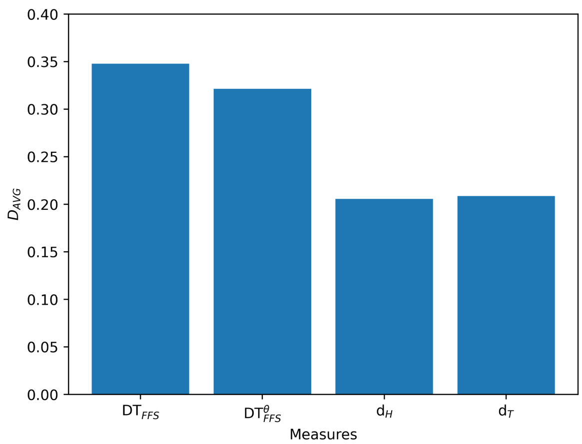

Figure 8. The average difference between the maximum and minimum values of each distance measure.

Figure 5 illustrates that the Tanimoto similarity measures tend to maximize differences of the similarity between different FFSs while Figure 6 illustrates that the Tanimoto similarity reveals that they produce the maximum average differences between the maximum and minimum similarity values. These findings suggest that our proposed similarity measures generate similarity scores with greater variation, thereby endowing the Tanimoto similarity measures with greater ability to discern differences across various levels of FFSs. Furthermore, it can exhibit better performance in discriminating FFSs with high similarity. The heightened differences in similarity scores assigned to different FFSs can bolster the trustworthiness of FFSs classification and facilitate more confident decision-making. However, S1 and S2 exhibit the smaller values of in Figure 5 and 6, indicating a tendency to assign similar values to diverse samples, potentially resulting in greater hesitation when making decisions within the same environment, thereby impeding efficient and confident decision-making. These outcomes underscore the superiority of our proposed similarity measures over the extant ones under scrutiny. Similarly, we performed the identical analysis on distance measures and achieved the same outcomes, as depicted

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in Figure 7 and 8, corroborating the superiority of our proposed distance measures. These properties will be further demonstrated by additional instances in the future examples.

4. Applications

This section presents two models that are designed to tackle pattern recognition, medical diagnosis and MADM based on the proposed measures. To substantiate the efficiency of the proposed models, a series of experiments comparing with the existing measures were conducted.

4.1. A novel model for pattern recognition and medical diagnosis

Given a set of attributes $Z = \{z_1, z_2, \ldots, z_n\}$, we aim to classify the test samples $S = \{S_1, S_2, \ldots, S_m\}$ based on k patterns $F = \{F_1, F_2, \ldots, F_k\}$. The patterns are represented by FFSs, denoted as $F_i = \{\langle z_l, \rho_{F_i}, \sigma_{F_i}\rangle : z_l \in Z\}(i = 1, 2, \ldots, k)$, while the test samples are expressed as FFSs, denoted as $S_j = \{\left\langle z_l, \rho_{S_j}, \sigma_{S_j}\right\rangle : z_l \in Z\}(j = 1, 2, \ldots, m)$. Our objective is to accurately classify the test samples according to the given patterns. The recognition process is outlined below:

Step 1 Calculate the Tanimoto similarity(or distance) between S_j $(j = 1, 2, \ldots, m)$ and F_i $(i = 1, 2, \ldots, k)$.

Step 2 Obtain the maximum Tanimoto similarity $s\left(F_o, S_j\right)$ using equation 33 or the minimum Tanimoto distance $d\left(F_o, S_j\right)$ using equation 34:

$$s\left(F_o, S_j\right) = \max\{T_{FFS}(F_i, S_j)\} \tag{33}$$

$$d\left(F_o, S_j\right) = \min\{DT_{FFS}(F_i, S_j)\} \tag{34}$$

Step 3 If any pattern F_o has the highest Tanimoto similarity between S_i, then, S_i and F_o belong to the same category:

$$o = \arg\max\{s\left(F_o, S_j\right)\}, S_j \to F_o \tag{35}$$

If distance measure is used as the standard of measure, then the following form would be applied:

$$o = \arg\min\{d\left(F_o, S_j\right)\}, S_j \to F_o \tag{36}$$

Example 7 ([70])**.** *This example pertains to the pattern recognition of unknown samples, where three known sample categories are represented by FFS $F_i(i = 1, 2, 3)$ and the universe of discourse consists of their attributes $z_i(i = 1, 2, 3)$. Specifically,*

$$F_1 = \{\langle z_1, 0.8, 0.3\rangle, \langle z_2, 0.7, 0.2\rangle, \langle z_3, 0.6, 0.4\rangle\},$$
$$F_2 = \{\langle z_1, 0.9, 0.2\rangle, \langle z_2, 1.0, 0.0\rangle, \langle z_3, 0.8, 0.1\rangle\},$$
$$F_3 = \{\langle z_1, 0.2, 0.7\rangle, \langle z_2, 0.4, 0.6\rangle, \langle z_3, 0.7, 0.5\rangle\}.$$

There exists a sample S with unknown category, defined as:

$$S = \{\langle z_1, 0.8, 0.1\rangle, \langle z_2, 0.9, 0.3\rangle, \langle z_3, 0.9, 0.0\rangle\},$$

and the proposed Tanimoto measures are employed to determine the category of the unknown sample S. The recognition process is as follows:

Step 1 *Calculate the Tanimoto similarity(or distance) between* $\mathrm{F}_i\ (i = 1,2,3)$ *and S:*

$$\begin{aligned} T_{FFS}\,(F_1,S) &= 0.665 \\ T_{FFS}\,(F_1,S) &= 0.894 \\ T_{FFS}\,(F_1,S) &= 0.238 \\ DT_{FFS}\,(F_1,S) &= 0.335 \\ DT_{FFS}\,(F_1,S) &= 0.106 \\ DT_{FFS}\,(F_1,S) &= 0.762 \end{aligned}$$

Step 2 *Obtain the maximum Tanimoto similarity* $s\left(F_o,S_j\right)$ *using equation 33 or the minimum Tanimoto distance* $d\left(F_o,S_j\right)$ *using equation 34:*

$$\begin{aligned} s\,(F_2,S) &= 0.894 \\ d\,(F_2,S) &= 0.1055 \end{aligned}$$

Step 3 *According to the equation 35 or equation 36, S and* F_2 *belongs to the same pattern.*

A comparative analysis is conducted with the results obtained by the existing similarity measures and distance measures mentioned in Section 2. The results are presented in Table 15 and Table 16.

Table 15. The results of different similarity measures.

Measures	$sm(F_1,S)$	$sm(F_2,S)$	$sm(F_3,S)$	Classification
T_{FFS}	0.665	0.894	0.238	F_2
T^{θ}_{FFS}	0.664	0.842	0.440	F_2
S^1_{FFS}	0.707	0.847	0.564	F_2
S^2_{FFS}	0.716	0.639	0.502	F_1
S^3_{FFS}	0.716	0.715	0.464	F_1
S^4_{FFS}	0.685	0.763	0.482	F_2
S^5_{FFS}	0.657	0.803	0.390	F_2
S^6_{FFS}	0.890	0.892	0.651	F_2

Table 16. The results of different distance measures.

Measures	$dm(F_1,S)$	$dm(F_2,S)$	$dm(F_3,S)$	Classification
DT_{FFS}	0.335	0.106	0.762	F_2
DT^{θ}_{FFS}	0.336	0.158	0.560	F_2
d_H	0.228	0.123	0.474	F_2
d_T	0.293	0.159	0.547	F_2

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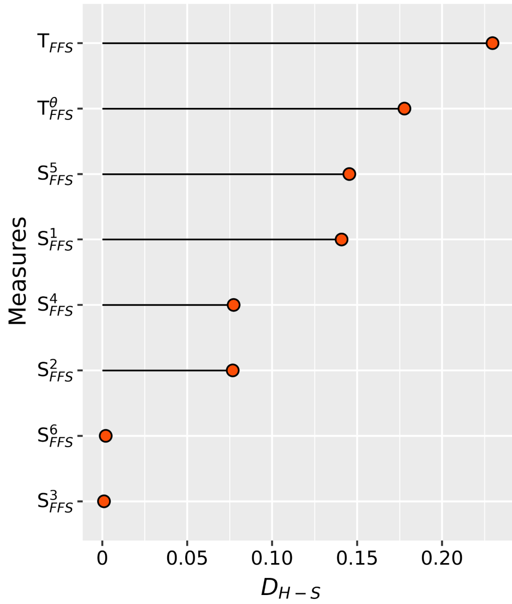

Figure 9. Difference between the highest and second-highest values obtained by different similarity measures.

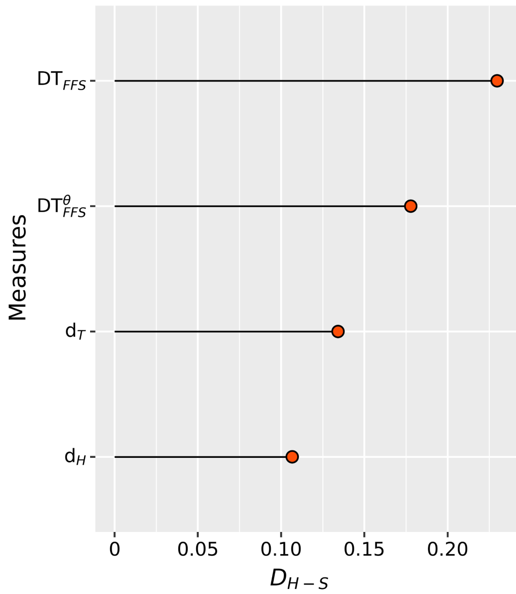

Figure 10. Difference between the highest and second-highest values obtained by different distance measures.

Based on the principle of minimum distance, it was determined that sample S is most similar to sample F_2, which is consistent with the findings in reference [70]. It is worth noting that other distance measures have also produced similar recognition results. Furthermore, based on the principle of maximum similarity, similarity measures S^1_{FFS}, S^4_{FFS}, S^5_{FFS} and S^6_{FFS} yielded the same recognition results, classifying sample S into

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F_2. Conversely, the results of S^2_{FFS} and S^3_{FFS} are different, as they classify sample S as F_1. Therefore, the accuracy of S^2_{FFS} and S^3_{FFS} remains to be discussed. These findings effectively demonstrates the efficacy of our proposed Tanimoto similarity measures and distance measures. Moreover, the difference between the highest similarity scores and second-highest similarity scores for each similarity measure was calculated which are record as D_{H-S}s. For instance, the D_{H-S} for T_{FFS} in this example can be calculated as

$$D_{H-S} = 0.894 - 0.238 = 0.656$$

. The D_{H-S}s for the other similarity measures are shown in Figure 9. The results of Tanimoto similarity measure show the largest difference between the highest and second-highest similarity scores, ranking second when considering hesitation. The similarity values between S and F_2, as determined via our proposed similarity measures, are noticeably distinct from those between S and other FFSs. Thus, we can confidently assert that F_2 is a more appropriate choice. However, the similarity values between S and known samples obtained from other similarity measures are more close to each other. Therefore, making decisions based on these similarity measures may lead to greater hesitation. Similarly, we performed the same operation on distance measurement, the D_{H-S}s for each distance measure are calculated and are shown in Figure 10. The results of Tanimoto distance measure show the largest difference between the highest and second-highest similarity scores, ranking second when considering hesitation. Therefore, we can obtain the same superiority conclusion as similarity measures above. These results reinforce the conclusion derived in Example 6, indicating that our proposed similarity measures and distance measures excel in differentiating between samples displaying high levels of similarity.

Example 8. *This example is the pattern recognition of mineral categories. Suppose that there are five typical mixed minerals represented by FFS $F_i (i = 1, 2, 3, 4, 5)$, and each mineral is composed of six basic minerals which form the universe of discourse $z_i (i = 1, 2, 3, 4, 5, 6)$. Our objective is to use the proposed measures to identify the category to which an unknown mixed mineral S belongs. Table ref tab7 shows the known FFSs and unknown S, while Tables 18 and 19 summarize the results obtained from similarity and distance measures, respectively.*

Table 17. Known FFSs and a simple S.

	z_1	z_2	z_3	z_4	z_5	z_6
F_1	$\langle 0.457, 0.284\rangle$	$\langle 0.036, 0.155\rangle$	$\langle 0.838, 0.377\rangle$	$\langle 0.032, 0.993\rangle$	$\langle 0.109, 0.618\rangle$	$\langle 0.266, 0.660\rangle$
F_2	$\langle 0.397, 0.726\rangle$	$\langle 0.073, 0.426\rangle$	$\langle 0.094, 0.308\rangle$	$\langle 0.678, 0.063\rangle$	$\langle 0.480, 0.373\rangle$	$\langle 0.243, 0.084\rangle$
F_3	$\langle 0.329, 0.542\rangle$	$\langle 0.390, 0.945\rangle$	$\langle 0.079, 0.967\rangle$	$\langle 0.475, 0.880\rangle$	$\langle 0.059, 0.763\rangle$	$\langle 0.965, 0.175\rangle$
F_4	$\langle 0.814, 0.424\rangle$	$\langle 0.026, 0.097\rangle$	$\langle 0.726, 0.319\rangle$	$\langle 0.415, 0.631\rangle$	$\langle 0.791, 0.289\rangle$	$\langle 0.762, 0.457\rangle$
F_5	$\langle 0.655, 0.629\rangle$	$\langle 0.306, 0.339\rangle$	$\langle 0.932, 0.013\rangle$	$\langle 0.266, 0.988\rangle$	$\langle 0.225, 0.007\rangle$	$\langle 0.673, 0.841\rangle$
S	$\langle 0.989, 0.199\rangle$	$\langle 0.138, 0.648\rangle$	$\langle 0.783, 0.232\rangle$	$\langle 0.882, 0.111\rangle$	$\langle 0.838, 0.267\rangle$	$\langle 0.370, 0.085\rangle$

Table 18. The results of different similarity measures.

Measures	$sm(F_1, S)$	$sm(F_2, S)$	$sm(F_3, S)$	$sm(F_4, S)$	$sm(F_5, S)$	**Classification**
T_{FFS}	0.184	0.267	0.104	0.480	0.225	F_4
T^{θ}_{FFS}	0.503	0.594	0.150	0.707	0.387	F_4
S^1_{FFS}	0.509	0.625	0.264	0.738	0.453	F_4
S^2_{FFS}	0.402	0.575	0.372	0.624	0.438	F_4
S^3_{FFS}	0.464	0.647	0.314	0.662	0.415	F_4
S^4_{FFS}	0.476	0.588	0.238	0.668	0.391	F_4
S^5_{FFS}	0.213	0.415	0.141	0.592	0.234	F_4
S^6_{FFS}	0.564	0.747	0.292	0.864	0.554	F_4

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Table 19. The results of different distance measures.

Measures	$dm(F_1, S)$	$dm(F_2, S)$	$dm(F_3, S)$	$dm(F_4, S)$	$dm(F_5, S)$	Classification
DT_{FFS}	0.816	0.733	0.896	0.520	0.775	F_4
DT^{θ}_{FFS}	0.497	0.406	0.850	0.293	0.613	F_4
d_H	0.513	0.371	0.607	0.305	0.505	F_4
d_T	0.559	0.433	0.679	0.368	0.565	F_4

Table 18 reveals that the Tanimoto similarity of mineral S and F_4 is the highest, suggesting that mineral S belongs to F_4. This is consistent with the results of other similarity measures. In addition, Table 19 shows that the Tanimoto distance between S and F_4 is the smallest, indicating that mineral S is the closest to F_4. These findings align with the results obtained from other distance measures. The results of our experiments provide evidence for the effectiveness of the proposed measures.

Example 9 ([75]). *Assuming the presence of four patients, namely Ragu, Mathi, Velu, and Karthi, denoted by $I = \{I_1, I_2, I_3, I_4\}$, and exhibiting symptoms including Headache, Acidity, Burning eyes, Back pain, and Depression, represented as $Z = \{Z_1, Z_2, Z_3, Z_4, Z_5\}$. The set of possible diagnoses is denoted by $D = \{D_1, D_2, D_3, D_4, D_5\}$, and includes: D_1: Stress; D_2: Ulcer; D_3: Vision problem; D_4: Spinal problem; D_5: Blood pressure. The relation $I \to Z$ is expressed by FFSs, as shown in Table 20, while the relation $Z \to D$ is represented by FFSs and listed in Table 21. Every entry in both tables is defined by the FFS, with the values indicating membership degree and non-membership degree, respectively. The proposed similarity and distance measures are employed to evaluate the similarity and distance between each patient and potential diagnosis. Based on the principle of maximum similarity or minimum distance, each patient is diagnosed accordingly. Tables 22, 23, and 24 present the similarity measure outcomes and distance of patient I towards each diagnosis D, alongside the ultimate diagnosis results.*

Table 20. Symptomatic characteristic of the patients.

	z_1	z_2	z_3	z_4	z_5
I_1	$\langle 0.9, 0.1\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.2, 0.8\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.2, 0.7\rangle$
I_2	$\langle 0.0, 0.7\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.6, 0.2\rangle$	$\langle 0.2, 0.7\rangle$	$\langle 0.1, 0.2\rangle$
I_3	$\langle 0.7, 0.1\rangle$	$\langle 0.7, 0.1\rangle$	$\langle 0.0, 0.5\rangle$	$\langle 0.1, 0.7\rangle$	$\langle 0.0, 0.6\rangle$
I_4	$\langle 0.5, 0.1\rangle$	$\langle 0.4, 0.3\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.8, 0.2\rangle$	$\langle 0.3, 0.4\rangle$

Table 21. Symptomatic characteristic of the diagnosis.

	z_1	z_2	z_3	z_4	z_5
D_1	$\langle 0.3, 0.0\rangle$	$\langle 0.3, 0.5\rangle$	$\langle 0.2, 0.8\rangle$	$\langle 0.7, 0.3\rangle$	$\langle 0.2, 0.6\rangle$
D_2	$\langle 0.0, 0.6\rangle$	$\langle 0.2, 0.6\rangle$	$\langle 0.2, 0.8\rangle$	$\langle 0.5, 0.0\rangle$	$\langle 0.1, 0.8\rangle$
D_3	$\langle 0.2, 0.2\rangle$	$\langle 0.5, 0.2\rangle$	$\langle 0.1, 0.7\rangle$	$\langle 0.2, 0.6\rangle$	$\langle 0.2, 0.8\rangle$
D_4	$\langle 0.2, 0.8\rangle$	$\langle 0.1, 0.5\rangle$	$\langle 0.7, 0.0\rangle$	$\langle 0.1, 0.0\rangle$	$\langle 0.2, 0.7\rangle$
D_5	$\langle 0.2, 0.8\rangle$	$\langle 0.0, 0.7\rangle$	$\langle 0.2, 0.8\rangle$	$\langle 0.1, 0.8\rangle$	$\langle 0.8, 0.1\rangle$

Table 22. Diagnostic results of the Tanimoto similarity measures.

Measures	$sm(I_i, D_1)$	$sm(I_i, D_2)$	$sm(I_i, D_3)$	$sm(I_i, D_4)$	$sm(I_i, D_5)$	Classification
I_1	0.588	0.472	0.447	0.206	0.209	D_1
I_2	0.208	0.314	0.244	0.501	0.439	D_4
I_3	0.300	0.176	0.470	0.167	0.234	D_3
I_4	0.421	0.176	0.270	0.124	0.090	D_1

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Table 23. Diagnostic results of the Tanimoto distance measures.

Measures	$dm(I_i, D_1)$	$dm(I_i, D_2)$	$dm(I_i, D_3)$	$dm(I_i, D_4)$	$dm(I_i, D_5)$	Classification
I_1	0.412	0.528	0.553	0.794	0.791	D_1
I_2	0.792	0.686	0.756	0.499	0.561	D_4
I_3	0.700	0.824	0.530	0.833	0.766	D_3
I_4	0.579	0.824	0.730	0.876	0.910	D_1

Table 24. The results of different measures.

Measures	I_1	I_2	I_3	I_4
T_{FFS}	Stress	Spinal problem	Vision problem	Stress
T^{θ}_{FFS}	Stress	Spinal problem	Vision problem	Stress
DT_{FFS}	Stress	Spinal problem	Vision problem	Stress
DT^{θ}_{FFS}	Stress	Spinal problem	Vision problem	Stress
Xiao and Ding	Stress	Spinal problem	Vision problem	Stress
Zhou	Stress	Spinal problem	Vision problem	Stress
Deng	Stress	Spinal problem	Vision problem	Stress

Based on the findings presented in Tables 22 and 23, it is observed that I_1 exhibits the highest Tanimoto similarity measure and the smallest Tanimoto distance measure towards D_1; I_2 displays the highest Tanimoto similarity measure and the smallest Tanimoto distance measure towards D_4; I_3 demonstrates the highest Tanimoto similarity measure and the smallest Tanimoto distance measure towards D_3; and I_4 showcases the highest Tanimoto similarity measure and the smallest Tanimoto distance measure towards D_1. Thus, we can conclude that Ragu is diagnosed with stress, Mathi with spinal problems, Velu with vision problems, and Karthi with stress.

In order to validate the effectiveness of our proposed measures, a comparative analysis was performed against other techniques, and the outcomes have been summarized in Table 24. It is observed from Table 24 that our proposed measures provide diagnostic outcomes that are consistent with those obtained using Xiao and Ding's method [75], and Deng's method [70] indicating the potential of our measures to address the medical diagnosis problem. The experimental results lend support to the practicability of our proposed similarity and distance measures.

4.2. A novel model for MADM

Suppose that $I = \{I_1, I_2 \ldots, I_m\}$ is a discrete set of alternatives, and $A = \{A_1, A_2 \ldots, A_n\}$ is the set of attributes, $\omega = \{\omega_1, \omega_2 \ldots, \omega_n\}$ is the weighting vector of the attribute $A_j (j = 1, 2, \ldots, n)$, where $\omega > 0$, $\sum_{j=1}^{n} w_j = 1$. Suppose that $R = (\rho_{ij}, \sigma_{ij})_{m\times n}$ is the Fermaten fuzzy matrix, where ρ_{ij} indicates the degree that the alternative I_i satisfies the attribute A_j and σ_{ij} indicates the degree that the alternative I_i does not satisfy the attribute A_j, $\rho_{ij} \subset [0,1]$, $\sigma_{ij} \subset [0,1]$, $(\rho_{ij})^3 + (\sigma_{ij})^3 \leq 1, i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. The proposed model is described below:

Step 1: Defining the Fermatean fuzzy positive ideal solution I^+:

$$(\rho^+, \sigma^+) = ((\rho_1^+, \sigma_1^+), (\rho_2^+, \sigma_2^+), \ldots, (\rho_j^+, \sigma_j^+)) = ((1,0), (1,0), \ldots (1,0)) \tag{37}$$

When an attribute A_j is a negative influence, we define the positive ideal solution as $(0,1)$.

Step 2: Calculating the weighted Tanimoto similarity measures(or the Tanimoto distance measures) between $I_i (i = 1, 2, \ldots, m)$ and I^+ as follows:

$$T^{\omega}_{FFS}(I_i, I^+) = \sum_{i=1}^{n} \frac{\omega_i(\rho_{ij}^3 (\rho_j^+)^3 + \sigma_{ij}^3 (\sigma_j^+)^3)}{\rho_{ij}^6 + \sigma_{ij}^6 + (\rho_j^+)^6 + (\sigma_j^+)^6 - \rho_{ij}^3 (\rho_j^+)^3 - \sigma_{ij}^3 (\sigma_j^+)^3} \tag{38}$$

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$$T^{\theta\omega}_{FFS}(I_i, I^+) = \sum_{i=1}^{n} \frac{\omega_i(\rho^3_{ij}(\rho^+_j)^3 + \sigma^3_{ij}(\sigma^+_j)^3 + \theta^3_{ij}(\theta^+_j)^3)}{\rho^6_{ij} + \sigma^6_{ij} + \theta^6_{ij} + (\rho^+_j)^6 + (\sigma^+_j)^6 + (\theta^+_j)^6 - \rho^3_{ij}(\rho^+_j)^3 - \sigma^3_{ij}(\sigma^+_j)^3 - \theta^3_{ij}(\theta^+_j)^3} \quad (39)$$

or

$$DT^{\omega}_{FFS}(I_i, I^+) = 1 - \sum_{i=1}^{n} \frac{\omega_i(\rho^3_{ij}(\rho^+_j)^3 + \sigma^3_{ij}(\sigma^+_j)^3)}{\rho^6_{ij} + \sigma^6_{ij} + (\rho^+_j)^6 + (\sigma^+_j)^6 - \rho^3_{ij}(\rho^+_j)^3 - \sigma^3_{ij}(\sigma^+_j)^3} \quad (40)$$

$$DT^{\theta\omega}_{FFS}(I_i, I^+) = 1 - \sum_{i=1}^{n} \frac{\omega_i(\rho^3_{ij}(\rho^+_j)^3 + \sigma^3_{ij}(\sigma^+_j)^3)}{\rho^6_{ij} + \sigma^6_{ij} + (\rho^+_j)^6 + (\sigma^+_j)^6 - \rho^3_{ij}(\rho^+_j)^3 - \sigma^3_{ij}(\sigma^+_j)^3} \quad (41)$$

Step 3: Rank all the alternatives $I_i(i = 1, 2, \ldots, m)$ and select the best one(s) in accordance with the weighted Tanimoto similarity measures or the weighted Tanimoto distance measures. If a alternative exhibits a higher weighted Tanimoto similarity or the smaller weighted Tanimoto distance, it is a more important alternative. If any alternative has the highest weighted Tanimoto similarity value or the smallest Tanimoto distance value, then, it is the most important alternative.

Example 10. *A certain company intends to procure a set of computers from a pool of five alternative model options, denoted as $I_i(i = 1, 2, 3, 4, 5)$. The company has identified four crucial attributes for selection, namely, manufacturing materials, response speed, service life, and after-sales quality $A_j(j = 1, 2, 3, 4, 5)$. These attributes are assigned weights, denoted as $\omega_j(j = 1, 2, 3, 4))$, where $\omega = (0.4, 0.1, 0.2, 0.3)$, which together form the Fermatean fuzzy decision matrix R:*

$$R = \begin{bmatrix} (0.6089, 0.5983) & (0.6236, 0.3782) & (0.3024, 0.3979) & (0.4229, 0.2788) \\ (0.7098, 0.2052) & (0.7238, 0.3782) & (0.6089, 0.2052) & (0.7238, 0.3782) \\ (0.1950, 0.7008) & (0.3224, 0.5780) & (0.4057, 0.5983) & (0.1189, 0.4780) \\ (0.6720, 0.5592) & (0.2213, 0.4780) & (0.4057, 0.4974) & (0.6236, 0.4780) \\ (0.3024, 0.2999) & (0.2213, 0.4780) & (0.5076, 0.2052) & (0.3224, 0.2788) \end{bmatrix}$$

. We propose the use of the MADM model to identify the most suitable solution.

Step 1: *Defining the Fermatean fuzzy positive ideal solution I^+:*

$$\begin{aligned}(\rho^+, \sigma^+) &= ((\rho^+_1, \sigma^+_1), (\rho^+_2, \sigma^+_2), (\rho^+_3, \sigma^+_3), (\rho^+_4, \sigma^+_4)) \\ &= ((1, 0), (1, 0), (1, 0), (1, 0)).\end{aligned}$$

Here, all attributes are considered positive attributes, so they are defined as (1, 0)

Step 2: *Calculating the weighted Tanimoto measures between $I_i(i = 1, 2, \ldots, m)$ and I^+ as Table 25 and Table 26:*

Table 25. The results of Tanimoto similarity measures.

Measures	$I_1 I^+$	$I_2 I^+$	$I_3 I^+$	$I_4 I^+$	$I_5 I^+$
T^{ω}_{FFS}	0.1662	0.4315	0.0274	0.1860	0.4270
$T^{\theta\omega}_{FFS}$	0.1107	0.2934	0.0132	0.1784	0.0280

Table 26. The results of Tanimoto distance measures.

Measures	$I_1 I^+$	$I_2 I^+$	$I_3 I^+$	$I_4 I^+$	$I_5 I^+$
DT^{ω}_{FFS}	0.8367	0.5620	0.9799	0.7489	0.9477
$DT^{\theta\omega}_{FFS}$	0.8893	0.7066	0.9867	0.8216	0.9720

Step 3: *We can find that the weighted similarity between I_2 and I^+ is the highest, similarly, the weighted distance between I_2 and I^+ is the smallest, so we choose I_2 the best alternative.*

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Furthermore, we gained the same selection result by the TOPSIS method proposed by Murat [71], and the calculation results are shown in Table 27. The closeness index γ and the positive ideal solution obtained by Murat U^+ are utilized in this study. As the γ gets smaller, I_2 is taken as the best alternative.

Table 27. The results of Murat.

Methods	I_1U^+	I_2U^+	I_3U^+	I_4U^+	I_5U^+
γ	0.5121	0.4412	0.5588	0.4913	0.5165

Example 11. *At present, there are five students $I_i(i = 1,2,3,4,5)$ available, from whom a certain company needs to select two interns. A set of selection criteria $A_j(j = 1,2,3,4,5)$ has been developed, which considers the students' academic performance, competition participation, school activities, mastery of professional knowledge related to the position and violations of discipline on campus. However, violations of discipline on campus are not considered positive attributes. Weights $\omega = (0.4, 0.2, 0.1, 0.2, 0.1)$ have been assigned to each attribute, resulting in the formation of a Fermatean fuzzy decision matrix R:*

$$R = \begin{bmatrix} (0.471,0.939) & (0.806,0.269) & (0.132,0.819) & (0.348,0.977) & (0.547,0.297) \\ (0.227,0.538) & (0.181,0.403) & (0.119,0.125) & (0.440,0.387) & (0.469,0.383) \\ (0.332,0.204) & (0.646,0.356) & (0.779,0.375) & (0.457,0.400) & (0.941,0.124) \\ (0.850,0.121) & (0.974,0.195) & (0.512,0.904) & (0.653,0.048) & (0.605,0.273) \\ (0.900,0.037) & (0.19,0.148) & (0.221,0.607) & (0.576,0.039) & (0.881,0.321) \end{bmatrix}$$

. The proposed MDAM model is being utilized for selecting the most suitable interns:

Step 1: *Defining the Fermatean fuzzy positive ideal solution I^+:*

$$\begin{aligned}(\rho^+,\sigma^+) &= ((\rho_1^+,\sigma_1^+),(\rho_2^+,\sigma_2^+),(\rho_3^+,\sigma_3^+),(\rho_4^+,\sigma_4^+)),(\rho_5^+,\sigma_5^+) \\ &= ((1,0),(1,0),(1,0),(1,0),(0,1)).\end{aligned}$$

Here, campus disciplinary behavior is not considered a positive factor and is therefore defined as (0, 1).

Step 2: *Calculating the weighted Tanimoto measures between $I_i(i = 1,2,\dots,m)$ and I^+ as Table 28 and Table 29:*

Table 28. The results of Tanimoto similarity measures.

Measures	I_1I^+	I_2I^+	I_3I^+	I_4I^+	I_5I^+
T^{ω}_{FFS}	0.1733	0.0303	0.1658	0.6012	0.4134
$T^{\theta\omega}_{FFS}$	0.1419	0.0171	0.1102	0.5196	0.3621

Table 29. The results of Tanimoto distance measures.

Measures	I_1I^+	I_2I^+	I_3I^+	I_4I^+	I_5I^+
DT^{ω}_{FFS}	0.8267	0.9697	0.8342	0.3988	0.5866
$DT^{\theta\omega}_{FFS}$	0.8581	0.9829	0.8898	0.4804	0.6379

Step 3: *Table 30 presents the sorted similarity measures and distance measures based on the obtained data, in accordance with the principles of maximum similarity and minimum distance. Based on the principles of maximum similarity and minimum distance, we choose I_4 and I_5 as the optimal choices.*

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Table 30. Ranking of Tanimoto measures results.

Measures	Ranking orders	Optimal choices
T^{ω}_{FFS}	$I_4 > I_5 > I_2 > I_1 > I_3$	$I_4\&I_5$
$T^{\theta\omega}_{FFS}$	$I_4 > I_5 > I_2 > I_1 > I_3$	$I_4\&I_5$
DT^{ω}_{FFS}	$I_3 > I_1 > I_2 > I_5 > I_4$	$I_4\&I_5$
$DT^{\theta\omega}_{FFS}$	$I_3 > I_1 > I_2 > I_5 > I_4$	$I_4\&I_5$

5. Conclusion

Fermatean fuzzy sets (FFSs) are an effective tool for representing uncertain information. However, the accurate measure of similarity and distance between two FFSs remains a long-term research problem. In this paper, we propose several new similarity and distance measures for FFSs based on the Tanimoto similarity concept. Numerical experiments demonstrate that the proposed measures not only avoid the counter-intuitive situations of many existing measures but also obtain more obvious results in distinguishing FFSs. Compared with some existing FFSs measures, the proposed measures are more effective and superior. We also applied our proposed measures to pattern recognition, medical diagnosis, and multiple-attribute decision-making tasks in Fermatean fuzzy environments, and obtained effective and reasonable results.Therefore, it can be easily used to measure feature similarity and support decision-making processes in complex and uncertain scenarios. In our future research, we plan to explore the full potential of our proposed similarity and distance measures by applying it to various problems in Fermatean fuzzy environments. By doing so, we aim to enhance its impact and validate its effectiveness in real-life scenarios. This will demonstrate its capability in providing practical solutions for complex and uncertain problems.

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