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Generalized Reynolds Operators on Hom-Lie Triple Systems

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Abstract: Uchino first initiated the study of generalized Reynolds operators on associative algebras. Recently, related research has become a hot topic. In this paper, we first introduce the notion of generalized Reynolds operators on Hom-Lie triple systems associated to a representation and a 3-cocycle. Then, we develop cohomology of generalized Reynolds operators on Hom-Lie triple systems with coefficients in a suitable representation. As applications, we use the first cohomology group to classify linear deformations and we study the obstruction class of an extendable order n deformation. Finally, we introduce and investigate Hom-NS-Lie triple system as the underlying structure of generalized Reynolds operators on Hom-Lie triple systems.

Keywords: Hom-Lie triple system; generalized Reynolds operator; cohomology; deformation; Hom-NS-Lie triple system

MSC: 17A30; 17B38; 17B56; 17B61

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1. Introduction

Lie triple system first appeared in Cartan's work [4] on Riemannian geometry. Since then, Jacobson [14,15] studied Lie triple systems from Jordan theory and quantum mechanics. Lie triple systems extend the classical theory of Lie algebras and Lie groups by introducing a trilinear product, capturing the interplay between three elements. After that, Lie triple systems have found applications in diverse fields such as quantum mechanics, differential geometry and numerical analysis of differential equations. As a Hom-type algebra [11] generalization of Lie triple system, Hom-Lie triple system was introduced by Yau in [29]. Further, Ma et al. [20] established the cohomology, central extensions and deformations of Hom-Lie triple systems. More research on Hom-Lie triple systems have been developed, see [1,13,24,30] and references cited therein.

The notion of Rota-Baxter operators on associative algebras was introduced by Baxter [3] in his study of the fluctuation theory. Later, the notion of a relative Rota-Baxter operator (also called an $\mathcal{O}$-operator) on a Lie algebra was independently introduced by Kupershmidt [17] to better understand the classical Yang-Baxter equation. Recently, relative Rota-Baxter operators were widely studied, see [2,5,6,18,21,26]. In addition, other operators related to (relative) Rota-Baxter operators are constantly emerging. Among them is Reynolds operator in fluid dynamics, which was originally proposed by Reynolds [23] in his renowned work on fluctuation theory. And then, Kampé de Fériet [16] created the notion of the Reynolds operator as a mathematical subject in general. Inspired by the twisted Poisson structure, Uchino [22] introduced the generalized Reynolds operators on associative algebras, also known as twisted Rota-Baxter operators, and studied its relationship with NS-algebras.

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In recent years, Das [8] introduced the cohomology of generalized Reynolds operators on associative algebras and considers NS-algebras as the underlying structure motivated by Uchino's work. He also developed the notions of generalized Reynolds operators on Lie algebras and NS-Lie algebras in [9]. Generalized Reynolds operators on other algebraic structures have also been widely studied, including 3-Lie algebras [7,12], 3-Hom-Lie algebras [19], Lie-Yamaguti algebras [25], Lie triple systems [10,25] and Lie supertriple systems [27].

Inspired by these works, we propose generalized Reynolds operators on Hom-Lie triple systems, investigate the corresponding cohomology theory which will be used to describe deformations and establish Hom-NS-Lie triple system as the underlying structure in present paper.

The paper is organized as follows. In Section 2, we recall some basic notions and facts about Hom-Lie triple systems. In Section 3, we introduce the notion of generalized Reynolds operators on a Lie triple system and we give some constructions. In Section 4, we develop the cohomologies of generalized Reynolds operators on Hom-Lie triple systems with coefficients in a suitable representation.. In Section 5, we study linear deformations and higher order deformations of generalized Reynolds operators on Hom-Lie triple systems via the cohomology theory. In Section 6, we introduce the notion of Hom-NS-Lie triple systems, which is the underlying algebraic structure of generalized Reynolds operators on Hom-Lie triple systems.

Throughout this paper, $\mathbb{K}$ denotes a field of characteristic zero. All the vector spaces and (multi)linear maps are taken over $\mathbb{K}$.

2. Preliminaires

In this section, we will briefly recall representations and cohomology of Hom-Lie triple systems from [29] and [20].

A Hom-Lie triple system (Hom-L.t.s.) is a triplet $(\mathfrak{L}, [-,-,-], \alpha)$ in which $\mathfrak{L}$ is a vector space together with a trilinear operation $[-,-,-] : \mathfrak{L} \times \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ and a linear map $\alpha : \mathfrak{L} \to \mathfrak{L}$, called the twisted map, satisfying $\alpha([a,b,c]) = [\alpha(a), \alpha(b), \alpha(c)]$ such that

$$[a,b,c] + [b,a,c] = 0, \tag{2.1}$$

$$[a,b,c] + [c,a,b] + [b,c,a] = 0, \tag{2.2}$$

$$[\alpha(a), \alpha(b), [x,y,z]] = [[a,b,x], \alpha(y), \alpha(z)] + [\alpha(x), [a,b,y], \alpha(z)] + [\alpha(x), \alpha(y), [a,b,z]], \tag{2.3}$$

where $x,y,z,a,b \in \mathfrak{L}$. In particular, $(\mathfrak{L}, [-,-,-], \alpha)$ is called regular Hom-Lie triple system if α is an algebraic automorphism of $\mathfrak{L}$.

A homomorphism between two Hom-Lie triple systems $(\mathfrak{L}_1, [-,-,-]_1, \alpha_1)$ and $(\mathfrak{L}_2, [-,-,-]_2, \alpha_2)$ is a linear map $\varphi : \mathfrak{L}_1 \to \mathfrak{L}_2$ satisfying

$$\varphi(\alpha_1(x)) = \alpha_2(\varphi(x)), \quad \varphi([x,y,z]_1) = [\varphi(x), \varphi(y), \varphi(z)]_2, \quad \forall\, x,y,z \in \mathfrak{L}_1.$$

Example 2.1. *Let $(\mathfrak{L}, [-,-], \alpha)$ be a Hom-Lie algebra, then $(\mathfrak{L}, [-,-,-], \alpha)$ is a Hom-Lie triple system, where $[x,y,z] = [[x,y], \alpha(z)], \forall x,y,z \in \mathfrak{L}$.*

Note that Yamaguti [28] introduced the representation and cohomology theory of Lie triple system. Furthermore, based on Yamaguti's work, the authors in [20] developed the representation and cohomology theory of Hom-Lie triple system, which can be described as follows.

Definition 2.2. *[20] A representation of a Hom-Lie triple system $(\mathfrak{L}, [-,-,-], \alpha)$ on a Hom-vector space (V, β) is a bilinear map $\theta : \mathfrak{L} \times \mathfrak{L} \to \mathrm{End}(V)$, such that for all $x,y,a,b \in \mathfrak{L}$*

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$$\theta(\alpha(x),\alpha(y))\circ\beta=\beta\circ\theta(x,y), \tag{2.4}$$
$$\theta(\alpha(a),\alpha(b))\theta(x,y)-\theta(\alpha(y),\alpha(b))\theta(x,a)-\theta(\alpha(x),[y,a,b])\circ\beta+D(\alpha(y),\alpha(a))\theta(x,b)=0, \tag{2.5}$$
$$\theta(\alpha(a),\alpha(b))D(x,y)-D(\alpha(x),\alpha(y))\theta(a,b)+\theta([x,y,a],\alpha(b))\circ\beta+\theta(\alpha(a),[x,y,b])\circ\beta=0, \tag{2.6}$$

where $D(x,y)=\theta(y,x)-\theta(x,y)$. We also denote a representation of $\mathfrak{L}$ on (V,β) by $(V,\beta;\theta)$. In particular, $(V,\beta;\theta)$ is called regular representation of $\mathfrak{L}$ if β is an automorphism of vector space V.

Example 2.3. *Let $(\mathfrak{L},[-,-,-],\alpha)$ be a Hom-Lie triple system. Define bilinear map*
$$\mathcal{R}:\mathfrak{L}\times\mathfrak{L}\to \mathrm{End}(\mathfrak{L}),(a_1,a_2)\mapsto(x\mapsto[x,a_1,a_2]),$$
with $\mathcal{L}(a_1,a_2)(x)=\mathcal{R}(a_2,a_1)x-\mathcal{R}(a_1,a_2)x=[a_1,a_2,x]$. Then, $(\mathfrak{L},\alpha;\mathcal{R})$ is a representation of the Hom-Lie triple system $\mathfrak{L}$, which is called the adjoint representation of $\mathfrak{L}$.

Let $(V,\beta;\theta)$ be a representation of a Hom-Lie triple system $(\mathfrak{L},[-,-,-],\alpha)$. Denote the $(2n+1)$-cochains of $\mathfrak{L}$ with coefficients in representation $(V,\beta;\theta)$ by
$$\begin{aligned}&\mathcal{C}^{2n+1}_{\mathrm{HLts}}(\mathfrak{L},V):=\{f\in \mathrm{Hom}(\mathfrak{L}^{\otimes 2n+1},V)\mid \beta(f(a_1,\cdots,a_{2n+1})=f(\alpha(a_1),\cdots,\alpha(a_{2n+1})),\\ &f(a_1,\cdots,a_{2n-2},a,b,c)+f(a_1,\cdots,a_{2n-2},b,a,c)=0,\ \circlearrowleft_{a,b,c} f(a_1,\cdots,a_{2n-2},a,b,c)=0\}.\end{aligned}$$

For $n\geq 1$, let $\delta:\mathcal{C}^{2n-1}_{\mathrm{HLts}}(\mathfrak{L},V)\to\mathcal{C}^{2n+1}_{\mathrm{HLts}}(\mathfrak{L},V)$ be the corresponding coboundary operator of the Hom-Lie triple system $(\mathfrak{L},[-,-,-[,\alpha)$ with coefficients in the representation $(V,\beta;\theta)$, More precisely, for $a_1,\cdots,a_{2n+1}\in\mathfrak{L}$ and $f\in\mathcal{C}^{2n-1}_{\mathrm{HLts}}(\mathfrak{L},V)$, as
$$\begin{aligned}&\delta f(a_1,\cdots,a_{2n+1})\\ =&\theta(\alpha^{n-1}(a_{2n}),\alpha^{n-1}(a_{2n+1}))f(a_1,\cdots,a_{2n-1})-\theta(\alpha^{n-1}(a_{2n-1}),\alpha^{n-1}(a_{2n+1}))f(a_1,\cdots,a_{2n-2},a_{2n})\\ &+\sum_{i=1}^{n}(-1)^{i+n}D(\alpha^{n-1}(a_{2i-1}),\alpha^{n-1}(a_{2i}))f(a_1,\cdots,a_{2i-2},a_{2i+1},\cdots,a_{2n+1})\\ &+\sum_{i=1}^{n}\sum_{j=2i+1}^{2n+1}(-1)^{i+n+1}f(\alpha(a_1),\cdots,\alpha(a_{2i-2}),\alpha(a_{2i+1}),\cdots,[a_{2i-1},a_{2i},a_j],\cdots,\alpha(a_{2n+1})).\end{aligned}$$

So $\delta\circ\delta=0$. See [20] for more details.

In particular, for $f\in\mathcal{C}^1_{\mathrm{HLts}}(\mathfrak{L},V)$, f is a 1-cocycle on $(\mathfrak{L},[-,-,-],\alpha)$ with coefficients in $(V,\beta;\theta)$ if $\delta f=0$, i.e.,
$$\theta(a_2,a_3)f(a_1)-\theta(a_1,a_3)f(a_2)+D(a_1,a_2)f(a_3)-f([a_1,a_2,a_3])=0.$$

A 3-cochain $\mathfrak{H}\in\mathcal{C}^3_{\mathrm{HLts}}(\mathfrak{L},V)$ is a 3-cocycle on $(\mathfrak{L},[-,-,-],\alpha)$ with coefficients in $(V,\beta;\theta)$ if $\delta\mathfrak{H}=0$, i.e.,
$$\begin{aligned}&\theta(\alpha(a_4),\alpha(a_5))\mathfrak{H}(a_1,a_2,a_3)-\theta(\alpha(a_3),\alpha(a_5))\mathfrak{H}(a_1,a_2,a_4)-D(\alpha(a_1),\alpha(a_2))\mathfrak{H}(a_3,a_4,a_5)\\ &+D(\alpha(a_3),\alpha(a_4))\mathfrak{H}(a_1,a_2,a_5)+\mathfrak{H}([a_1,a_2,a_3],\alpha(a_4),\alpha(a_5))+\mathfrak{H}(\alpha(a_3),[a_1,a_2,a_4],\alpha(a_5))\\ &+\mathfrak{H}(\alpha(a_3),\alpha(a_4),[a_1,a_2,a_5])-\mathfrak{H}(\alpha(a_1),\alpha(a_2),[a_3,a_4,a_5])=0.\end{aligned}$$

3. Generalized Reynolds operators on Hom-Lie triple systems

In this section, we introduce the notion of generalized Reynolds operators on Hom-Lie triple systems, which can be regarded as the generalization of relative Rota-Baxter operators on Hom-Lie triple systems [18,26] and generalized Reynolds operators on Lie triple systems [10,25]. We give its characterization by a graph and provide some examples.

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Definition 3.1. *(i) Let $(\mathfrak{L}, [-,-,-], \alpha)$ be a Hom-Lie triple system and $(V, \beta; \theta)$ be a representation of $\mathfrak{L}$. A linear operator $R : V \to \mathfrak{L}$ is called a generalized Reynolds operator on Hom-Lie triple system associated to $(V, \beta; \theta)$ and 3-cocycle $\mathfrak{H}$ if R satisfies:*

$$\alpha(Ru) = R\beta(u), \tag{3.1}$$
$$[Ru, Rv, Rw] = R(\theta(Rv, Rw)u + D(Ru, Rv)w - \theta(Ru, Rw)v + \mathfrak{H}(Ru, Rv, Rw)), \tag{3.2}$$

where $u, v, w \in V$.

(ii) A morphism of generalized Reynolds operators from R to R' consists of a pair (η, ζ) of a Hom-Lie triple system morphism $\eta : (\mathfrak{L}, [-,-,-], \alpha) \to (\mathfrak{L}', [-,-,-]', \alpha')$ and a linear map $\zeta : V \to V'$ satisfying

$$\alpha' \circ \eta = \eta \circ \alpha, \quad \beta' \circ \zeta = \zeta \circ \beta, \tag{3.3}$$
$$\eta \circ R = R' \circ \zeta, \tag{3.4}$$
$$\zeta(\theta(a, b)u) = \theta'(\eta(a), \eta(b))\zeta(u), \tag{3.5}$$
$$\zeta(\mathfrak{H}(a, b, c)) = \mathfrak{H}'(\eta(a), \eta(b), \eta(c)), \tag{3.6}$$

for$a, b, c \in \mathfrak{L}, u \in V$.

Remark 3.2. *(i) A generalized Reynolds operator R on Hom-Lie triple system $(\mathfrak{L}, [-,-,-], \alpha)$ with $\alpha = \mathrm{id}$ is nothing but a generalized Reynolds operator R on Lie triple system $(\mathfrak{L}, [-,-,-])$. See [10,25] for more details about generalized Reynolds operators on Lie triple systems.*

(ii) Any relative Rota-Baxter operator (in particular, Rota-Baxter operator of weight 0) on a Hom-Lie triple system is a generalized Reynolds operator with $\mathfrak{H} = 0$. See [18,26] for more details about relative Rota-Baxter operators on Hom-Lie triple systems.

Example 3.3. *Let $(V, \beta; \theta)$ be a representation of a Hom-Lie triple system $(\mathfrak{L}, [-,-,-], \alpha)$. Suppose that $f : \mathfrak{L} \to V$ is an invertible linear map and f satisfies $\beta \circ f = f \circ \alpha$, take $\mathfrak{H} = -\delta f$. Then $R = f^{-1} : V \to \mathfrak{L}$ is a generalized Reynolds operator.*

Example 3.4. *In [13], Hou, Ma and Chen introduced the notion of Nijenhuis operator by the 2-order deformation of Hom-Lie triple system $(\mathfrak{L}, [-,-,-], \alpha)$. More precisely, a linear map $N : \mathfrak{L} \to \mathfrak{L}$ is called a Nijenhuis operator if for all $a, b, c \in \mathfrak{L}$, the following equations hold:*

$$\begin{aligned} N \circ \alpha =& \alpha \circ N, \\ [Na, Nb, Nc] =& N([a, Nb, Nc] + [Na, b, Nc] + [Na, Nb, c]) - N^2([Na, b, c] + [a, Nb, c] + [a, b, Nc]) \\ &+ N^3[a, b, c]. \end{aligned}$$

In this case, the Hom-vector space $(\mathfrak{L}, \alpha)$ carries a new Hom-Lie triple system structure with bracket

$$\begin{aligned} [a, b, c]_N =& [a, Nb, Nc] + [Na, b, Nc] + [Na, Nb, c] - N([Na, b, c] + [a, Nb, c] + [a, b, Nc]) \\ &+ N^2[a, b, c], \quad \forall a, b, c \in \mathfrak{L}. \end{aligned}$$

This deformed Hom-Lie triple system $\mathfrak{L}_N = (\mathfrak{L}, [-,-,-]_N, \alpha)$ has a representation on $(\mathfrak{L}, \alpha)$ by $\theta_N(a, b)c = [c, Na, Nb]$, for $a, b \in \mathfrak{L}_N, c \in \mathfrak{L}$. The map $\mathfrak{H} : \mathfrak{L}_N \times \mathfrak{L}_N \times \mathfrak{L}_N \to \mathfrak{L}, \mathfrak{H}(a, b, c) = -N([Na, b, c] + [a, Nb, c] + [a, b, Nc]) + N^2[a, b, c]$ is a 3-cocycle with coefficients in $(L, \alpha; \theta_N)$. Moreover, the identity map $id : \mathfrak{L} \to \mathfrak{L}_N$ is a generalized Reynolds operator.

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Example 3.5. *Let $(\mathfrak{L}, [-,-,-], \alpha)$ be a Hom-Lie triple system and $(\mathfrak{L}, \alpha; \mathcal{R})$ the adjoint representation. Set the 3-cocycle $\mathfrak{H}(a,b,c) = -[a,b,c]$, for $a,b,c \in \mathfrak{L}$, then, a linear operator $T : \mathfrak{L} \to \mathfrak{L}$ defined by Eqs. (3.1) and (3.2) is called a Reynolds operator on $(\mathfrak{L}, [-,-,-], \alpha)$, more specifically, T satisfies:*

$$\alpha(Ta) = T\alpha(a), \tag{3.7}$$
$$[Ta, Tb, Tc] = T([Ta, Tb, c] + [a, Tb, Tc] + [Ta, b, Tc] - [Ta, Tb, Tc]), \tag{3.8}$$

where $a,b,c \in \mathfrak{L}$.

Example 3.6. *Let $D : \mathfrak{L} \to \mathfrak{L}$ be a derivation on a Hom-Lie triple system $(\mathfrak{L}, [-,-,-], \alpha)$. If $D + \frac{1}{2}id$ is invertible, then $(D + \frac{1}{2}id)^{-1}$ is a Reynolds operator on $(\mathfrak{L}, [-,-,-], \alpha)$.*

Given a 3-cocycle $\mathfrak{H}$ in the cochain complex of $\mathfrak{L}$ with coefficients in V, one can construct the twisted semidirect product Hom-Lie triple system. More precisely, the direct sum $\mathfrak{L} \oplus V$ carries a Hom-Lie triple system structure with the bracket given by

$$[a+u, b+v, c+w]_{\ltimes_{\mathfrak{H}}} = [a,b,c] + D(a,b)w - \theta(a,c)v + \theta(b,c)u + \mathfrak{H}(a,b,c),$$
$$(\alpha \oplus \beta)(a+u) = \alpha(a) + \beta(u), \quad \forall a,b,c \in \mathfrak{L}, \quad u,v,w \in V.$$

We denote this twisted semidirect product Hom-Lie triple system by $\mathfrak{L} \ltimes_{\mathfrak{H}} V$.

Proposition 3.7. *A linear map $R : V \to \mathfrak{L}$ is a generalized Reynolds operator on $\mathfrak{L}$ if and only if the graph of R*

$$Gr(R) = \{Ru + u \mid u \in V\}$$

is a subalgebra of the twisted semidirect product Hom-Lie triple system by $\mathfrak{L} \ltimes_{\mathfrak{H}} V$.

Proof. Let $R : V \to \mathfrak{L}$ be a linear map, then for any $u,v,w \in V$, we have

$$\alpha \oplus \beta(Ru + u) = \alpha \circ R(u) + \beta(u),$$
$$[Ru+u, Rv+v, Rw+w]_{\ltimes_{\mathfrak{H}}} = [Ru, Rv, Rw] + D(Ru, Rv)w - \theta(Ru, Rw)v + \theta(Rv, Rw)u + \mathfrak{H}(Ru, Rv, Rw),$$

which implies that the graph $Gr(R)$ is a subalgebra of the twisted semidirect product Hom-Lie triple system $\mathfrak{L} \ltimes_{\mathfrak{H}} V$ if and only if R satisfies Eqs. (3.1) and (3.2), which means that R is a generalized Reynolds operator. □

Since $Gr(R)$ is isomorphic to V as a vector space. Define a trilinear operation on V by

$$[u,v,w]_R = D(Ru, Rv)w - \theta(Ru, Rw)v + \theta(Rv, Rw)u + \mathfrak{H}(Ru, Rv, Rw), \ \forall\, u,v,w \in V.$$

By Proposition 3.7, we get $(V, [-,-,-]_R, \beta)$ is a Hom-Lie triple system.

4. Cohomologies of generalized Reynolds operators on Hom-Lie triple systems

In this section, first, we construct a representation of the Hom-Lie triple system $(V, [-,-,-]_R, \beta)$ on the Hom-vector space $(\mathfrak{L}, \alpha)$. Then we develop the cohomology of generalized Reynolds operator on the Hom-Lie triple system.

Lemma 4.1. *Let $R : V \to \mathfrak{L}$ be a generalized Reynolds operator on a Hom-Lie triple system $(\mathfrak{L}, [-,-,-], \alpha)$ associated to $(V, \beta; \theta)$ and 3-cocycle $\mathfrak{H}$. For any $u,v \in V$, $a \in \mathfrak{L}$, define $\theta_R : V \otimes V \to$ End$(\mathfrak{L})$ by*

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$$\theta_R(u,v)(a) = [a, Ru, Rv] + R(\theta(a, Rv)u - D(a, Ru)v - \mathfrak{H}(a, Ru, Rv)), \tag{4.1}$$

then $(\mathfrak{L}, \alpha; \theta_R)$ *is a representation of the Hom-Lie triple system* $(V, [-,-,-]_R, \beta)$.

Proof. For any $u, v, s, t \in V, a \in \mathfrak{L}$, note that

$$\begin{aligned}
&D_R(u,v)(a)\\
=&\theta_R(v,u)(a) - \theta_R(u,v)(a)\\
=&[a, Rv, Ru] + R(\theta(a, Ru)v - D(a, Rv)u - \mathfrak{H}(a, Rv, Ru)) - [a, Ru, Rv]\\
&- R(\theta(a, Rv)u - D(a, Ru)v - \mathfrak{H}(a, Ru, Rv))\\
=&[Ru, Rv, a] + R(\theta(Ru, a)v - \theta(Rv, a)u - \mathfrak{H}(Ru, Rv, a)).
\end{aligned} \tag{4.2}$$

Further, we obtain that

$$\begin{aligned}
&\theta_R(\beta(u), \beta(v))\alpha(a)\\
=&[\alpha(a), R\beta(u), R\beta(v)] + R(\theta(\alpha(a), R\beta(v))\beta(u) - D(\alpha(a), R\beta(u))\beta(v) - \mathfrak{H}(\alpha(a), R\beta(u), R\beta(v)))\\
=&[\alpha(a), \alpha(Ru), \alpha(Rv)] + R(\theta(\alpha(a), \alpha(Rv))\beta(u) - D(\alpha(a), \alpha(Ru))\beta(v) - \mathfrak{H}(\alpha(a), \alpha(Ru), \alpha(Rv)))\\
=&\alpha([a, Ru, Rv] + R(\theta(a, Rv)u - D(a, Ru)v - \mathfrak{H}(a, Ru, Rv)))\\
=&\alpha(\theta_R(u,v)(a)),\\
&\theta_R(\beta(u), \beta(v))\theta_R(s,t)a - \theta_R(\beta(t), \beta(v))\theta_R(s,u)a - \theta_R(\beta(s), [t,u,v]_R)\alpha(a) + D_R(\beta(t), \beta(u))\theta_R(s,v)a\\
=&[[a, Rs, Rt], R\beta(u), R\beta(v)] + R\theta([a, Rs, Rt], R\beta(v))\beta(u) - RD([a, Rs, Rt], R\beta(u))\beta(v)\\
&- R\mathfrak{H}([a, Rs, Rt], R\beta(u), R\beta(v)) + [R\theta(a, Rt)s, R\beta(u), R\beta(v)] + R\theta(R\theta(a, Rt)s, R\beta(v))\beta(u)\\
&- RD(R\theta(a, Rt)s, R\beta(u))\beta(v) - R\mathfrak{H}(R\theta(a, Rt)s, R\beta(u), R\beta(v)) - [RD(a, Rs)t, R\beta(u), R\beta(v)]\\
&- R\theta(RD(a, Rs)t, R\beta(v))\beta(u) + RD(RD(a, Rs)t, R\beta(u))\beta(v) + R\mathfrak{H}(RD(a, Rs)t, R\beta(u), R\beta(v))\\
&- [R\mathfrak{H}(a, Rs, Rt), R\beta(u), R\beta(v)] - R\theta(R\mathfrak{H}(a, Rs, Rt), R\beta(v))\beta(u) + RD(R\mathfrak{H}(a, Rs, Rt), R\beta(u))\beta(v)\\
&+ R\mathfrak{H}(R\mathfrak{H}(a, Rs, Rt), R\beta(u), R\beta(v)) - [[a, Rs, Ru], R\beta(t), R\beta(v)] - R\theta([a, Rs, Ru], R\beta(v))\beta(t)\\
&+ RD([a, Rs, Ru], R\beta(t))\beta(v) + R\mathfrak{H}([a, Rs, Ru], R\beta(t), R\beta(v)) - [R\theta(a, Ru)s, R\beta(t), R\beta(v)]\\
&- R\theta(R\theta(a, Ru)s, R\beta(v))\beta(t) + RD(R\theta(a, Ru)s, R\beta(t))\beta(v) + R\mathfrak{H}(R\theta(a, Ru)s, R\beta(t), R\beta(v))\\
&+ [RD(a, Rs)u, R\beta(t), R\beta(v)] + R\theta(RD(a, Rs)u, R\beta(v))\beta(t) - RD(RD(a, Rs)u, R\beta(t))\beta(v)\\
&- R\mathfrak{H}(RD(a, Rs)u, R\beta(t), R\beta(v)) + [R\mathfrak{H}(a, Rs, Ru), R\beta(t), R\beta(v)] + R\theta(R\mathfrak{H}(a, Rs, Ru), R\beta(v))\beta(t)\\
&- RD(R\mathfrak{H}(a, Rs, Ru), R\beta(t))\beta(v) - R\mathfrak{H}(R\mathfrak{H}(a, Rs, Ru), R\beta(t), R\beta(v)) - [\alpha(a), R\beta(s), R[t,u,v]_R]\\
&- R\theta(\alpha(a), R[t,u,v]_R)\beta(s) + RD(\alpha(a), R\beta(s))[t,u,v]_R + R\mathfrak{H}(\alpha(a), R\beta(s), R[t,u,v]_R)\\
&+ [R\beta(t), R\beta(u), [a, Rs, Rv]] + R\theta(R\beta(t), [a, Rs, Rv])\beta(u) - R\theta(R\beta(u), [a, Rs, Rv])\beta(t)\\
&- R\mathfrak{H}(R\beta(t), R\beta(u), [a, Rs, Rv]) + [R\beta(t), R\beta(u), R\theta(a, Rv)s] + R\theta(R\beta(t), R\theta(a, Rv)s)\beta(u)\\
&- R\theta(R\beta(u), R\theta(a, Rv)s)\beta(t) - R\mathfrak{H}(R\beta(t), R\beta(u), R\theta(a, Rv)s) - [R\beta(t), R\beta(u), RD(a, Rs)v]\\
&- R\theta(R\beta(t), RD(a, Rs)v)\beta(u) + R\theta(R\beta(u), RD(a, Rs)v)\beta(t) + R\mathfrak{H}(R\beta(t), R\beta(u), RD(a, Rs)v)\\
&- [R\beta(t), R\beta(u), R\mathfrak{H}(a, Rs, Rv)] - R\theta(R\beta(t), R\mathfrak{H}(a, Rs, Rv))\beta(u) + R\theta(R\beta(u), R\mathfrak{H}(a, Rs, Rv))\beta(t)\\
&+ R\mathfrak{H}(R\beta(t), R\beta(u), R\mathfrak{H}(a, Rs, Rv))\\
=&0.
\end{aligned}$$

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Similarly, we also have

$$\theta_R(\beta(u),\beta(v))D_R(s,t)a - D_R(\beta(s),\beta(t))\theta_R(u,v)a + \theta_R([s,t,u]_R,\beta(v))\alpha(a) + \theta_R(\beta(u),[s,t,v]_R)\alpha(a) = 0.$$

Therefore, $(\mathfrak{L},\alpha;\theta_R)$ is a representation of $(V,[-,-,-]_R,\beta)$. □

Denote by $\delta_R : \mathcal{C}^{2n-1}_{\mathrm{HLts}}(V,\mathfrak{L}) \to \mathcal{C}^{2n+1}_{\mathrm{HLts}}(V,\mathfrak{L})$ the coboundary operator of the Hom-Lie triple system $(V,[-,-,-]_R,\beta)$ with coefficients in the representation $(\mathfrak{L},\alpha;\theta_R)$. More precisely, for $v_1,\cdots,v_{2n+1} \in V$ and $f \in \mathcal{C}^{2n-1}_{\mathrm{HLts}}(V,\mathfrak{L})$, as

$$\begin{aligned}
&\delta_R f(v_1,\cdots,v_{2n+1})\\
=&\theta_R(\beta^{n-1}(v_{2n}),\beta^{n-1}(v_{2n+1}))f(v_1,\cdots,v_{2n-1}) - \theta_R(\beta^{n-1}(v_{2n-1}),\beta^{n-1}(v_{2n+1}))f(v_1,\cdots,v_{2n-2},v_{2n})\\
&+\sum_{i=1}^{n}(-1)^{i+n}D_R(\beta^{n-1}(v_{2i-1}),\beta^{n-1}(v_{2i}))f(v_1,\cdots,v_{2i-2},v_{2i+1},\cdots,v_{2n+1})\\
&+\sum_{i=1}^{n}\sum_{j=2i+1}^{2n+1}(-1)^{i+n+1}f(\beta(v_1),\cdots,\beta(v_{2i-2}),\beta(v_{2i+1}),\cdots,[v_{2i-1},v_{2i},v_j]_R,\cdots,\beta(v_{2n+1})).
\end{aligned}$$

In particular, for $f \in \mathcal{C}^{1}_{\mathrm{HLts}}(V,\mathfrak{L})$, f is a 1-cocycle on $(V,[-,-,-]_R,\beta)$ with coefficients in $(\mathfrak{L},\alpha;\theta_R)$ if $\delta_R f(v_1,v_2,v_3) = 0$, i.e.,

$$\theta_R(v_2,v_3)f(v_1) - \theta_R(v_1,v_3)f(v_2) + D_R(v_1,v_2)f(v_3) - f([v_1,v_2,v_3]_R) = 0$$

and

$$\alpha(f(v_1)) - f(\beta(v_1)) = 0.$$

Proposition 4.2. *Let $R : V \to \mathfrak{L}$ be a generalized Reynolds operator on a Hom-Lie triple system $(\mathfrak{L},[-,-,-],\alpha)$ associated to $(V,\beta;\theta)$ and 3-cocycle $\mathfrak{H}$. If there exist two elements $a,b \in \mathfrak{L}$ such that $\alpha(a) = a, \alpha(b) = b$, we define $\wp(a,b) : V \to \mathfrak{L}$ by*

$$\wp(a,b)u = RD(a,b)\beta^{-1}(u) - [a,b,R\beta^{-1}(u)] + R\mathfrak{H}(a,b,R\beta^{-1}(u)),$$

for any $u \in V$. Then $\wp(a,b)$ is a 1-cocycle on $(V,[-,-,-]_R,\beta)$ with coefficients in $(\mathfrak{L},\alpha;\theta_R)$.

Proof. For any $v_1,v_2,v_3 \in V$, first, obviously $\alpha(\wp(a,b)(v_1)) - \wp(a,b)(\beta(v_1)) = 0$. Next, by Eqs. (3.6), (4.1) and (4.2), we have

$$\begin{aligned}
&\delta_R\wp(a,b)(v_1,v_2,v_3)\\
=&\theta_R(v_2,v_3)\wp(a,b)(v_1) - \theta_R(v_1,v_3)\wp(a,b)(v_2) + D_R(v_1,v_2)\wp(a,b)(v_3) - \wp(a,b)([v_1,v_2,v_3]_R)\\
=&\theta_R(v_2,v_3)RD(a,b)\beta^{-1}(v_1) - \theta_R(v_2,v_3)[a,b,R\beta^{-1}(v_1)] + \theta_R(v_2,v_3)R\mathfrak{H}(a,b,R\beta^{-1}(v_1))\\
&-\theta_R(v_1,v_3)RD(a,b)\beta^{-1}(v_2) + \theta_R(v_1,v_3)[a,b,R\beta^{-1}(v_2)] - \theta_R(v_1,v_3)R\mathfrak{H}(a,b,R\beta^{-1}(v_2))\\
&+D_R(v_1,v_2)RD(a,b)\beta^{-1}(v_3) - D_R(v_1,v_2)[a,b,R\beta^{-1}(v_3)] + D_R(v_1,v_2)R\mathfrak{H}(a,b,R\beta^{-1}(v_3))\\
&-RD(a,b)\beta^{-1}([v_1,v_2,v_3]_R) - [a,b,R\beta^{-1}([v_1,v_2,v_3]_R)] + R\mathfrak{H}(a,b,R\beta^{-1}([v_1,v_2,v_3]_R))\\
=&[RD(a,b)\beta^{-1}(v_1),Rv_2,Rv_3] + R\theta(RD(a,b)\beta^{-1}(v_1),Rv_3)v_2 - RD(RD(a,b)\beta^{-1}(v_1),Rv_2)v_3\\
&-R\mathfrak{H}(RD(a,b)\beta^{-1}(v_1),Rv_2,Rv_3) - [[a,b,R\beta^{-1}(v_1)],Rv_2,Rv_3] - R\theta([a,b,R\beta^{-1}(v_1)],Rv_3)v_2\\
&+RD([a,b,R\beta^{-1}(v_1)],Rv_2)v_3 + R\mathfrak{H}([a,b,R\beta^{-1}(v_1)],Rv_2,Rv_3) + [R\mathfrak{H}(a,b,R\beta^{-1}(v_1)),Rv_2,Rv_3]
\end{aligned}$$

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$$
\begin{aligned}
&+R\theta(R\mathfrak{H}(a,b,R\beta^{-1}(v_1)),Rv_3)v_2-RD(R\mathfrak{H}(a,b,R\beta^{-1}(v_1)),Rv_2)v_3-R\mathfrak{H}(R\mathfrak{H}(a,b,R\beta^{-1}(v_1)),Rv_2,Rv_3)\\
&-[RD(a,b)\beta^{-1}(v_2),Rv_1,Rv_3]-R\theta(RD(a,b)\beta^{-1}(v_2),Rv_3)v_1+RD(RD(a,b)\beta^{-1}(v_2),Rv_1)v_3\\
&+R\mathfrak{H}(RD(a,b)\beta^{-1}(v_2),Rv_1,Rv_3)+[[a,b,R\beta^{-1}(v_2)],Rv_1,Rv_3]+R\theta([a,b,R\beta^{-1}(v_2)],Rv_3)v_1\\
&-RD([a,b,R\beta^{-1}(v_2)],Rv_1)v_3-R\mathfrak{H}([a,b,R\beta^{-1}(v_2)],Rv_1,Rv_3)-[R\mathfrak{H}(a,b,R\beta^{-1}(v_2)),Rv_1,Rv_3]\\
&-R\theta(R\mathfrak{H}(a,b,R\beta^{-1}(v_2)),Rv_3)v_1+RD(R\mathfrak{H}(a,b,R\beta^{-1}(v_2)),Rv_1)v_3+R\mathfrak{H}(R\mathfrak{H}(a,b,R\beta^{-1}(v_2)),Rv_1,Rv_3)\\
&+[Rv_1,Rv_2,RD(a,b)\beta^{-1}(v_3)]+R\theta(Rv_1,RD(a,b)\beta^{-1}(v_3))v_2-R\theta(Rv_2,RD(a,b)\beta^{-1}(v_3))v_1\\
&-R\mathfrak{H}(Rv_1,Rv_2,RD(a,b)\beta^{-1}(v_3))-[Rv_1,Rv_2,[a,b,R\beta^{-1}(v_3)]]-R\theta(Rv_1,[a,b,R\beta^{-1}(v_3)])v_2\\
&+R\theta(Rv_2,[a,b,R\beta^{-1}(v_3)])v_1+R\mathfrak{H}(Rv_1,Rv_2,[a,b,R\beta^{-1}(v_3)])+[Rv_1,Rv_2,R\mathfrak{H}(a,b,R\beta^{-1}(v_3))]\\
&+R\theta(Rv_1,R\mathfrak{H}(a,b,R\beta^{-1}(v_3)))v_2-R\theta(Rv_2,R\mathfrak{H}(a,b,R\beta^{-1}(v_3)))v_1-R\mathfrak{H}(Rv_1,Rv_2,R\mathfrak{H}(a,b,R\beta^{-1}(v_3)))\\
&-RD(a,b)\beta^{-1}(D(Rv_1,Rv_2)v_3)+RD(a,b)\beta^{-1}(\theta(Rv_1,Rv_3)v_2)-RD(a,b)\beta^{-1}(\theta(Rv_2,Rv_3)v_1)\\
&-RD(a,b)\beta^{-1}(\mathfrak{H}(Rv_1,Rv_2,Rv_3))-[a,b,R\beta^{-1}(D(Rv_1,Rv_2)v_3)]+[a,b,R\beta^{-1}(\theta(Rv_1,Rv_3)v_2)]\\
&-[a,b,R\beta^{-1}(\theta(Rv_2,Rv_3)v_1)]-[a,b,R\beta^{-1}(\mathfrak{H}(Rv_1,Rv_2,Rv_3))]+R\mathfrak{H}(a,b,R\beta^{-1}(D(Rv_1,Rv_2)v_3))\\
&-R\mathfrak{H}(a,b,R\beta^{-1}(\theta(Rv_1,Rv_3)v_2))+R\mathfrak{H}(a,b,R\beta^{-1}(\theta(Rv_2,Rv_3)v_1))+R\mathfrak{H}(a,b,R\beta^{-1}(\mathfrak{H}(Rv_1,Rv_2,Rv_3)))\\
=&0.
\end{aligned}
$$

Therefore, $\delta_R\wp(a,b)=0$. □

Definition 4.3. *Let $R:V\to\mathfrak{L}$ be a generalized Reynolds operator on a Hom-Lie triple system $(\mathfrak{L},[-,-,-],\alpha)$ associated to $(V,\beta;\theta)$ and 3-cocycle $\mathfrak{H}$. Define the set of n-cochains by*

$$
\mathcal{C}^n_R(V,\mathfrak{L})=\begin{cases}\mathcal{C}^{2n-1}_{\text{HLts}}(V,\mathfrak{L}), & n\geq 1;\\ \{(a,b)\in\wedge^2\mathfrak{L}\mid \alpha(a)=a,\alpha(b)=b\}, & n=0.\end{cases}
$$

Define $\partial_R:\mathcal{C}^n_R(V,\mathfrak{L})\to\mathcal{C}^{n+1}_R(V,\mathfrak{L})$ by

$$
\partial_R=\begin{cases}\delta_R, & n\geq 1;\\ \wp, & n=0.\end{cases}
$$

By $\delta\circ\delta=0$ and Proposition 4.2, we know that $(\oplus_{n=0}^{+\infty}\mathcal{C}^n_R(V,\mathfrak{g}),\partial)$ is a cochain complex. Denote the set of n-cocycles by $\mathcal{Z}^n_R(V,\mathfrak{L})$, the set of n-coboundaries by $\mathcal{B}^n_R(V,\mathfrak{L})$, and n-th cohomology group by

$$
\mathcal{H}^n_R(V,\mathfrak{L})=\frac{\mathcal{Z}^n_R(V,\mathfrak{L})}{\mathcal{B}^n_R(V,\mathfrak{L})},n\geq 1,
$$

which is taken to be the n-th cohomology group for the generalized Reynolds operator R.

Remark 4.4. *The cohomology theory for generalized Reynolds operators on Hom-Lie triple systems enjoys certain functorial properties. Let $R,R':V\to\mathfrak{L}$ be two generalized Reynolds operators on a Hom-Lie triple system $(\mathfrak{L},[-,-,-],\alpha)$ associated to $(V,\beta;\theta)$ and 3-cocycle $\mathfrak{H}$, and (η,ζ) be a homomorphism from R to R′ in which ζ is invertible. Define linear map $\Phi:\mathcal{C}^n_R(V,\mathfrak{L})\to\mathcal{C}^n_{R'}(V,\mathfrak{L})$ by*

$$
\Phi(f)(v_1,\cdots,v_{2n-1})=\eta(f(\zeta^{-1}(v_1),\cdots,\zeta^{-1}(v_{2n-1}))),
$$

for any $f\in\mathcal{C}^n_R(V,\mathfrak{L})$ and $v_1,\cdots,v_{2n-1}\in V$. Then it is straightforward to deduce that Φ is a cochain map from the cochain complex $(\oplus_{n=1}^{+\infty}\mathcal{C}^n_R(V,\mathfrak{L}),\partial_R)$ to the cochain complex $(\oplus_{n=1}^{+\infty}\mathcal{C}^n_{R'}(V,\mathfrak{L}),\partial_{R'})$. Consequently, it induces a homomorphism Φ^* from the cohomology group $\mathcal{H}^n_R(V,\mathfrak{L})$ to $\mathcal{H}^n_{R'}(V,\mathfrak{L})$.

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5. Deformatons of generalized Reynolds operators on Hom-Lie triple systems

In this section, we study linear deformations and higher order deformations of generalized Reynolds operators on Hom-Lie triple systems via the cohomology theory established in the former section.

First, we use the cohomology constructed to characterize the linear deformations of generalized Reynolds operators on Hom-Lie triple systems.

Definition 5.1. *Let $R : V \to \mathfrak{L}$ be a generalized Reynolds operator on a Hom-Lie triple system $(\mathfrak{L}, [-,-,-], \alpha)$ associated to $(V, \beta; \theta)$ and 3-cocycle $\mathfrak{H}$. A linear deformation of R is a generalized Reynolds operator of the form $R_t = R + tR_1$, where $R_1 : V \to \mathfrak{L}$ is a linear map and t is a parameter with $t^2 = 0$.*

Suppose $R + tR_1$ is a linear deformation of R, direct deduction shows that $R_1 \in \mathcal{C}^1_R(V, \mathfrak{L})$ is a 1-cocycle on $(V, [-,-,-]_R, \beta)$ with coefficients in $(\mathfrak{L}, \alpha; \theta_R)$. So the cohomology class of R_1 defines an element in $\mathcal{H}^1_R(V, \mathfrak{L})$. Furthermore, the 1-cocycle R_1 is called the infinitesimal of the linear deformation R_t of R.

Definition 5.2. *Let $R : V \to \mathfrak{L}$ be a generalized Reynolds operator on a regular Hom-Lie triple system $(\mathfrak{L}, [-,-,-], \alpha)$ associated to regular representation $(V, \beta; \theta)$ and 3-cocycle $\mathfrak{H}$. Two linear deformations $R_t = R + tR_1$ and $R'_t = R + tR'_1$ are called equivalent if there exist two elements $a, b \in \mathfrak{L}$ such that $\alpha(a) = a, \alpha(b) = b$ and the pair $(Id_{\mathfrak{L}} + t\alpha^{-1}(\mathcal{L}(a,b)-), Id_V + t\beta^{-1}(D(a,b)-) + t\beta^{-1}(\mathfrak{H}(a,b,R-)))$ is a homomorphism from R_t to R'_t.*

Suppose R_t and R'_t are equivalent, then Eq. (3.4) yields that

$$(Id_{\mathfrak{L}} + t\alpha^{-1}(\mathcal{L}(a,b)-))R_t u = R'_t(Id_V + t\beta^{-1}(D(a,b)-) + t\beta^{-1}(\mathfrak{H}(a,b,R-)))u, \forall u \in V.$$

which means that

$$\begin{aligned} R_1 u - R'_1 u =& R\beta^{-1}(D(a,b)u) - \alpha^{-1}([a,b,Ru]) + R\beta^{-1}(\mathfrak{H}(a,b,Ru)) \\ =& RD(a,b)\beta^{-1}(u) - [a,b,R\beta^{-1}(u)] + R\mathfrak{H}(a,b,R\beta^{-1}(u)). \end{aligned}$$

By Proposition 4.2, we have $R_1 - R'_1 = \wp(a,b) = \partial_R(a,b)$. So their cohomology classes are the same in $\mathcal{H}^1_R(V, \mathfrak{L})$.

Conversely, any 1-cocycle R_1 gives rise to the linear deformation $R + tR_1$. To sum up, we have the following result.

Proposition 5.3. *Let $R : V \to \mathfrak{L}$ be a generalized Reynolds operator on a regular Hom-Lie triple system $(\mathfrak{L}, [-,-,-], \alpha)$ associated to regular representation $(V, \beta; \theta)$ and 3-cocycle $\mathfrak{H}$. Then there is a bijection between the set of all equivalence classes of linear deformation of R and the first cohomology group $\mathcal{H}^1_R(V, \mathfrak{L})$.*

Next, we introduce a special cohomology class associated to an order n deformation of a generalized Reynolds operator, and show that an order n deformation of a generalized Reynolds operator is extendable if and only if this cohomology class in the second cohomology group vanishes.

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Definition 5.4. *Let $R : V \to \mathfrak{L}$ be a generalized Reynolds operator on a Hom-Lie triple system $(\mathfrak{L}, [-,-,-], \alpha)$ associated to $(V, \beta; \theta)$ and 3-cocycle $\mathfrak{H}$. If $R_t = \sum_{i=0}^{n} t^i R_i$ with $R_0 = R, R_i \in \mathrm{Hom}(V, \mathfrak{L}), i = 1, \cdots, n$, defines a $\mathbb{K}[[t]]/(t^{n+1})$-module map from $V[[t]]/(t^{n+1})$ to the Hom-Lie triple system $\mathfrak{L}[[t]]/(t^{n+1})$ satisfying*

$$
\begin{aligned}
&R_t \circ \beta = \alpha \circ R_t, \\
&[R_t u, R_t v, R_t w] = R_t(\theta(R_t v, R_t w)u + D(R_t u, R_t v)w - \theta(R_t u, R_t w)v + \mathfrak{H}(R_t u, R_t v, R_t w)),
\end{aligned}
$$

for any $u, v, w \in V$, we say that R_t is an order n deformation of R.

Definition 5.5. *Let $R : V \to \mathfrak{L}$ be a generalized Reynolds operator on a Hom-Lie triple system $(\mathfrak{L}, [-,-,-], \alpha)$ associated to $(V, \beta; \theta)$ and 3-cocycle $\mathfrak{H}$. Let $R_t = \sum_{i=0}^{n} t^i R_i$ be an order n deformation of R. If there is a $R_{n+1} \in \mathcal{C}_R^1(V, \mathfrak{L})$ such that $R'_t = R_t + t^{n+1} R_{n+1}$ is an order $(n+1)$ deformation of R, then we say that R_t is extendable.*

Proposition 5.6. *Let $R : V \to \mathfrak{L}$ be a generalized Reynolds operator on a Hom-Lie triple system $(\mathfrak{L}, [-,-,-], \alpha)$ associated to $(V, \beta; \theta)$ and 3-cocycle $\mathfrak{H}$. Let $R_t = \sum_{i=0}^{n} t^i R_i$ be an order n deformation of R. Then R_t is extendable if and only if the cohomology class $[Obs^n] \in \mathcal{H}_R^2(V, \mathfrak{L})$ vanishes, where*

$$
\begin{aligned}
Obs^n(u_1, u_2, u_3) = &\sum_{\substack{i+j+k=n+1 \\ 0 \le i,j,k \le n}} ([R_i u_1, R_j u_2, R_k u_3] - R_i(D(R_j u_1, R_k u_2)u_3 \\
&- \theta(R_j u_1, R_k u_3)u_2 + \theta(R_j u_2, R_k u_3)u_1)) - \sum_{\substack{i+j+k+l=n+1 \\ 0 \le i,j,k,l \le n}} R_i \mathfrak{H}(R_j u_1, R_k u_2, R_l u_3).
\end{aligned}
$$

Proof. Let $R'_t = R_t + t^{n+1} R_{n+1}$ be the extension of R_t, then for all $u_1, u_2, u_3 \in V$

$$
[R'_t u_1, R'_t u_2, R'_t u_3] = R'_t(D(R'_t u_1, R'_t u_2)u_3 - \theta(R'_t u_1, R'_t u_3)u_2 + \theta(R'_t u_2, R'_t u_3)u_1 + \mathfrak{H}(R'_t u_1, R'_t u_2, R'_t u_3)).
$$

Expanding the equation and comparing the coefficients of t^{n+1} yields that:

$$
\begin{aligned}
&\sum_{\substack{i+j+k=n+1 \\ 0 \le i,j,k \le n+1}} ([R_i u_1, R_j u_2, R_k u_3] - R_i(D(R_j u_1, R_k u_2)u_3 - \theta(R_j u_1, R_k u_3)u_2 + \theta(R_j u_2, R_k u_3)u_1)) \\
&- \sum_{\substack{i+j+k+l=n+1 \\ 0 \le i,j,k,l \le n+1}} R_i \mathfrak{H}(R_j u_1, R_k u_2, R_l u_3) = 0,
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&\sum_{\substack{i+j+k=n+1 \\ 0 \le i,j,k \le n}} ([R_i u_1, R_j u_2, R_k u_3] - R_i(D(R_j u_1, R_k u_2)u_3 - \theta(R_j u_1, R_k u_3)u_2 + \theta(R_j u_2, R_k u_3)u_1)) \\
&- \sum_{\substack{i+j+k+l=n+1 \\ 0 \le i,j,k,l \le n}} R_i \mathfrak{H}(R_j u_1, R_k u_2, R_l u_3) + [R_{n+1} u_1, R u_2, R u_3] + [R u_1, R_{n+1} u_2, R u_3] \\
&+ [R u_1, R u_2, R_{n+1} u_3] - R_{n+1}(D(R u_1, R u_2)u_3 - \theta(R u_1, R u_3)u_2 + \theta(R u_2, R u_3)u_1 + \mathfrak{H}(R u_1, R u_2, R u_3)) \\
&- R\big(D(R_{n+1} u_1, R u_2)u_3 + D(R u_1, R_{n+1} u_2)u_3 - \theta(R_{n+1} u_1, R u_3)u_2 - \theta(R u_1, R_{n+1} u_3)u_2 \\
&+ \theta(R_{n+1} u_2, R u_3)u_1 + \theta(R u_2, R_{n+1} u_3)u_1 + \mathfrak{H}(R_{n+1} u_1, R u_2, R u_3) \\
&+ \mathfrak{H}(R u_1, R_{n+1} u_2, R u_3) + \mathfrak{H}(R u_1, R u_2, R_{n+1} u_3)\big) = 0,
\end{aligned}
$$

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that is $Obs^n(u_1,u_2,u_3)+\partial_R R_{n+1}(u_1,u_2,u_3)=0$. Hence $Obs^n=-\partial_R R_{n+1}$, further $\partial_R Obs^n=0$, which implies that the cohomology class $[Obs^n]\in\mathcal{H}^2_R(V,\mathfrak{L})$ vanishes.

Conversely, suppose that the cohomology class $[Obs^n]$ vanishes, then there exists a 1-cochain $R_{n+1}\in\mathcal{C}^1_R(V,\mathfrak{L})$ such that $Obs^n=-\partial_R R_{n+1}$. Set $R'_t=R_t+t^{n+1}R_{n+1}$. Then R'_t satisfies

$$\begin{aligned}&\sum_{i+j+k=d}\big([\mathcal{A}_iu_1,\mathcal{A}_ju_2,\mathcal{A}_ku_3]_{\mathfrak{g}}-\mathcal{A}_i(D(\mathcal{A}_ju_1,\mathcal{A}_ku_2)u_3-\theta(\mathcal{A}_ju_1,\mathcal{A}_ku_3)u_2+\theta(\mathcal{A}_ju_2,\mathcal{A}_ku_3)u_1)\big)\\&-\sum_{i+j+k+l=d}R_i\mathfrak{H}(R_ju_1,R_ku_2,R_lu_3)=0,\quad 0\le d\le n+1,\end{aligned}$$

which implies that R'_t is an order $(n+1)$ deformation of R. Hence it is an extension of R_t. □

6. Hom-NS-Lie triple systems

In this section, we introduce the notion of Hom-NS-Lie triple system, which is the underlying algebraic structure of generalized Reynolds operators. Moreover, we show that there exists a Hom-Lie triple system structure on a Hom-NS-Lie triple system.

Definition 6.1. *(i) A Hom-NS-Lie triple system $(\mathfrak{L},\{-,-,-\},[-,-,-],\alpha)$ consists of a vector space $\mathfrak{L}$ with trilinear products $\{-,-,-\},[-,-,-]:\mathfrak{L}\otimes\mathfrak{L}\otimes\mathfrak{L}\to\mathfrak{L}$ and an algebra morphism $\alpha:\mathfrak{L}\to\mathfrak{L}$ such that*

$$\begin{aligned}
[a_1,a_2,a_3]&=-[a_2,a_1,a_3], &(6.1)\\
\circlearrowleft_{a_1,a_2,a_3}[a_1,a_2,a_3]&=0, &(6.2)\\
\{\alpha(b_1),\alpha(b_2),[[a_1,a_2,a_3]]\}&=\{\{b_1,b_2,a_1\},\alpha(a_2),\alpha(a_3)\}-\{\{b_1,b_2,a_2\},\alpha(a_1),\alpha(a_3)\}\\
&\quad+\{\alpha(a_1),\alpha(a_2),\{b_1,b_2,a_3\}\}^*, &(6.3)\\
\{\alpha(b_1),\alpha(b_2),\{a_1,a_2,a_3\}\}^*&=\{\{b_1,b_2,a_1\}^*,\alpha(a_2),\alpha(a_3)\}+\{\alpha(a_1),[[b_1,b_2,a_2]],\alpha(a_3)\}\\
&\quad+\{\alpha(a_1),\alpha(a_2),[[b_1,b_2,a_3]]\}, &(6.4)\\
[\alpha(b_1),\alpha(b_2),[[a_1,a_2,a_3]]]&=[[[b_1,b_2,a_1]],\alpha(a_2),\alpha(a_3)]+[\alpha(a_1),[[b_1,b_2,a_2]],\alpha(a_3)]\\
&\quad+[\alpha(a_1),\alpha(a_2),[[b_1,b_2,a_3]]]+\{[b_1,b_2,a_1],\alpha(a_2),\alpha(a_3)\}\\
&\quad-\{[b_1,b_2,a_2],\alpha(a_1),\alpha(a_3)\}+\{\alpha(a_1),\alpha(a_2),[b_1,b_2,a_3]\}^*\\
&\quad-\{\alpha(b_1),\alpha(b_2),[a_1,a_2,a_3]\}^*, &(6.5)
\end{aligned}$$

where $a_1,a_2,a_3,b_1,b_2\in\mathfrak{L}$, $\{-,-,-\}^*$ and $[[-,-,-]]$ are defined to be

$$\begin{aligned}
\{a_1,a_2,a_3\}^*&=\{a_3,a_2,a_1\}-\{a_3,a_1,a_2\}, &(6.6)\\
[[a_1,a_2,a_3]]&=\{a_1,a_2,a_3\}^*+\{a_1,a_2,a_3\}-\{a_2,a_1,a_3\}+[a_1,a_2,a_3]. &(6.7)
\end{aligned}$$

(ii) A homomorphism between two Hom-NS-Lie triple systems $(\mathfrak{L}_1,\{-,-,-\}_1,[-,-,-]_1,\alpha_1)$ and $(\mathfrak{L}_2,\{-,-,-\}_2,[-,-,-]_2,\alpha_2)$ is a linear map $\varphi:\mathfrak{L}_1\to\mathfrak{L}_2$ satisfying $\varphi(\alpha_1(a_1))=\alpha_2(\varphi(a_1))$, $\varphi(\{a_1,a_2,a_3\}_1)=\{\varphi(a_1),\varphi(a_2),\varphi(a_3)\}_2$, $\varphi([a_1,a_2,a_3]_1)=[\varphi(a_1),\varphi(a_2),\varphi(a_3)]_2$.

Remark 6.2. *(i) Let $(\mathfrak{L},\{-,-,-\},[-,-,-],\alpha)$ be a Hom-NS-Lie triple system. If the bracket $\{-,-,-\}=0$, then we get $(\mathfrak{L},[-,-,-],\alpha)$ is a Hom-Lie triple system.*

(ii) A NS-Lie triple system is a Hom-NS-Lie triple system with $\alpha=\mathrm{id}_{\mathfrak{L}}$. See [10] for more details about NS-Lie triple systems.

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Proposition 6.3. *Let* $(\mathfrak{L}, \{-,-,-\}, [-,-,-], \alpha)$ *be a Hom-NS-Lie triple system. Then,*

(i) the triple $(\mathfrak{L}, [[-,-,-]], \alpha)$ *is a Hom-Lie triple system, which is called the adjacent Hom-Lie triple system.*

(ii) the triple $(\mathfrak{L}, \alpha; \vartheta)$ *is a representation of the adjacent Hom-Lie triple system* $(\mathfrak{L}, [[-,-,-]], \alpha)$*, where*

$$\vartheta : \mathfrak{L} \otimes \mathfrak{L} \to \mathrm{End}(\mathfrak{L}),\ (a_1, a_2) \mapsto (b \mapsto \{b, a_1, a_2\}),\ \forall a_1, a_2, b \in \mathfrak{L}.$$

Proof. (i) Obviously, for any $a_1, a_2, a_3 \in \mathfrak{L}$, by Eqs. (6.1), (6.2), (6.6) and (6.7), we have $[[a_1, a_2, a_3]] = -[[a_2, a_1, a_3]]$ and $\circlearrowleft_{a_1,a_2,a_3} [[a_1, a_2, a_3]] = 0$. Further, for any $a_1, a_2, a_3, b_1, b_2 \in \mathfrak{L}$, by Eqs. (6.3)-(6.7), we have

$$\begin{aligned}
&[[\alpha(b_1), \alpha(b_2), [[a_1, a_2, a_3]]]] - [[[[b_1, b_2, a_1]], \alpha(a_2), \alpha(a_3)]] - [[\alpha(a_1), [[b_1, b_2, a_2]], \alpha(a_3)]] \\
&- [[\alpha(a_1), \alpha(a_2), [[b_1, b_2, a_3]]]] \\
=&\{\alpha(b_1), \alpha(b_2), [[a_1, a_2, a_3]]\}^* + \{\alpha(b_1), \alpha(b_2), [[a_1, a_2, a_3]]\} - \{\alpha(b_2), \alpha(b_1), [[a_1, a_2, a_3]]\} \\
&+ [\alpha(b_1), \alpha(b_2), [[a_1, a_2, a_3]]] - \{[[b_1, b_2, a_1]], \alpha(a_2), \alpha(a_3)\}^* - \{[[b_1, b_2, a_1]], \alpha(a_2), \alpha(a_3)\} \\
&+ \{\alpha(a_2), [[b_1, b_2, a_1]], \alpha(a_3)\} - [[[b_1, b_2, a_1]], \alpha(a_2), \alpha(a_3)] - \{\alpha(a_1), [[b_1, b_2, a_2]], \alpha(a_3)\}^* \\
&- \{\alpha(a_1), [[b_1, b_2, a_2]], \alpha(a_3)\} + \{[[b_1, b_2, a_2]], \alpha(a_1), \alpha(a_3)\} - [\alpha(a_1), [[b_1, b_2, a_2]], \alpha(a_3)] \\
&- \{\alpha(a_1), \alpha(a_2), [[b_1, b_2, a_3]]\}^* - \{\alpha(a_1), \alpha(a_2), [[b_1, b_2, a_3]]\} + \{\alpha(a_2), \alpha(a_1), [[b_1, b_2, a_3]]\} \\
&- [\alpha(a_1), \alpha(a_2), [[b_1, b_2, a_3]]] \\
=&0.
\end{aligned}$$

Hence, $(\mathfrak{L}, [[-,-,-]], \alpha)$ is a Hom-Lie triple system.

(ii) For all $a_1, a_2, a_3 \in \mathfrak{L}$, we have

$$\mathfrak{D}(a_1, a_2)a_3 = \vartheta(a_2, a_1)a_3 - \vartheta(a_1, a_2)a_3 = \{a_3, a_2, a_1\} - \{a_3, a_1, a_2\} = \{a_1, a_2, a_3\}^*.$$

Obviously, $\vartheta(\alpha(a_1), \alpha(a_2))\alpha(a_3) = \alpha(\vartheta(a_1, a_2)a_3)$. Further, for any $a_1, a_2, a_3, b_1, b_2 \in \mathfrak{L}$, by Eqs. (6.3) and (6.4), we get

$$\begin{aligned}
&\vartheta(\alpha(b_1), \alpha(b_2))\vartheta(a_1, a_2)a_3 - \vartheta(\alpha(a_2), \alpha(b_2))\vartheta(a_1, b_1)a_3 - \vartheta(\alpha(a_1), [[a_2, b_1, b_2]])\alpha(a_3) \\
&+ \mathfrak{D}(\alpha(a_2), \alpha(b_1))\vartheta(a_1, b_2)a_3 \\
=&\{\{a_3, a_1, a_2\}, \alpha(b_1), \alpha(b_2)\} - \{\{a_3, a_1, b_1\}, \alpha(a_2), \alpha(b_2)\} - \{\alpha(a_3), \alpha(a_1), [[a_2, b_1, b_2]]\} \\
&+ \{\alpha(a_2), \alpha(b_1), \{a_3, a_1, b_2\}\}^* \\
=&0, \\
&\vartheta(\alpha(b_1), \alpha(b_2))\mathfrak{D}(a_1, a_2)a_3 - \mathfrak{D}(\alpha(a_1), \alpha(a_2))\vartheta(b_1, b_2)a_3 + \vartheta([[a_1, a_2, b_1]], \alpha(b_2))\alpha(a_3) \\
&+ \vartheta(\alpha(b_1), [[a_1, a_2, b_2]])\alpha(a_3) \\
=&\{\{a_1, a_2, a_3\}^*, \alpha(b_1), \alpha(b_2)\} - \{\alpha(a_1), \alpha(a_2), \{a_3, b_1, b_2\}\}^* + \{\alpha(a_3), [[a_1, a_2, b_1]], \alpha(b_2)\} \\
&+ \{\alpha(a_3), \alpha(b_1), [[a_1, a_2, b_2]]\} \\
=&0.
\end{aligned}$$

Therefore, $(\mathfrak{L}, \alpha; \vartheta)$ is a representation of the adjacent Hom-Lie triple system $(\mathfrak{L}, [[-,-,-]], \alpha)$. □

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Corollary 6.4. *Let* $\varphi : (\mathfrak{L}_1, \{-,-,-\}_1, [-,-,-]_1, \alpha_1) \to (\mathfrak{L}_2, \{-,-,-\}_2, [-,-,-]_2, \alpha_2)$ *be a Hom-NS-Lie triple system homomorphism. Then,* φ *is also a Hom-Lie triple system homomorphism between the subadjacent Hom-Lie triple system from* $(\mathfrak{L}_1, [[-,-,-]]_1, \alpha_1)$ *to* $(\mathfrak{L}_2, [[-,-,-]]_2, \alpha_2)$.

The following proposition illustrate that Hom-NS-Lie triple systems can be viewed as the underlying algebraic structures of generalized Reynolds operators on Hom-Lie triple systems.

Proposition 6.5. *Let* $R : V \to \mathfrak{L}$ *be a generalized Reynolds operator on a Hom-Lie triple system* $(\mathfrak{L}, [-,-,-], \alpha)$ *associated to* $(V, \beta; \theta)$ *and 3-cocycle* $\mathfrak{H}$*. Then, the 4-tuple* $(V, \{-,-,-\}_\theta, [-,-,-]_{\mathfrak{H}}, \beta)$ *is a Hom-NS-Lie triple system, where*

$$\{u,v,w\}_\theta = \theta(Rv, Rw)u, \ [u,v,w]_{\mathfrak{H}} = \mathfrak{H}(Ru, Rv, Rw), \ \forall u,v,w \in V.$$

Proof. For any $u,v,w,s,t \in V$, first, obviously, we have $[u,v,w]_{\mathfrak{H}} = -[v,u,w]_{\mathfrak{H}}$ and $\circlearrowleft_{u,v,w} [u,v,w]_{\mathfrak{H}} = 0$. On the one hand

$$\begin{aligned}
\{u,v,w\}^*_\theta =& \{w,v,u\}_\theta - \{w,u,v\}_\theta = \theta(Rv,Ru)w - \theta(Ru,Rv)w = D(Ru,Rv)w,\\
[[u,v,w]]_V =& \{u,v,w\}^*_\theta + \{u,v,w\}_\theta - \{v,u,w\}_\theta + [u,v,w]_{\mathfrak{H}}\\
=& D(Ru,Rv)w + \theta(Rv,Rw)u - \theta(Ru,Rw)v + \mathfrak{H}(Ru,Rv,Rw).
\end{aligned}$$

On the other hand, by Eqs. (2.5),(2.6), (3.1)-(3.2), we get

$$\begin{aligned}
&\{\beta(s),\beta(t),[[u,v,w]]_V\}_\theta - \{\{s,t,u\}_\theta,\beta(v),\beta(w)\}_\theta + \{\{s,t,v\}_\theta,\beta(u),\beta(w)\}_\theta\\
&- \{\beta(u),\beta(v),\{s,t,w\}_\theta\}^*_\theta\\
=&\theta(R\beta(t),R[[u,v,w]]_V)\beta(s) - \theta(R\beta(v),R\beta(w))\theta(Rt,Ru)s + \theta(R\beta(u),R\beta(w))\theta(Rt,Rv)s\\
&- D(R\beta(u),R\beta(v))\theta(Rt,Rw)s\\
=&0,\\
&\{\{s,t,u\}^*_\theta,\beta(v),\beta(w)\}_\theta + \{\beta(u),[[s,t,v]]_V,\beta(w)\}_\theta + \{\beta(u),\beta(v),[[s,t,w]]_V\}_\theta\\
&- \{\beta(s),\beta(t),\{u,v,w\}_\theta\}^*_\theta\\
=&\theta(R\beta(v),R\beta(w))D(Rs,Rt)u + \theta(R[[s,t,v]]_V,R\beta(w))\beta(u) + \theta(R\beta(v),R[[s,t,w]]_V)\beta(u)\\
&- D(R\beta(s),R\beta(t))\theta(Rv,Rw)u\\
=&0,\\
&[[[s,t,u]]_V,\beta(v),\beta(w)]_{\mathfrak{H}} + [\beta(u),[[s,t,v]]_V,\beta(w)]_{\mathfrak{H}} + [\beta(u),\beta(v),[[s,t,w]]_V]_{\mathfrak{H}}\\
&+ \{[s,t,u]_{\mathfrak{H}},\alpha(v),\alpha(w)\}_\theta - \{[s,t,v]_{\mathfrak{H}},\beta(u),\beta(w)\}_\theta + \{\beta(u),\beta(v),[s,t,w]_{\mathfrak{H}}\}^*_\theta\\
&- \{\beta(s),\beta(t),[u,v,w]_{\mathfrak{H}}\}^*_\theta - [\beta(s),\beta(t),[[u,v,w]]_V]_{\mathfrak{H}}\\
=&\mathfrak{H}(R[[s,t,u]]_V,R\beta(v),R\beta(w)) + \mathfrak{H}(R\beta(u),R[[s,t,v]]_V,R\beta(w)) + \mathfrak{H}(R\beta(u),R\beta(v),R[[s,t,w]]_V)\\
&+ \theta(R\alpha(v),R\alpha(w))\mathfrak{H}(Rs,Rt,Ru) - \theta(R\beta(u),R\beta(w))\mathfrak{H}(Rs,Rt,Rv) + D(R\beta(u),R\beta(v))\mathfrak{H}(Rs,Rt,Rw)\\
&- D(R\beta(s),R\beta(t))\mathfrak{H}(Ru,Rv,Rw) - \mathfrak{H}(R\beta(s),R\beta(t),R[[u,v,w]]_V)\\
=&0.
\end{aligned}$$

Thus $(V, \{-,-,-\}_\theta, [-,-,-]_{\mathfrak{H}}, \beta)$ is a Hom-NS-Lie triple system. □

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Example 6.6. *Let* $(\mathfrak{L}, [-,-,-], \alpha)$ *be a Hom-Lie triple system and* $N : \mathfrak{L} \to \mathfrak{L}$ *be a Nijenhuis operator. Then* $(\mathfrak{L}, \{-,-,-\}_\theta, [-,-,-]_{\mathfrak{H}}, \alpha)$ *is a Hom-NS-Lie triple system, where*

$$\begin{aligned}\{a,b,c\}_\theta =&[a, Nb, Nc],\\ [a,b,c]_{\mathfrak{H}} =& -N([Na,b,c] + [a,Nb,c] + [a,b,Nc]) + N^2[a,b,c], \ \ \forall a,b,c \in \mathfrak{L}.\end{aligned}$$

Proposition 6.7. *Let* $R_1 : V_1 \to \mathfrak{L}_1$ *(resp.* $R_2 : V_2 \to \mathfrak{L}_2$*) be a generalized Reynolds operators on a Hom-Lie triple system* $(\mathfrak{L}_1, [-,-,-]_1, \alpha_1)$ *(resp.* $(\mathfrak{L}_2, [-,-,-]_2, \alpha_2)$ *) associated to* $(V_1, \beta_1; \theta_1)$ *(resp.* $(V_2, \beta_2; \theta_2)$*) and 3-cocycle* $\mathfrak{H}_1$ *(resp.* $\mathfrak{H}_2$*), and* (η, ζ) *be a homomorphism from* R_1 *to* R_2*. Let* $(V_1, \{-,-,-\}_{\theta_1}, [-,-,-]_{\mathfrak{H}_1}, \beta_1)$ *and* $(V_2, \{-,-,-\}_{\theta_2}, [-,-,-]_{\mathfrak{H}_2}, \beta_2)$ *be the induced Hom-NS-Lie triple systems respectively. Then,* ζ *is a homomorphism from the Hom-NS-Lie triple system* $(V_1, \{-,-,-\}_{\theta_1}, [-,-,-]_{\mathfrak{H}_1}, \beta_1)$ *to* $(V_2, \{-,-,-\}_{\theta_2}, [-,-,-]_{\mathfrak{H}_2}, \beta_2)$*.*

Proof. For any $u, v, w \in V$, by Eqs. (3.3)-(3.6), we have

$$\begin{aligned}\zeta(\{u,v,w\}_{\theta_1}) =&\zeta(\theta_1(R_1 v, R_1 w)u) = \theta_2(\eta(R_1 v), \eta(R_1 w))\zeta(u)\\ =&\theta_2(R_2\zeta(v), R_2\zeta(w))\zeta(u)\\ =&\{\zeta(u), \zeta(v), \zeta(w)\}_{\theta_2},\\ \zeta([u,v,w]_{\mathfrak{H}_1}) =&\zeta(\mathfrak{H}_1(R_1 u, R_1 v, R_1 w)) = \mathfrak{H}_2(\eta(R_1 u), \eta(R_1 v), \eta(R_1 w))\\ =&\mathfrak{H}_2(R_2\zeta(u), R_2\zeta(v), R_2\zeta(w))\\ =&[\zeta(u), \zeta(v), \zeta(w)]_{\mathfrak{H}_2}.\end{aligned}$$

Hence, $\varphi_{\mathfrak{h}}$ is a homomorphism from $(V_1, \{-,-,-\}_{\theta_1}, [-,-,-]_{\mathfrak{H}_1}, \beta_1)$ to $(V_2, \{-,-,-\}_{\theta_2}, [-,-,-]_{\mathfrak{H}_2}, \beta_2)$. □

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