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A New Highly Accurate Approach to Compact Operators on Some Generalized Fibonacci Difference Sequence Spaces

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Abstract: The current research examines the many characteristics of the $\ell_p(\widetilde{F}(\widetilde{r},\widetilde{s}))$ $(1 \leq p < \infty)$ and $\ell_\infty(\widetilde{F}(\widetilde{r},\widetilde{s}))$ spaces which are the generalized forms of those by Candan in 2024 using Fibonacci numbers and two non-zero real numbers in accordance with a predetermined rule, we have made an effort to go through all the characteristics and features which the author of earlier versions thought are the most valuable. This manuscript contains all the information required to describe the matrix class $(\ell_1, \ell_p(\widetilde{F}(\widetilde{r},\widetilde{s})))$ $(1 \leq p < \infty)$ which are going to be given in detail in the following sections of the manuscript. We are going to offer estimates for the norms of the bounded linear operators L_A formed by those matrix transformations utilizing the Hausdorff measure of non-compactness and identify prerequisites to derive the relevant subclasses of compact matrix operators. When the findings of the current study are compared to those found in the literature, it can be said that the newly found ones are more inclusive and comprehensive.

Keywords: sequence spaces; fibonacci numbers; compact operators; hausdorff measure of non-compactness

MSC: 46A45; 11B39; 46B50

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1. Elementary Classical Concepts

Our goal is to continue using the matrix domain while also reminding readers of the knowledge they will need to properly apply calculus in their work in following parts. To do this, we continued to modify some of the measurement theory techniques while maintaining the paper's mathematical quality, the new sequence space's orientation to the Hausdorff measure, its focus on earlier research, and the diversity of the theorems. The degree of rigor is essentially the same, despite the fact that many of the presentations in this new work are notably more generic than those in preceding pieces. It will be beneficial for novices to examine the five important books listed in References [1–5] with accessible content as part of the entire review strategy, without sacrificing the standards or breadth their consumers want to see. Let's attempt to clarify some of the fundamentals first without going overboard with the obvious. The history of numbers dates back almost as far as humankind itself, and they were developed to satisfy the universal and scientific demand for mathematics. This still holds true today as it did when the topic first emerged. The sequences in the majority of our work will have sets of natural numbers $\mathbb{N} = \{0, 1, 2, ...\}$ and real numbers $\mathbb{R}$ as their domains and ranges, respectively. Instead of writing $\lim_{k\to\infty}$, $\sup_{k\in\mathbb{N}}$, $\inf_{k\in\mathbb{N}}$ and $\sum_{k=0}^{\infty}$, respectively, we will write $\lim_k$, $\sup_k$, $\inf_k$ and $\sum_k$.

Infinite sequences and infinite series are two related issues that will be discussed in the following sections. A function whose domain is the set of natural numbers is one that produces an infinite series of numbers. The word "series" always suggests an infinite amount of terms that may be added together in a certain order. All real sequences' vector spaces are symbolized by the symbol ω. We are well aware that each subspace of ω is referred to as a sequence space. A few more notations regarding sequences are required for use in this work. φ, ℓ_∞, c and c_0. should be used to indicate the sets of all finite sequences, bounded sequences, convergent sequences, and null sequences, respectively.

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The notation ℓ_p stands for the sequence space $\{x \in \omega : \sum_k |x_k|^p < \infty\}$ for any real integer p with $1 \leq p < \infty$. Additionally, the notations e and $e^{(n)}$, respectively, are used to represent the sequence $(1, 1, ...)$ and the sequence with 1 only in the n^{th} term and 0 in all other terms for each natural integer n. The n-section of any sequence x is the sum $\sum_{k=0}^{n} x_k e^{(k)}$ denoted by $x^{[n]}$ and is known as the n-section of any sequence x. Besides, cs and bs notations are used to display series whose partial summation sequence are converging and constrained, respectively. The term $B - space$ refers to a complete normed space. While all coordinate functionals π_k, denoted by $\pi_k(x) = x_k$, are continuous in a $K - space$, all coordinate functionals π_k are continious in a topological sequence space. In essence, a $BK - space$ is a Banach space with continuous coordinates that satisfies the criteria for both a $K - space$ and a $B - space$. It is claimed that a $BK - space$ labeled as $X \supset \varphi$ has AK if all sequences $x = (x_k) \in X$ have the same representation, in which $x = \sum_k x_k e^{(k)}$. As an illustration, the sequence space ℓ_p $(1 \leq p < \infty)$ may be thought of as a $BK - space$ with the norm $\|x\|_p = \left(\sum_k |x_k|^p\right)^{1/p}$. Plus, because they have the norm $\|x\|_\infty = \sup_k |x_k|$, c_0, c, and ℓ_∞ likewise qualify as $BK - spaces$. Furthermore, the $BK - spaces$ c_0 and ℓ_p exhibit AK, in which $1 \leq p < \infty$.

If there exists a singular sequence (α_n) consisting of scalars such that $x = \sum_n \alpha_n b_n$, meaning that $\lim_m \|x - \sum_{n=0}^{m} \alpha_n b_n\| = 0$, then the sequence (b_n) in a normed space X is referred to as a Schauder basis for all $x \in X$.

The following is the definition of the β-dual of a sequence space X:

$$X^\beta = \{a = (a_k) \in \omega : ax = (a_k x_k) \in cs \text{ for all } x = (x_k) \in X\}.$$

An infinite matrix of real numbers, denoted by $A = (a_{nk})_{n,k=0}^{\infty}$, where $n, k \in \mathbb{N}$, can be represented as A_n, which denotes the sequence in the n^{th} row of A. Furthermore, if $x = (x_k)_{k=0}^{\infty} \in \omega$, the A-transform of x is defined as the sequence $Ax = \{A_n(x)\}_{n=0}^{\infty}$, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k; \quad (n \in \mathbb{N}) \tag{1}$$

provided that the series on the right-hand side converges for each $n \in \mathbb{N}$.

(X, Y) refers to the class of all infinite matrices that map from X to Y, where X and Y are subsets of ω. To put it another way, $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for every $n \in N$ and $Ax \in Y$ for every $x \in X$.

The matrix domain is one method for creating a new sequence space, and a comprehensive understanding of it needs significant knowledge. Assume X is any sequence space

$$X_A = \{x = (x_k) \in \omega : Ax \in X\}, \tag{2}$$

therefore defines the domain X_A of an infinite matrix A in X. Let us also point out that X_A is a sequence space. The reader is encouraged to consult the recent publications [6–10] on the domains of certain triangles in classical sequence spaces and related subjects.

The following outcomes are basic and frequently used [11,12].

Lemma 1. *Let $X \supset \phi$ and Y be a BK–space.*
(a) Therefore, for any matrix $A \in (X, Y)$, we get $(X, Y) \subset B(X, Y)$, so indicating that for any $x \in X$, $L_A(x) = Ax$ describes an operator $L_A \in B(X, Y)$.
(b) If X has AK, and after that $B(X, Y) \subset (X, Y)$, meaning that there is a $A \in (X, Y)$ with any operator having $L \in B(X, Y)$ and $L(x) = Ax$ for every $x \in X$.

2. The Hausdorff measure of non-compactness

The goal of this section is to discuss the Hausdorff measure, which is used in theory and practice to characterize compact operators between Banach spaces. This part begins with

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unambiguous formulations of relative definitions, rules, and theorems, as well as explanatory and other demonstrative topic matter. It is organized around proved and additional theorems. The proven theorems help to demonstrate and amplify the theory, as well as to restate the key concepts required for effective learning. In several parts of mathematics, the idea of Hausdorff measure of non-compactness emerges. Under specific conditions, this idea has recently been utilized to describe compact matrix operators between BK–spaces.

The notion of the Hausdorff measure of non-compactness is essentially derived from the research of Goldenštein, Gohberg and Markus [13], and it was later picked up and investigated by Goldenštein and Markus [14]. Nonetheless, some of its concepts date back to the time of Kuratowski [15]. Afterwards, Darbo [16] expanded additional idea in addition to the conventional Schauder fixed point principle.

It is necessary to rephrase the notion of a compact operator in the context of infinite-dimensional Banach spaces X and Y. A linear operator L that maps from X to Y is thought to be compact if it covers the full domain of X, and furthermore, if the sequence $(L(x_n))$ that represents the images of all bounded sequences (x_n) in X under L has a convergent subsequence. In the discipline of functional analysis, $C(X,Y)$ denotes the collection of all compact operators in $B(X,Y)$.

Assume (X,d) is a metric space. The open ball $B(x,t)$ is defined as the set $\{x \in X : d(x,x_0) < t\}$, where t represents the radius and x_0 stands for the center.

Moreover, let us describe $M(X)$ as the collection of all bounded subsets of X. In case of $Q \in M(X)$, the Hausdorff measure of the given set Q non-compactness, represented by $\chi(Q)$, is described in the following form:

$$\chi(Q) = \inf\left\{\epsilon > 0 : Q \subset \bigcup_{k=1}^{n} B(x_k,t_k), x_k \in X,\ t_k < \epsilon\ (k = 1,2,...),\ n \in \mathbb{N}\right\}.$$

The function $\chi : M_X \to [0,\infty)$ is the definition of the Hausdorff measure of non-compactness.

The applicability of the Hausdorff measure theorems to condensing operators, compact matrix operators on certain BK-spaces, and measures of non-compactness in Banach spaces can be found examined more thoroughly in the earlier works such as those and therein [11,17–20].

This paragraph aims to give a brief explanation of the Hausdorff measure of non-compactness operators between Banach spaces. The Hausdorff measures of non-compactness on X and Y, respectively, are χ_1 and χ_2 in the case of X and Y are being Banach spaces. If $L(Q) \in M(Y)$ for all $Q \in M(X)$, and if there exists $C \geq 0$ such that $\chi_2(L(Q)) \leq C\chi_1(Q)$ for all $Q \in M(X)$, then the operator $L : X \to Y$ is called as (χ_1,χ_2)-bounded. The quantity

$$|L|(\chi_1,\chi_2) = \inf\{C \geq 0 : \chi_2(L(Q)) \leq C\chi_1(Q) \text{ for all } Q \in \mathcal{M}X\}$$

is described as the (χ_1,χ_2)–measure of non-compactness of L if the operator L is (χ_1, χ_2)-bounded.

It is vital to note the fact that if both χ_1 and χ_2 are denoted as χ, then $|L|(\chi_1,\chi_2) = |L|_\chi$.

In the present setting, our main goal is to give a through explanation of how the Hausdorff measure of non-compactness may be used to describe compact operators between Banach spaces. Let X and Y be Banach spaces, and let L be an element of $B(X,Y)$, illustrating that L is a bounded linear operator from X to Y. If L is non-compact, the Hausdorff measure of non-compactness of L, denoted as $\|L\|_\chi$, is described in the following form ([20], [Theorem 2.25])

$$\|L\|\chi = \chi(L(SX)). \tag{3}$$

Additionally, the Hausdorff measure of non-compactness $\|L\|_\chi$, which is denoted by the expression given in Ref. ([20], [Corollary 2.26]), characterizes L as a compact operator if and only if it equals

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zero. Furthermore, L is considered a compact operator if and only if the Hausdorff measure of non-compactness $\|L\|_\chi$ is equal to zero, as stated in ([20], [Corollary 2.26])

$$\|L\|_\chi = 0. \tag{4}$$

The identities stated in (3) and (4) form the basis for calculating the Hausdorff measure of non-compactness, abbreviated as $\chi(Q)$, for bounded sets Q in a Banach space X. The characterisation of compact operators $L \in B(X,Y)$ is made easier by these identities. When X has a Schauder basis, estimates or even identities for $\chi(Q)$ can be derived.

Theorem 1. *([13] or [20], [Theorem 2.23]) Let X be a Banach space with a Schauder basis $(b_k)_{k=0}^{\infty}$, $Q \in M_X$, $P_n : X \to X$ will be the projectors onto the linear span of $\{b_0, b_1, \ldots, b_n\}$ and $R_n = I - P_n$ for $n = 0, 1, \ldots,$ in which I indicates the identity map on X. Under these conditions, the following inequality is satisfied*

$$\frac{1}{a} \cdot \limsup_{n\to\infty} \left(\sup_{x\in Q} \|\mathcal{R}_n(x)\| \right) \le \chi(Q) \le \limsup_{n\to\infty} \left(\sup_{x\in Q} \|\mathcal{R}_n(x)\| \right),$$

in which $a = \limsup_{n\to\infty} \|R_n\|$.

In particular, the following result shows how to derive the Hausdorff measure of non-compactness in the BK-spaces with AK, c_0, and ell_p $(1 \le p < \infty)$.

Theorem 2. *([20], [Theorem 2.15]) A bounded subset of the normed space X, in which X is ℓ_p for $1 \le p < \infty$ or c_0, is defined as Q. We can have*

$$\chi(Q) = \lim_{n\to\infty} \left(\sup_{x\in Q} \|\mathcal{R}_n(x)\| \right). \tag{5}$$

if $P_n : X \to X$ is the operator described by $P_n(x) = x^{[n]}$ for every $x = (x_k)_{k=0}^{\infty} \in X$ and $R_n = I - P_n$ for $n = 0, 1, \ldots$.

The above results, together with the Hausdorff measure of non-compactness, make it quite plausible to derive both necessary and sufficient requirements for matrix operators between a Schauder basis and a BK-space. By presenting bounded linear operators between these Banach spaces as a result of matrix mappings over BK-spaces, AK operators are transformed into compact operators. Currently, many academics have used this strategy in a variety of research articles (see, for example, [21–31]). The importance of these ideas will become clear in later conversations. In this paper, we are going to give an explanation of the matrix classes $(\ell_1, \ell_p(\widetilde{F}(\widetilde{r},\widetilde{s})))$ $(1 \le p < \infty)$. Furthermore, by using the Hausdorff measure of non-compactness, we develop requirements for obtaining the pertinent subclasses of compact matrix operators. In addition, we develop an identity for the matrix transformation-determined norms of the bounded linear operators L_A.

3. The Fibonacci Difference Sequence Spaces $\ell_p(\widetilde{F}(\widetilde{r},\widetilde{s}))$ and $\ell_\infty(\widetilde{F}(\widetilde{r},\widetilde{s}))$

Infinite sequences have been employed in mathematics since antiquity, but they were particularly prevalent in the early history of the calculus. The mathematician Fibonacci utilized a series of integers $1, 1, 2, 3, 5, \ldots$ in his work Liber Abaci (1202) during the middle times. Fibonacci sequences may already be familiar to you, but even if not, you will be able to easily comprehend the following formula. The stages in the series are often marked $1, 1, 2, 3, 5, dots$, and so on, for simplicity. The *Fibonacci sequences* $f = (f_n)$ provide a much better explanation. Using the recursion formula

$$f_n = f_{n-1} + f_{n-2}; \; n \ge 2,$$

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begins with the expression $f_0 = f_1 = 1$.

The usage of Fibonacci sequences is widespread and provides opportunities for practical application. The Fibonacci numbers were discovered to be connected to the most dramatic distinctions in plants, certain living things in nature, and art and architecture. A number of Fibonacci sequence applications are outside the purview of this work, but the information in this part can help you get ready for more study and provide you knowledge you can use when necessary. There is a wealth of knowledge on Fibonacci numbers, especially the Golden ratio, in the reference number [32].

Let $1 \leq p \leq \infty$ and q represent the conjugate of p throughout, that is, $q = p/(p-1)$ for $1 < p < \infty$, that is, $q = p/(p-1)$ for $1 < p < \infty$, $q = \infty$ for $p = 1$ or $q = 1$ for $p = \infty$.

In 2015, right after Kara [33], Candan and Kara [34] introduced the generalized Fibonacci difference sequence spaces $\ell_p(\widehat{F}(r,s))$ and $\ell_\infty(\widehat{F}(r,s))$. Very recently, Candan [35] has described the matrix class $(\ell_1, \ell_p(\widehat{F}(r,s)))$ $(1 \leq p < \infty)$. Now, we introduce the generalized Fibonacci difference sequence spaces $\ell_p(\widetilde{F}(\widetilde{r},\widetilde{s}))$ and $\ell_\infty(\widetilde{F}(\widetilde{r},\widetilde{s}))$, as follows;

$$\ell_p(\widetilde{F}(\widetilde{r},\widetilde{s})) = \left\{ x = (x_n) \in \omega : \sum_n \left| r_n \frac{f_n}{f_{n+1}} x_n + s_{n-1} \frac{f_{n+1}}{f_n} x_{n-1} \right|^p < \infty \right\}; \ 1 \leq p < \infty$$

and

$$\ell_\infty(\widetilde{F}(\widetilde{r},\widetilde{s})) = \left\{ x = (x_n) \in \omega : \sup_{n\in\mathbb{N}} \left| r_n \frac{f_n}{f_{n+1}} x_n + s_{n-1} \frac{f_{n+1}}{f_n} x_{n-1} \right| < \infty \right\}.$$

We note here that, the generalization is made using two real number sequences that do not convergence to zero.

When we use the equivalent notation of (2) for the sequence spaces $\ell_p(\widetilde{F}(\widetilde{r},\widetilde{s}))$ and $\ell_\infty(\widetilde{F}(\widetilde{r},\widetilde{s}))$, related sequence spaces become

$$\ell_p(\widetilde{F}(\widetilde{r},\widetilde{s})) = (\ell_p)_{\widetilde{F}(\widetilde{r},\widetilde{s})} \ (1 \leq p < \infty) \ \text{and also} \ \ell_\infty(\widetilde{F}(\widetilde{r},\widetilde{s})) = (\ell_\infty)_{\widetilde{F}(\widetilde{r},\widetilde{s})}, \tag{6}$$

in which the matrix $\widetilde{F}(\widetilde{r},\widetilde{s}) = (\widehat{f}_{nk}(\widetilde{r},\widetilde{s}))$ is described by

$$\widehat{f}_{nk}(\widetilde{r},\widetilde{s}) = \begin{cases} s_n \frac{f_{n+1}}{f_n} & (k = n-1) \\ r_n \frac{f_n}{f_{n+1}} & (k = n) \\ 0 & (0 \leq k < n-1) \text{ or } (k > n) \end{cases} ; \ (n, k \in \mathbb{N}). \tag{7}$$

To signal the fact that the sequence spaces $\ell_p(\widetilde{F}(\widetilde{r},\widetilde{s}))$ and $\ell_\infty(\widetilde{F}(\widetilde{r},\widetilde{s}))$ are BK-spaces according to the

$$\|x\|_{\ell_p(\widetilde{F}(\widetilde{r},\widetilde{s}))} = \left(\sum_n |y_n(x)|^p \right)^{1/p} ; \ (1 \leq p < \infty) \ \text{and} \ \|x\|_{\ell_\infty(\widetilde{F}(\widetilde{r},\widetilde{s}))} = \sup_{n\in\mathbb{N}} |y_n(x)|, \tag{8}$$

norms defined on them, respectively, in which the sequence $y = (y_n) = (\widetilde{F}(\widetilde{r},\widetilde{s})_n(x))$ which is the $\widetilde{F}(\widetilde{r},\widetilde{s})$-transform of any sequence $x = (x_n)$ is used. That is

$$y_n = \widetilde{F}(\widetilde{r},\widetilde{s})_n(x) = \begin{cases} r_0 \frac{f_0}{f_1} x_0 = r_0 x_0 & (n = 0) \\ r_n \frac{f_n}{f_{n+1}} x_n + s_{n-1} \frac{f_{n+1}}{f_n} x_{n-1} & (n \geq 1) \end{cases} ; \ (n \in \mathbb{N}). \tag{9}$$

The fact that the results of this present study are more through than those of Alotaibi et al. [36] in 2015 and Candan [35] in 2024 should be noted.

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4. Main results

Although this article cannot cover all possible uses of compact operators, the information in this part can help you grasp the topic and help you retain knowledge that you can need later on. From a historical standpoint, the current understanding of the Hausdorff measure is the result of the combined work of many people. But Goldenštein, Gohberg, and Markus proposed the idea of non-compactness' Hausdorff measure in 1957, and Goldenštein and Markus went on to further develop it. In the study [37], the sequence spaces Y, ℓ_∞, c_0, and c are considered, letting the characterization of the classes $(\ell_p(\widehat{F}), Y)$, $(\ell_\infty(\widehat{F}), Y)$, $(\ell_1(\widehat{F}), Y)$, as well as the compact operators $(\ell_p(\widehat{F}), \ell_1)$ and $(\ell_1(\widehat{F}), \ell_p)$. In this study, we introduce the classes $B(\ell_1, \ell_p^\lambda)$ for $1 \le p < \infty$ and compute the operator norms in $B(\ell_1, \ell_p^\lambda)$. Furthermore, leveraging the findings from the earlier parts of the present study, we describe the classes $C(\ell_1, \ell_p)$ for $1 \le p < \infty$ and determine the Hausdorff measure of non-compactness for operators in $B(\ell_1, \ell_p^\lambda)$.

Again it is assumed that $1 \le p < \infty$. Now a characterization of $B(\ell_1, \ell_p(\widetilde{F}(\widetilde{r}, \widetilde{s})))$ is provided together with the computation of the operator norms in $B(\ell_1, \ell_p(\widehat{F}(\widetilde{r}, \widetilde{s})))$. Additionally, one can use those results given in the earlier sections to both specify the Hausdorff measure of non-compactness for operators in $B(\ell_1, \ell_p(\widetilde{F}(\widetilde{r}, \widetilde{s})))$ and characterize the classes $C(\ell_1, \ell_p(\widetilde{F}(\widetilde{r}, \widetilde{s})))$ for $1 \le p < \infty$.

It is important to note that the following result has a particular advantage in specific proofs.

Lemma 2. *([20], [Theorem 3.8]) T is a triangular matrix and with it X and Y being any two sequence spaces; for the matrix A to be an element of the (X, Y_T) class, the necessary and sufficient condition is that $C = T \cdot A$ and the matrix C belongs to the class (X, Y). In addition, if the X and Y are BK-spaces, and also if the matrix A is an element of the class (X, Y_T), then*

$$\|L_A\| = \|L_C\|. \tag{10}$$

Next, the identities for those operator norm and the characterizations of the classes $B(\ell_1, \ell_p(\widetilde{F}(\widetilde{r}, \widetilde{s})))$ for $(1 \le p < \infty)$ are described.

Theorem 3. *Let $1 \le p < \infty$.*
(a) We have $L \in B(\ell_1, \ell_p(\widetilde{F}(\widetilde{r}, \widetilde{s})))$ if and only if there exists an infinite matrix $A \in (\ell_1, \ell_p(\widetilde{F}(\widetilde{r}, \widetilde{s})))$ such that

$$\|A\| = \sup_k \left(\sum_n \left| r_n \frac{f_n}{f_{n+1}} a_{nk} + s_{n-1} \frac{f_{n+1}}{f_n} a_{n-1,k} \right|^p \right)^{1/p} < \infty \tag{11}$$

and

$$L(x) = Ax \text{ for all } x \in \ell_1. \tag{12}$$

(b) If $L \in B(\ell_1, \ell_p(\widetilde{F}(\widetilde{r}, \widetilde{s})))$ then

$$\|L\| = \|A\|. \tag{13}$$

Proof. For (a), if one keeps in mind the fact that ℓ_1 is a BK-space with AK, for $L \in B(\ell_1, \ell_p(\widetilde{F}(\widetilde{r}, \widetilde{s})))$ from Lemma 1 when the condition $1 \le p < \infty$ hypothesis condition; the necessary and sufficient condition is that there is an infinite matrix A such that $A \in (\ell_1, \ell_p(\widetilde{F}(\widetilde{r}, \widetilde{s})))$ provided that the condition (12) is satisfied. If we denote the product of the matrices $\widetilde{F}(\widetilde{r}, \widetilde{s}) = (\widetilde{f}_{nk}(\widetilde{r}, \widetilde{s}))$ and $A = (a_{nk})$ by $C = (c_{nk})$, one can express it neatly in the following form

$$c_{nk} = r_n \frac{f_n}{f_{n+1}} a_{nk} + s_{n-1} \frac{f_{n+1}}{f_n} a_{n-1,k}.$$

At the moment, it is quiet easy to tell the fact that using Lemma 2(a) that the necessary and sufficient condition $A \in (\ell_1, \ell_p(\widetilde{F}(\widetilde{r}, \widetilde{s})))$ is $C \in (\ell_1, \ell_p)$. If the [Example 8.4.1D] in the reference [12] is utilized at

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this level of the proof, one can obviously see that the necessary and sufficient condition for $C \in (\ell_1, \ell_p)$ is

$$\|C\| = \sup_k \left(\sum_{n=0}^{\infty} |c_{nk}|^p \right)^{1/p} < \infty,$$

which proves the claim. □

(b) First of all, we are going to illustrate that $\|L\| \leq \|A\|$. Let $L \in B(\ell_1, \ell_p^\lambda)$. Again, one can see from (10) that $\|L\| = \|L_C\|$ for $L_C \in B(\ell_1, \ell_p)$ is given by the equation $L_C(x) = Cx$ for every $x \in \ell_1$. At the moment, one can state using the Minkowsky's inequality that we can write the following expressions

$$\begin{aligned} \|L_C(x)\|_p &= \left(\sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} c_{nk} x_k \right|^p \right)^{1/p} \\ &\leq \sum_{k=0}^{\infty} |x_k| \left(\sum_{n=0}^{\infty} |c_{nk}|^p \right)^{1/p} \\ &\leq \|C\| \cdot \|x\| \\ &= \|A\| \cdot \|x\|, \end{aligned}$$

and from here one can state the following inequality

$$\|L\| \leq \|A\| \tag{14}$$

for the norms of L and A. Now, it is time to prove the other side of this inequality. For this purpose, if the condition $e^{(k)} \in S_{\ell_1}$ $(k \in N)$ is considered, one sees that

$$\|L\| \geq \|A\| \tag{15}$$

from the equation below

$$\left\| L_C(e^{(k)}) \right\| = \left(\sum_{n=0}^{\infty} |c_{nk}|^p \right)^p.$$

When (14) and (15) are considered together, it is proved that (13).

The Hausdorff measure of the non-compactness of operators in $B(\ell_1, \ell_p(\widetilde{F}(\widetilde{r}, \widetilde{s})))$ are going to be given as stated in the following expression. Another closely related result which is going to be utilized in the first proof will be given below.

Lemma 3. *([38], [Theorem 4.2]) Let T be a triangle and χ and χ_T be the Hausdorff measures of non-compactness on M_X and M_{X_T}, respectively. Assume that X is a linear metric space with a translation invariant metric. If $Q \in M_{X_T}$, then $\chi_T(Q) = \chi(TQ)$.*

Theorem 4. *Let $L \in B(\ell_1, \ell_p(\widetilde{F}(\widetilde{r}, \widetilde{s})))$ with $(1 \leq p < \infty)$ and A demonstrate the matrix which stands for L. In that case we get*

$$\|L\|_{\chi_{\ell_p(\widetilde{F}(\widetilde{r},\widetilde{s}))}} = \lim_{m\to\infty} \left(\sup_k \sum_{n=m}^{\infty} \left| r_n \frac{f_n}{f_{n+1}} a_{jk} + s_{n-1} \frac{f_{n+1}}{f_n} a_{j-1,k} \right|^p \right)^{1/p}.$$

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Proof. Firstly, one can write in a brief manner $S = S_{\ell_1}$, also $C^{[m]}$ $(m \in \mathbb{N})$ for the matrix having the rows $C_n^{[m]} = 0$ for $0 \leq n \leq m$ and $C_n^{[m]} = C_n$ for $n \geq m+1$. Under these conditions, when both Lemma 3 and together with (3), (5), (11) and (13) are used, the following equations may easily be computed as

$$\begin{aligned}
\|L\|_{\chi_{\ell_p(\widehat{F}(\widetilde{r},\widetilde{s}))}} &= \chi_{\ell_p(\widehat{F}(\widetilde{r},\widetilde{s}))}(L(S)) \\
&= \chi_{\ell_p}(L_C(S)) \\
&= \lim_{m\to\infty}\left(\sup_{x\in S}\|\mathcal{R}_m(Cx)\|_p\right) \\
&= \lim_{m\to\infty}\left(\sup_{x\in S}\|C^{[m]}x\|_p\right) \\
&= \lim_{m\to\infty}\left\|C^{[m]}\right\| \\
&= \lim_{m\to\infty}\left(\sup_k \sum_{n=m}^{\infty}\left|r_n\frac{f_n}{f_{n+1}}a_{jk} + s_{n-1}\frac{f_{n+1}}{f_n}a_{j-1,k}\right|^p\right)^{1/p}.
\end{aligned}$$

□

One can see that in fact this is the required result.

Now it is high time to present the following theorem, which gives the characterization of $C(\ell_1, \ell_p(\widetilde{F}(\widetilde{r},\widetilde{s})))$ by coordinating the condition presented in (4) and Theorem 4.

Theorem 5. $(1 \leq p < \infty)$, *if* $L \in B(\ell_1, \ell_p(\widetilde{F}(\widetilde{r},\widetilde{s})))$ *and at the same time the matrix* A *is the matrix representing* L*, a necessary and sufficient condition for* L *to be compact is that the following limit is equal to zero, that is*

$$\lim_{m\to\infty}\left(\sup_k \sum_{n=m}^{\infty}\left|r_n\frac{f_n}{f_{n+1}}a_{jk} + s_{n-1}\frac{f_{n+1}}{f_n}a_{j-1,k}\right|^p\right) = 0.$$

References

1. F. Başar, Summability Theory and Its Applications, 2^{nd} ed., CRC Press/Taylor & Francis Group, Boca Raton. London. New York, 2022.
2. F. Başar & H. Dutta, Summable Spaces and Their Duals, Matrix Transformations and Geometric Properties, CRC Press, Taylor & Francis Group, Monographs and Research Notes in Mathematics, Boca Raton · London · New York, 2020. ISBN: 978-0-8153-5177-1.
3. M. Mursaleen, F. Başar, Sequence Spaces: Topics in Modern Summability Theory, CRC Press, Taylor & Francis Group, Series: Mathematics and Its Applications, Boca Raton · London · New York, 2020.
4. M. Mursaleen, Applied Summability Methods, Springer Briefs, 2014.
5. B. de Malafosse, E.Malkowsky, and V. Rakocevic, Operators Between Sequence Spaces and Applications, Springer Nature Singapore, 152 Beach Road, Singapore 18972, Singapore.
6. F. Başar, B. Altay, On the space of sequences of $p-$bounded variation and related matrix mappings (English, Ukrainian summary) Ukrain. Mat. Zh. 55(1) (2003), 136–147.
7. B. Altay, F. Başar, M. Mursaleen, On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ I Informs. Sci. 176(10) (2006), 1450-1462.
8. B. Altay, F. Başar, Certain topological properties and duals of the matrix domain of a triangle matrix in a sequence space, J.Math. Anal Appl. 336(1) (2007), 632–645.
9. F. Başar, E. Malkowsky, B. Altay, Matrix trasformations on the matrix domains of triangles in the spaces of strongly C_1-summable and bounded sequences, Publ. Math. Debrecen 73(1-2) (2008), 193–213.
10. M. Başarır, F. Başar, E.E. Kara, On the spaces of Fibonacci difference absolutely $p-$summable, null and convergent sequences, Sarajevo J. Math. 12(25) (2016), 167–182.

11. J. Banaś and M. Mursaleen, Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations, Springer, New Delhi, 2014.
12. A. Wilansky, Summability through Functional Analysis, North-Holland Mathematics Studies 85, Elsevier Science Publishers, Amesterdam-New York-Oxford, 1984.
13. L. S. Goldenštein, I. T. Gohberg and A. S. Markus, Investigations of some properties of bounded linear operators with their q-norms, Učen. Zap. Kishinevsk. Univ., 29 (1957) 29–36.
14. L. S. Goldenštein and A. S. Markus, On a measure of noncompactness of bounded sets and linear operators, in: Studies in Algebra and Mathematical Analysis, Kishinev, 1965, pp. 45–54.
15. K. Kuratowski, Sur les espaces complets, Fund. Math., 15 (1930) 301–309.
16. G. Darbo, Punti uniti in transformazioni a condominio non compatto, Rend. Semin. Mat. Univ. Padova, 24 (1955), 84–92.
17. R. R. Akhmerov, M. I. Kamenskij , A. S. Potapov, A. E. Rodkina and B. N. Sadovskii, Measures of Noncompactness and Condensing Operators, Operator Theory: Advances and Applications, Vol. 55, Birkhäuser, Basel, 1992.
18. J. M. Ayerbe Toledano, T. Domínguez Benavides, G. López Azedo, Measures of Noncompactness in Metric Fixed Point Theory, Birkhäuser, Basel, 1997.
19. J. Banaś and K. Goebel, Measures of Noncompactness in Banach Spaces, Lecture Notes in Pure and Applied Mathematics, Vol. 60, Marcel Dekker, New York 1980.
20. E. Malkowsky and V. Rakočević, An introduction into the theory of sequence spaces and measures of noncompactness, Zbornik radova, Mat. institut SANU (Beograd), 9(17) (2000) 143–234.
21. A. Alotaibi, E. Malkowsky and M. Mursaleen, Measure of noncompactness for compact matrix operators on some BK spaces, Filomat, 28 (2014) 1081–1086.
22. M. Başarır, E.E. Kara, On compact operators on the Riesz $B^{(m)}$-difference sequence spaces, Iran. J. Sci. Technol. 35 (A4) (2011) 279–285.
23. M. Başarır, E.E. Kara, On some difference sequence spaces of weighted means and compact operators, Ann. Funct. Anal. 2 (2011) 114–129.
24. M. Başarır, E.E. Kara, On the B-difference sequence space derived by generalized weighted mean and compact operators, J. Math. Anal. Appl. 391 (2012) 67–81.
25. E.E. Kara and M. Başarır, On compact operators and some Euler $B(m)$-difference sequence spaces, J. Math. Anal. Appl., 379 (2011) 499–511.
26. B. de Malafosse, E. Malkowsky and V. Rakočević, Measure of noncompactness of operators and matrices on the spaces c and c_0, Int. J. Math. Math. Sci., 2006 (2006) 1–5.
27. B. de Malafosse and V. Rakočević, Applications of measure of noncompactness in operators on the spaces s_α, s_α^0, $s_\alpha^{(c)}$, ℓ_α^p, J. Math. Anal. Appl., 323(1) (2006) 131–145.
28. M. Mursaleen, V. Karakaya, H. Polat and N. Simsek, Measure of noncompactness of matrix operators on some difference sequence spaces of weighted means, Comput. Math. Appl., 62 (2011) 814–820.
29. M. Mursaleen and S. A. Mohiuddine, Applications of measures of noncompactness to the infinite system of differential equations in ℓ_p spaces, Nonlinear Analysis, 75 (2012) 2111–2115.
30. M. Mursaleen and A. K. Noman, Compactness by the Hausdorff measure of noncompactness, Nonlinear Anal, 73 (2010) 2541–2557.
31. M. Mursaleen and A. K. Noman, Compactness of matrix operators on some new difference sequence spaces, Linear Algebra Appl., 436(1) (2012) 41–52.
32. T. Koshy, Fibonacci and Lucas Numbers with Applications, Wiley, 2001.
33. E.E. Kara, Some topological and geometrical properties of new Banach sequence spaces, J. Inequal. Appl. 2013 (38) (2013), 15 pages.
34. M. Candan, E.E. Kara, A study on topological and geometrical characteristics of new Banach sequence spaces, Gulf J. Math 3(4),(2015) 67–84.
35. M. Candan, A robust approach about compact operators on some generalized Fibonacci difference sequence spaces, FCMS∞ doi:/fcmathsci.1303769 (2024) 5(1) 1-12.
36. A. Alotaibi, M. Mursaleen, B.AS Alamri and S.A. Mohiuddine, Compact operators on some Fibonacci difference sequence spaces, J. Inequal. Appl. 2015 (203) (2015), 8 pages.

 doi:10.20944/preprints202401.0143.v1

37. E.E. Kara, M. Başarır and M. Mursaleen,Compactness of matrix operators on some sequence spaces derived by Fibonacci numbers, arXiv:1309.0152v1 [math.FA] 31 Aug 2013.
38. E. Malkowsky, V. Rakočević, On matrix domains of triangles, Appl. Math. Comput. 187 (2007), 1146–1163.