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Functional Kuppinger-Durisi-Bölcskei Uncertainty Principle

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Functional Kuppinger-Durisi-Bölcskei Uncertainty Principle

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Abstract: Let $\mathcal{X}$ be a Banach space. Let $\{\tau_j\}_{j=1}^n, \{\omega_k\}_{k=1}^m \subseteq \mathcal{X}$ and $\{f_j\}_{j=1}^n$, $\{g_k\}_{k=1}^m \subseteq \mathcal{X}^*$ satisfy $|f_j(\tau_j)| \geq 1$ for all $1 \leq j \leq n$, $|g_k(\omega_k)| \geq 1$ for all $1 \leq k \leq m$. If $x \in \mathcal{X} \setminus \{0\}$ is such that $x = \theta_\tau \theta_f x = \theta_\omega \theta_g x$, then we show that $\|\theta_f x\|_0 \|\theta_g x\|_0 \geq \frac{\left[1-(\|\theta_f x\|_0-1)\max\limits_{1\leq j,r\leq n, j\neq r}|f_j(\tau_r)|\right]^+\left[1-(\|\theta_g x\|_0-1)\max\limits_{1\leq k,s\leq m, k\neq s}|g_k(\omega_s)|\right]^+}{\left(\max\limits_{1\leq j\leq n, 1\leq k\leq m}|f_j(\omega_k)|\right)\left(\max\limits_{1\leq j\leq n, 1\leq k\leq m}|g_k(\tau_j)|\right)}$. (1) We call Inequality (1) as **Functional Kuppinger-Durisi-Bölcskei Uncertainty Principle**. Inequality (1) improves the uncertainty principle obtained by Kuppinger, Durisi and Bölcskei *[IEEE Trans. Inform. Theory (2012)]* (which improved the Donoho-Stark-Elad-Bruckstein uncertainty principle *[SIAM J. Appl. Math. (1989), IEEE Trans. Inform. Theory (2002)]*). We also derive functional form of the uncertainity principle obtained by Studer, Kuppinger, Pope and Bölcskei *[EEE Trans. Inform. Theory (2012)]*.

Keywords: uncertainty principle; hilbert space; banach space

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1. Introduction

Let $\hat{}: \mathbb{C}^d \to \mathbb{C}^d$ be the Fourier transform. For $h \in \mathbb{C}^d$, let $\|h\|_0$ be the number of nonzero entries in h. It is correct to say that the progress of today's world is not possible without the following result of Donoho and Stark [3].

Theorem 1.1 ([3] **Donoho-Stark Uncertainty Principle**). *For every $d \in \mathbb{N}$,*

$$\left(\frac{\|h\|_0 + \|\widehat{h}\|_0}{2}\right)^2 \geq \|h\|_0\|\widehat{h}\|_0 \geq d, \quad \forall h \in \mathbb{C}^d \setminus \{0\}. \tag{2}$$

Given a collection $\{\tau_j\}_{j=1}^n$ in a finite dimensional Hilbert space $\mathcal{H}$ over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), we define

$$\theta_\tau : \mathcal{H} \ni h \mapsto \theta_\tau h := (\langle h, \tau_j\rangle)_{j=1}^n \in \mathbb{K}^n.$$

Elad and Bruckstein generalized Inequality (2) to pairs of orthonormal bases [4].

Theorem 1.2 ([4] **Elad-Bruckstein Uncertainty Principle**). *Let $\{\tau_j\}_{j=1}^n$, $\{\omega_j\}_{j=1}^n$ be two orthonormal bases for a finite dimensional Hilbert space $\mathcal{H}$. Then*

$$\left(\frac{\|\theta_\tau h\|_0 + \|\theta_\omega h\|_0}{2}\right)^2 \geq \|\theta_\tau h\|_0\|\theta_\omega h\|_0 \geq \frac{1}{\max\limits_{1\leq j,k\leq n}|\langle \tau_j, \omega_k\rangle|^2}, \quad \forall h \in \mathcal{H} \setminus \{0\}.$$

For $a \in \mathbb{R}$, set $a^+ := \max\{0, a\}$. Kuppinger, Durisi and Bölcskei showed that Theorem 1.2 can be improved to unit norm vectors [12].

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Theorem 1.3 ([12] **Kuppinger-Durisi-Bölcskei Uncertainty Principle).** *Let $\{\tau_j\}_{j=1}^n$, $\{\omega_k\}_{k=1}^m$ be two collections of unit vectors in a finite dimensional Hilbert space $\mathcal{H}$. If $h \in \mathcal{H} \setminus \{0\}$ is such that*

$$h = \theta_\tau^* \theta_\tau h = \theta_\omega^* \theta_\omega h, \tag{3}$$

then

$$\|\theta_\tau h\|_0 \|\theta_\omega h\|_0 \geq \frac{\left[1 - (\|\theta_\tau h\|_0 - 1) \max\limits_{1 \leq j, r \leq n, j \neq r} |\langle \tau_j, \tau_r \rangle|\right]^+ \left[1 - (\|\theta_\omega h\|_0 - 1) \max\limits_{1 \leq k, s \leq m, k \neq s} |\langle \omega_k, \omega_s \rangle|\right]^+}{\max\limits_{1 \leq j \leq n, 1 \leq k \leq m} |\langle \tau_j, \omega_k \rangle|^2}.$$

Let $0 \leq \varepsilon < 1$. Recall that [3] a vector $(a_j)_{j=1}^n \in \mathbb{K}^n$ is said to be **ε-concentrated on a subset** $M \subseteq \{1, \dots, n\}$ **w.r.t. 1-norm** if

$$\sum_{j \in M} |a_j| \geq (1 - \varepsilon) \sum_{j=1}^n |a_j| \iff \varepsilon \sum_{j=1}^n |a_j| \geq \sum_{j \in M^c} |a_j|.$$

Theorem 1.3 has been improved by Studer, Kuppinger, Pope and Bölcskei [17]. In the following theorem and in rest of the paper, given a subset $M \subseteq \mathbb{N}$, the number of elements in M is denoted by $o(M)$.

Theorem 1.4 ([17] **Studer-Kuppinger-Pope-Bölcskei Uncertainty Principle).** *Let $\{\tau_j\}_{j=1}^n$, $\{\omega_k\}_{k=1}^m$ be two collections of unit vectors in a finite dimensional Hilbert space $\mathcal{H}$. Let $h \in \mathcal{H} \setminus \{0\}$ be such that*

$$h = \theta_\tau^* \theta_\tau h = \theta_\omega^* \theta_\omega h.$$

If $\theta_\tau h$ is ε-concentrated on a subset $M \subseteq \{1, \dots, n\}$ w.r.t. 1-norm and $\theta_\omega h$ is δ-concentrated on a subset $N \subseteq \{1, \dots, n\}$ w.r.t. 1-norm, then

$$o(M) o(N) \geq \frac{\left[1 - \varepsilon - (o(M) - 1 + \varepsilon) \max\limits_{1 \leq j, r \leq n, j \neq r} |\langle \tau_j, \tau_r \rangle|\right]^+ \left[1 - \delta - (o(N) - 1 + \delta) \max\limits_{1 \leq k, s \leq m, k \neq s} |\langle \omega_k, \omega_s \rangle|\right]^+}{\max\limits_{1 \leq j \leq n, 1 \leq k \leq m} |\langle \tau_j, \omega_k \rangle|^2}.$$

When $\varepsilon = 0$, Theorem 1.4 reduces to Theorem 1.3. In this paper, we derive both finite and infinite dimensional Banach space versions of Theorems 1.3 and 1.4. It is reasonable to note that Theorem 1.2 has been improved using Parseval frames for Hilbert spaces by Ricaud and Torrésani [16] and later extended to Banach spaces in the paper [10]. Most important thing to keep in mind is that uncertainty principle derived in [16] is for Parseval frames (which says vectors have norm less than or equal to one) which is not required in Theorem 1.3 (but with the condition that vectors are unit vectors). Also note that it is not required the validity of Equation (3) for all $h \in \mathcal{H}$ (in that case, both will become orthonormal bases).

2. Functional Kuppinger-Durisi-Bölcskei Uncertainty Principle

In the paper, $\mathbb{K}$ denotes $\mathbb{C}$ or $\mathbb{R}$ and $\mathcal{X}$ denotes a Banach space over $\mathbb{K}$. Dual of $\mathcal{X}$ is denoted by $\mathcal{X}^*$. Given a collection $\{\tau_j\}_{j=1}^n$ in $\mathcal{X}$ and a collection $\{f_j\}_{j=1}^n$ in $\mathcal{X}^*$ we define

$$\theta_f : \mathcal{X} \ni x \mapsto \theta_f x := (f_j(x))_{j=1}^n \in \mathbb{K}^n,$$

$$\theta_\tau : \mathbb{K}^n \ni (a_j)_{j=1}^n \mapsto \sum_{j=1}^n a_j \tau_j \in \mathcal{X}.$$

Following is the Banach space generalization of Theorem 1.3.

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Theorem 2.1 (Functional Kuppinger-Durisi-Bölcskei Uncertainty Principle). *Let $\{\tau_j\}_{j=1}^n$, $\{\omega_k\}_{k=1}^m$ be two collections in a finite dimensional Banach space $\mathcal{X}$ and $\{f_j\}_{j=1}^n$, $\{g_k\}_{k=1}^m$ be two collections in $\mathcal{X}^*$ satisfying*

$$|f_j(\tau_j)| \geq 1, \ \forall 1 \leq j \leq n, \quad |g_k(\omega_k)| \geq 1, \ \forall 1 \leq k \leq m.$$

If $x \in \mathcal{X} \setminus \{0\}$ is such that

$$x = \theta_\tau \theta_f x = \theta_\omega \theta_g x, \tag{4}$$

then

$$\|\theta_f x\|_0 \|\theta_g x\|_0 \geq \frac{\left[1 - (\|\theta_f x\|_0 - 1) \max\limits_{1 \leq j, r \leq n, j \neq r} |f_j(\tau_r)|\right]^+ \left[1 - (\|\theta_g x\|_0 - 1) \max\limits_{1 \leq k, s \leq m, k \neq s} |g_k(\omega_s)|\right]^+}{\left(\max\limits_{1 \leq j \leq n, 1 \leq k \leq m} |f_j(\omega_k)|\right) \left(\max\limits_{1 \leq j \leq n, 1 \leq k \leq m} |g_k(\tau_j)|\right)}.$$

Proof. Let $1 \leq j \leq n$. Then using Equation (4)

$$\begin{aligned}
|f_j(x)| &= |f_j(\theta_\tau \theta_f x)| = \left| f_j \left(\sum_{r=1}^n f_r(x) \tau_r \right) \right| = \left| \sum_{r=1}^n f_r(x) f_j(\tau_r) \right| \\
&= \left| f_j(x) f_j(\tau_j) + \sum_{r=1, r \neq j}^n f_r(x) f_j(\tau_r) \right| \geq |f_j(x) f_j(\tau_j)| - \left| \sum_{r=1, r \neq j}^n f_r(x) f_j(\tau_r) \right| \\
&\geq |f_j(x)| - \left| \sum_{r=1, r \neq j}^n f_r(x) f_j(\tau_r) \right| \geq |f_j(x)| - \sum_{r=1, r \neq j}^n |f_r(x) f_j(\tau_r)| \\
&\geq |f_j(x)| - \left(\sum_{r=1, r \neq j}^n |f_r(x)| \right) \max_{1 \leq j, r \leq n, j \neq r} |f_j(\tau_r)| \\
&= |f_j(x)| - \left(\sum_{r=1}^n |f_r(x)| - |f_j(x)| \right) \max_{1 \leq j, r \leq n, j \neq r} |f_j(\tau_r)| \\
&= |f_j(x)| - \left(\|\theta_f x\|_1 - |f_j(x)| \right) \max_{1 \leq j, r \leq n, j \neq r} |f_j(\tau_r)| \\
&= \left(1 + \max_{1 \leq j, r \leq n, j \neq r} |f_j(\tau_r)| \right) |f_j(x)| - \|\theta_f x\|_1 \max_{1 \leq j, r \leq n, j \neq r} |f_j(\tau_r)|.
\end{aligned}$$

On the other hand, again using Equation (4)

$$\begin{aligned}
|f_j(x)| &= |f_j(\theta_\omega \theta_g x)| = \left| f_j \left(\sum_{k=1}^m g_k(x) \omega_k \right) \right| = \left| \sum_{k=1}^m g_k(x) f_j(\omega_k) \right| \\
&\leq \sum_{k=1}^m |g_k(x) f_j(\omega_k)| \leq \left(\sum_{k=1}^m |g_k(x)| \right) \max_{1 \leq j \leq n, 1 \leq k \leq m} |f_j(\omega_k)| \\
&= \|\theta_g x\|_1 \max_{1 \leq j \leq n, 1 \leq k \leq m} |f_j(\omega_k)|
\end{aligned}$$

Therefore we have

$$\left(1 + \max_{1 \leq j, r \leq n, j \neq r} |f_j(\tau_r)| \right) |f_j(x)| - \|\theta_f x\|_1 \max_{1 \leq j, r \leq n, j \neq r} |f_j(\tau_r)| \leq \|\theta_g x\|_1 \max_{1 \leq j \leq n, 1 \leq k \leq m} |f_j(\omega_k)| \tag{5}$$

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Summing Inequality (5) on the support of $\theta_f x$ we get

$$\left(1+\max_{1\leq j,r\leq n, j\neq r}|f_j(\tau_r)|\right)\sum_{j\in \operatorname{supp}(\theta_f x)}|f_j(x)|-\|\theta_f x\|_1\left(\max_{1\leq j,r\leq n, j\neq r}|f_j(\tau_r)|\right)\sum_{j\in \operatorname{supp}(\theta_f x)}1\leq$$
$$\|\theta_g x\|_1\left(\max_{1\leq j\leq n, 1\leq k\leq m}|f_j(\omega_k)|\right)\sum_{j\in \operatorname{supp}(\theta_f x)}1,$$

i.e.,

$$\left(1+\max_{1\leq j,r\leq n, j\neq r}|f_j(\tau_r)|\right)\|\theta_f x\|_1-\|\theta_f x\|_1\left(\max_{1\leq j,r\leq n, j\neq r}|f_j(\tau_r)|\right)\|\theta_f x\|_0\leq$$
$$\|\theta_g x\|_1\left(\max_{1\leq j\leq n, 1\leq k\leq m}|f_j(\omega_k)|\right)\|\theta_f x\|_0,$$

i.e.,

$$\left[1-(\|\theta_f x\|_0-1)\max_{1\leq j,r\leq n, j\neq r}|f_j(\tau_r)|\right]\|\theta_f x\|_1\leq\|\theta_g x\|_1\left(\max_{1\leq j\leq n, 1\leq k\leq m}|f_j(\omega_k)|\right)\|\theta_f x\|_0.$$

Since the right side of previous inequality is non negative, we have

$$\left[1-(\|\theta_f x\|_0-1)\max_{1\leq j,r\leq n, j\neq r}|f_j(\tau_r)|\right]^+\|\theta_f x\|_1\leq\|\theta_g x\|_1\left(\max_{1\leq j\leq n, 1\leq k\leq m}|f_j(\omega_k)|\right)\|\theta_f x\|_0. \quad (6)$$

Similarly

$$\left[1-(\|\theta_g x\|_0-1)\max_{1\leq k,s\leq m, k\neq s}|g_k(\omega_s)|\right]^+\|\theta_g x\|_1\leq\|\theta_f x\|_1\left(\max_{1\leq j\leq n, 1\leq k\leq m}|g_k(\tau_j)|\right)\|\theta_g x\|_0. \quad (7)$$

Multiplying Inequalities (6) and (7) we get

$$\left[1-(\|\theta_f x\|_0-1)\max_{1\leq j,r\leq n, j\neq r}|f_j(\tau_r)|\right]^+\left[1-(\|\theta_g x\|_0-1)\max_{1\leq k,s\leq m, k\neq s}|g_k(\omega_s)|\right]^+\|\theta_f x\|_1\|\theta_g x\|_1\leq$$
$$\|\theta_g x\|_1\|\theta_f x\|_1\|\theta_f x\|_0\|\theta_g x\|_0\left(\max_{1\leq j\leq n, 1\leq k\leq m}|f_j(\omega_k)|\right)\left(\max_{1\leq j\leq n, 1\leq k\leq m}|g_k(\tau_j)|\right).$$

A cancellation of $\|\theta_f x\|_1\|\theta_g x\|_1$ gives the required inequality. □

Next we derive Banach space version of Theorem 1.4.

Theorem 2.2 (Functional Studer-Kuppinger-Pope-Bölcskei Uncertainty Principle). *Let $\{\tau_j\}_{j=1}^n$, $\{\omega_k\}_{k=1}^m$ be two collections in a finite dimensional Banach space $\mathcal{X}$ and $\{f_j\}_{j=1}^n$, $\{g_k\}_{k=1}^m$ be two collections in $\mathcal{X}^*$ satisfying*

$$|f_j(\tau_j)|\geq 1,\ \forall 1\leq j\leq n,\quad |g_k(\omega_k)|\geq 1,\ \forall 1\leq k\leq m. \quad (8)$$

Let $x\in\mathcal{X}\setminus\{0\}$ be such that

$$x=\theta_\tau\theta_f x=\theta_\omega\theta_g x. \quad (9)$$

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If $\theta_f x$ is ε-concentrated on a subset $M \subseteq \{1, \ldots, n\}$ w.r.t. 1-norm and $\theta_g x$ is δ-concentrated on a subset $N \subseteq \{1, \ldots, n\}$ w.r.t. 1-norm, then

$$o(M)o(N) \geq \frac{\left[1 - \varepsilon - (o(M) - 1 + \varepsilon) \max_{1 \leq j,r \leq n, j \neq r} |f_j(\tau_r)|\right]^+ \left[1 - \delta - (o(N) - 1 + \delta) \max_{1 \leq k,s \leq m, k \neq s} |g_k(\omega_s)|\right]^+}{\left(\max_{1 \leq j \leq n, 1 \leq k \leq m} |f_j(\omega_k)|\right) \left(\max_{1 \leq j \leq n, 1 \leq k \leq m} |g_k(\tau_j)|\right)}. \quad (10)$$

Proof. We start by using Equation (5). Let $1 \leq j \leq n$. Then

$$\left(1 + \max_{1 \leq j,r \leq n, j \neq r} |f_j(\tau_r)|\right) |f_j(x)| - \|\theta_f x\|_1 \max_{1 \leq j,r \leq n, j \neq r} |f_j(\tau_r)| \leq \|\theta_g x\|_1 \max_{1 \leq j \leq n, 1 \leq k \leq m} |f_j(\omega_k)| \quad (11)$$

Summing Inequality (11) on the support of M we get

$$\left(1 + \max_{1 \leq j,r \leq n, j \neq r} |f_j(\tau_r)|\right) \sum_{j \in M} |f_j(x)| - \|\theta_f x\|_1 \left(\max_{1 \leq j,r \leq n, j \neq r} |f_j(\tau_r)|\right) \sum_{j \in M} 1 \leq$$
$$\|\theta_g x\|_1 \left(\max_{1 \leq j \leq n, 1 \leq k \leq m} |f_j(\omega_k)|\right) \sum_{j \in M} 1,$$

i.e.,

$$\left(1 + \max_{1 \leq j,r \leq n, j \neq r} |f_j(\tau_r)|\right) \sum_{j \in M} |f_j(x)| - \|\theta_f x\|_1 \left(\max_{1 \leq j,r \leq n, j \neq r} |f_j(\tau_r)|\right) o(M) \leq$$
$$\|\theta_g x\|_1 \left(\max_{1 \leq j \leq n, 1 \leq k \leq m} |f_j(\omega_k)|\right) o(M). \quad (12)$$

Since $\theta_f x$ is ε-concentrated on M we are given with

$$\sum_{j \in M} |f_j(x)| \geq (1 - \varepsilon) \sum_{j=1}^{n} |f_j(x)|. \quad (13)$$

Using Inequality (13) in Inequality (12) we get

$$\left(1 + \max_{1 \leq j,r \leq n, j \neq r} |f_j(\tau_r)|\right) (1 - \varepsilon) \sum_{j=1}^{n} |f_j(x)| - \|\theta_f x\|_1 \left(\max_{1 \leq j,r \leq n, j \neq r} |f_j(\tau_r)|\right) o(M) \leq$$
$$\left(1 + \max_{1 \leq j,r \leq n, j \neq r} |f_j(\tau_r)|\right) \sum_{j \in M} |f_j(x)| - \|\theta_f x\|_1 \left(\max_{1 \leq j,r \leq n, j \neq r} |f_j(\tau_r)|\right) o(M) \leq$$
$$\|\theta_g x\|_1 \left(\max_{1 \leq j \leq n, 1 \leq k \leq m} |f_j(\omega_k)|\right) o(M),$$

i.e.,

$$\left(1 + \max_{1 \leq j,r \leq n, j \neq r} |f_j(\tau_r)|\right) (1 - \varepsilon) \|\theta_f x\|_1 - \|\theta_f x\|_1 \left(\max_{1 \leq j,r \leq n, j \neq r} |f_j(\tau_r)|\right) o(M) \leq$$
$$\|\theta_g x\|_1 \left(\max_{1 \leq j \leq n, 1 \leq k \leq m} |f_j(\omega_k)|\right) o(M),$$

i.e.,

$$\left[1 - \varepsilon - (o(M) - 1 + \varepsilon) \max_{1 \leq j,r \leq n, j \neq r} |f_j(\tau_r)|\right] \|\theta_f x\|_1 \leq \|\theta_g x\|_1 \left(\max_{1 \leq j \leq n, 1 \leq k \leq m} |f_j(\omega_k)|\right) o(M).$$

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Since the right side of previous inequality is non negative, we have

$$\left[1-\varepsilon-(o(M)-1+\varepsilon)\max_{1\leq j,r\leq n, j\neq r}|f_j(\tau_r)|\right]^+\|\theta_f x\|_1 \leq \|\theta_g x\|_1\left(\max_{1\leq j\leq n, 1\leq k\leq m}|f_j(\omega_k)|\right)o(M). \quad (14)$$

Similarly

$$\left[1-\delta-(o(N)-1+\delta)\max_{1\leq k,s\leq m, k\neq s}|g_k(\omega_s)|\right]^+\|\theta_g x\|_1 \leq \|\theta_f x\|_1\left(\max_{1\leq j\leq n, 1\leq k\leq m}|g_k(\tau_j)|\right)o(N). \quad (15)$$

Multiplying Inequalities (14) and (15) we get

$$\left[1-\varepsilon-(o(M)-1+\varepsilon)\max_{1\leq j,r\leq n, j\neq r}|f_j(\tau_r)|\right]^+\left[1-\delta-(o(N)-1+\delta)\max_{1\leq k,s\leq m, k\neq s}|g_k(\omega_s)|\right]^+\|\theta_f x\|_1\|\theta_g x\|_1 \leq$$
$$\|\theta_g x\|_1\|\theta_f x\|_1\|o(M)o(N)\left(\max_{1\leq j\leq n, 1\leq k\leq m}|f_j(\omega_k)|\right)\left(\max_{1\leq j\leq n, 1\leq k\leq m}|g_k(\tau_j)|\right).$$

By canceling $\|\theta_f x\|_1\|\theta_g x\|_1$ we get the inequality in the statement of theorem. □

Note that $\theta_f x$ (resp. $\theta_g x$) is 0-supported on $\text{supp}(\theta_f x)$ (resp. $\text{supp}(\theta_g x)$). Hence Theorem 2.1 follows from Theorem 2.2.

Corollary 2.3. *Theorem 1.4 follows from Theorem 2.2.*

Proof. Given two collections $\{\tau_j\}_{j=1}^n$, $\{\omega_k\}_{k=1}^m$ of unit vectors in a finite dimensional Hilbert space $\mathcal{H}$, by defining

$$f_j : \mathcal{H} \ni h \mapsto \langle h, \tau_j\rangle \in \mathbb{K}; \quad \forall 1 \leq j \leq n, \quad g_k : \mathcal{H} \ni h \mapsto \langle h, \omega_k\rangle \in \mathbb{K}, \quad \forall 1 \leq k \leq m$$

we get the result. □

Theorem 2.2 brings the following question.

Question 2.4. *Given a Banach space $\mathcal{X}$ for which subsets $M, N \subseteq \mathbb{N}$ and pairs $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$, $(\{g_k\}_{k=1}^m, \{\omega_k\}_{k=1}^m)$ satisfying (8) and (9) we have equality in Inequality (10)?*

3. Infinite dimensional Functional Kuppinger-Durisi-Bölcskei Uncertainty Principle

In this section we derive infinite dimensional versions of Theorem 2.1 and Theorem 2.2. Unlike finite dimensions, we cannot start with arbitrary infinite collection of elements in a Banach space. Following restricted class of collection has to be used.

Definition 3.1 ([11]). *Let $\mathcal{X}$ be a Banach space, $\{\tau_j\}_{j=1}^\infty \subseteq \mathcal{X}$ and $\{f_j\}_{j=1}^\infty \subseteq \mathcal{X}^*$. The pair $(\{f_j\}_{j=1}^\infty, \{\tau_j\}_{j=1}^\infty)$ is said to be a* ***1-approximate Bessel sequence*** *(1-ABS) for $\mathcal{X}$ if following conditions hold.*

(i) *The map*

$$\theta_f : \mathcal{X} \ni x \mapsto \theta_f x := \{f_j(x)\}_{j=1}^\infty \in \ell^1(\mathbb{N})$$

is a well-defined bounded linear operator.

(ii) *The map*

$$\theta_\tau : \ell^1(\mathbb{N}) \ni \{a_j\}_{j=1}^\infty \mapsto \sum_{j=1}^\infty a_j\tau_j \in \mathcal{X}$$

is a well-defined bounded linear operator.

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Theorem 3.2. *Let $(\{f_j\}_{j=1}^{\infty}, \{\tau_j\}_{j=1}^{\infty})$ and $(\{g_k\}_{k=1}^{\infty}, \{\omega_k\}_{k=1}^{\infty})$ be two 1-ABS for a Banach space $\mathcal{X}$ satisfying*

$$|f_j(\tau_j)| \geq 1, \ \forall j \in \mathbb{N}, \quad |g_k(\omega_k)| \geq 1, \ \forall k \in \mathbb{N}.$$

If $x \in \mathcal{X} \setminus \{0\}$ is such that

$$x = \theta_\tau \theta_f x = \theta_\omega \theta_g x,$$

then

$$\|\theta_f x\|_0 \|\theta_g x\|_0 \geq \frac{\left[1 - (\|\theta_f x\|_0 - 1) \sup_{j,r \in \mathbb{N}, j \neq r} |f_j(\tau_r)|\right]^+ \left[1 - (\|\theta_g x\|_0 - 1) \sup_{k,s \in \mathbb{N}, k \neq s} |g_k(\omega_s)|\right]^+}{\left(\sup_{j,k \in \mathbb{N}} |f_j(\omega_k)|\right) \left(\sup_{j,k \in \mathbb{N}} |g_k(\tau_j)|\right)}.$$

Proof. Let $j \in \mathbb{N}$. Then

$$\begin{aligned}
|f_j(x)| &= |f_j(\theta_\tau \theta_f x)| = \left| f_j\left(\sum_{r=1}^{\infty} f_r(x)\tau_r\right)\right| = \left|\sum_{r=1}^{\infty} f_r(x) f_j(\tau_r)\right| \\
&= \left| f_j(x) f_j(\tau_j) + \sum_{r=1, r\neq j}^{\infty} f_r(x) f_j(\tau_r)\right| \geq |f_j(x) f_j(\tau_j)| - \left|\sum_{r=1, r\neq j}^{\infty} f_r(x) f_j(\tau_r)\right| \\
&\geq |f_j(x)| - \left|\sum_{r=1, r\neq j}^{\infty} f_r(x) f_j(\tau_r)\right| \geq |f_j(x)| - \sum_{r=1, r\neq j}^{\infty} |f_r(x) f_j(\tau_r)| \\
&\geq |f_j(x)| - \left(\sum_{r=1, r\neq j}^{\infty} |f_r(x)|\right) \sup_{j,r \in \mathbb{N}, j \neq r} |f_j(\tau_r)| \\
&= |f_j(x)| - \left(\sum_{r=1}^{\infty} |f_r(x)| - |f_j(x)|\right) \sup_{j,r \in \mathbb{N}, j \neq r} |f_j(\tau_r)| \\
&= |f_j(x)| - \left(\|\theta_f x\|_1 - |f_j(x)|\right) \sup_{j,r \in \mathbb{N}, j \neq r} |f_j(\tau_r)| \\
&= \left(1 + \sup_{1 \leq j,r \leq n, j \neq r} |f_j(\tau_r)|\right) |f_j(x)| - \|\theta_f x\|_1 \sup_{j,r \in \mathbb{N}, j \neq r} |f_j(\tau_r)|.
\end{aligned}$$

We also find

$$\begin{aligned}
|f_j(x)| &= |f_j(\theta_\omega \theta_g x)| = \left| f_j\left(\sum_{k=1}^{\infty} g_k(x)\omega_k\right)\right| = \left|\sum_{k=1}^{\infty} g_k(x) f_j(\omega_k)\right| \\
&\leq \sum_{k=1}^{\infty} |g_k(x) f_j(\omega_k)| \leq \left(\sum_{k=1}^{\infty} |g_k(x)|\right) \sup_{j,k \in \mathbb{N}} |f_j(\omega_k)| \\
&= \|\theta_g x\|_1 \sup_{j,k \in \mathbb{N}} |f_j(\omega_k)|.
\end{aligned}$$

Now by doing a similar type of calculation as in the proof of Theorem 2.1 we get the result. □

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We recall that a vector $\{a_j\}_{j=1}^{\infty} \in \ell^1(\mathbb{N})$ is said to be ε-concentrated on a subset $M \subseteq \mathbb{N}$ w.r.t. 1-norm if

$$\sum_{j\in M} |a_j| \geq (1-\varepsilon)\sum_{j=1}^{\infty} |a_j| \iff \varepsilon \sum_{j=1}^{\infty} |a_j| \geq \sum_{j\in M^c} |a_j|.$$

It is a easy to see the following infinite dimensional version of Theorem 2.2.

Theorem 3.3. *Let $(\{f_j\}_{j=1}^{\infty}, \{\tau_j\}_{j=1}^{\infty})$ and $(\{g_k\}_{k=1}^{\infty}, \{\omega_k\}_{k=1}^{\infty})$ be two 1-ABS for a Banach space $\mathcal{X}$ satisfying*

$$|f_j(\tau_j)| \geq 1, \ \forall j \in \mathbb{N}, \quad |g_k(\omega_k)| \geq 1, \ \forall k \in \mathbb{N}.$$

Let $x \in \mathcal{X} \setminus \{0\}$ be such that

$$x = \theta_\tau \theta_f x = \theta_\omega \theta_g x.$$

If $\theta_f x$ is ε-concentrated on a subset $M \subseteq \mathbb{N}$ w.r.t. 1-norm and $\theta_g x$ is δ-concentrated on a subset $N \subseteq \mathbb{N}$ w.r.t. 1-norm, then

$$o(M)o(N) \geq \frac{\left[1-\varepsilon-(o(M)-1+\varepsilon)\sup\limits_{j,r\in\mathbb{N}, j\neq r} |f_j(\tau_r)|\right]^+ \left[1-\delta-(o(N)-1+\delta)\sup\limits_{k,s\in\mathbb{N}, k\neq s} |g_k(\omega_s)|\right]^+}{\left(\sup\limits_{j,k\in\mathbb{N}} |f_j(\omega_k)|\right)\left(\sup\limits_{j,k\in\mathbb{N}} |g_k(\tau_j)|\right)}.$$

The techniques used in [10] have been extended to derive continuous versions of uncertainty principles for Banach spaces using Lebesgue function spaces [9]. However, it seems that the techniques used in this paper cannot be extended to get continuous versions of the results derived in this paper.

We end the paper with the following two interesting and important questions.

Question 3.4. (i) *Can Theorem 2.2 be improved using divisors of the dimension of the space (like Roy uncertainty principle [13,14], Murty-Whang uncertainty principle [15]). In particular, for prime dimensional Banach spaces (like Tao uncertainty principle [18])?*

(ii) *What are the versions Theorem 2.2 and the results in [10] for vector spaces over finite fields (like Goldstein-Guralnick-Isaacs uncertainty principle [8], Evra-Kowalski-Lubotzky uncertainty principle [5], Borello-Willems-Zini uncertainty principle [2], Feng-Hollmann-Xiang uncertainty principle [6], Garcia-Karaali-Katz uncertainty principle [7] and Borello-Solé uncertainty principle [1])?*

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