Communication

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Metallic Ratios are Uniquely Defined by an Acute Angle of a Right Triangle

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Posted Date: 8 January 2024

doi: 10.20944/preprints202401.0551.v1

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Article

Metallic Ratios Are Uniquely Defined by an Acute Angle of a Right Triangle

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Abstract: We show that metallic ratios for $k \in \mathbb{R}$ are uniquely defined by an acute angle of a right triangle, where for $k \neq \{0, \pm 2\}$ this angle is the argument of a normalized complex number, and for $k \in \mathbb{Z}$ this number is defined by a Pythagorean triple.

Keywords: metallic ratios; Pythagorean triples; emergent dimensionality

1. Introduction

Metallic ratios of $k \in \mathbb{N}_0$ are defined as

$$M_{k\pm} := \frac{k \pm \sqrt{k^2+4}}{2}, \tag{1}$$

being the roots of the quadratic equation

$$M_{k\pm}^2 - kM_{k\pm} - 1 = 0, \tag{2}$$

with the property $M_{k-} = -1/M_{k+} = k - M_{k+}$. They are shown in Figure 1.

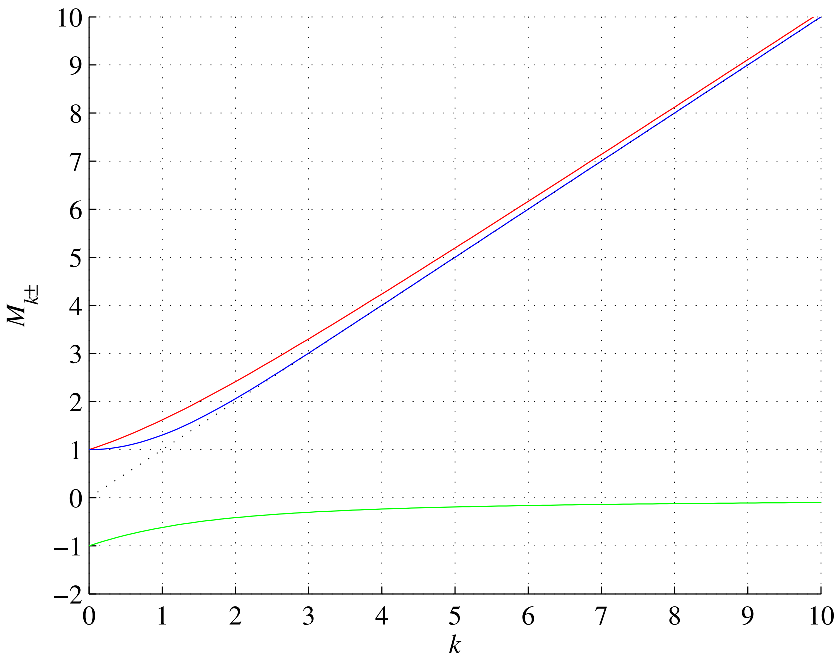

Figure 1. Metallic ratios: positive M_{k+} (red), negative M_{k-} (green) as continuous functions of $0 \leq k \leq 10$ and Ł-K metric $D_{NN}(k)$, $s = \sqrt{\pi}/2$ (blue).

The positive metallic ratio M_{k+} as a continuous function of $k \in \mathbb{R}$ has the same property as the Łukaszyk-Karmowski (Ł-K) metric [1] between two independent continuous random variables:

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It becomes asymptotic to $M_{k+} = k$ as k goes to ifinity, as the $+4$ factor in the square root becomes negligible and $M_{k\pm} \approx \{k, 0\}$ for large k. Because the ratios (1) are usually visualized as ratios of the edge lengths of a rectangle and these are assumed to be nonnegative, usually only the positive principal square root M_{k+} of (2) is considered, where for $k = 1$ the golden ratio is obtained, for $k = 2$ the silver ratio, for $k = 3$ the bronze ratio, etc. However, distance nonnegativity does not hold for the Ł–K metric [1], for example; such axiomatization may be misleading [2].

It was shown [3] that for $k \neq \{0, 2\}$ the metallic ratios (1) can be expressed by primitive Pythagorean triples, as

$$M_{k+} = \cot\left(\frac{\theta}{4}\right), \tag{3}$$

and for $k \geq 3$

$$k = 2\sqrt{\frac{c+b}{c-b}}, \tag{4}$$

where θ is the angle between a longer cathetus b and hypotenuse c of a right triangle defined by a Pythagorean triple, as shown in Figure 2, whereas for $k = \{3, 4\}$ it is the angle between a hypotenuse and a shorter cathetus a ($\{M_{3+}, M_{10+}\}$ and $\{M_{4+}, M_{6+}\}$ are defined by the same Pythagorean triples, respectively, $(5, 12, 13)$ and $(3, 4, 5)$), and

$$M_{1+} = \cot\left(\frac{\pi - \theta_{(3,5)}}{4}\right). \tag{5}$$

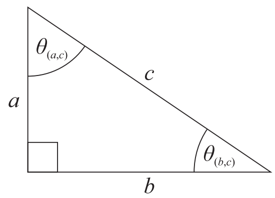

Figure 2. Right triangle showing a longer (b), shorter (a) hypotenuse, catheti (c) and angles $\theta = \theta_{(b,c)}$ and $\theta_{(a,c)}$.

For example the Pythagorean triple $(20, 21, 29)$ defines M_{5+}, the Pythagorean triple $(3, 4, 5)$ defines M_{6+}, the Pythagorean triple $(28, 45, 53)$ defines M_{7+}, and so on.

2. Results

Theorem 1. *Metallic ratios are uniquely defined by an acute angle of a right triangle.*

Proof. We express the RHS of the equation (3) using half-angle formulae for sine and cosine, and substituting $\varphi := \theta/2$

$$\cot\left(\frac{\theta}{4}\right) = \cot\left(\frac{\varphi}{2}\right) = \frac{1+\cos\varphi}{\sin\varphi} = \frac{1+\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} = \\ = \frac{1+\sqrt{\frac{1+\cos\theta}{2}}}{\sqrt{\frac{1-\cos\theta}{2}}} = M_k, \tag{6}$$

since $0 < \theta < \pi/2$, so $\operatorname{sgn}(\sin(\theta/2) = \operatorname{sgn}(\cos(\theta/2) = 1$.

Multiplying the numerator and denominator of (6) by $\sqrt{(1+\cos\theta)/2}$ and performing some basic algebraic manipulations, we arrive at the quadratic equation for M_k

$$\sin(\theta)^2 M_k^2 - 2\sin(\theta)\left[1+\cos(\theta)\right] M_k - \sin(\theta)^2 = 0, \tag{7}$$

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having roots

$$M_{\theta\pm} = \frac{1+\cos(\theta) \pm \sqrt{2(1+\cos(\theta))}}{\sin(\theta)}, \tag{8}$$

corresponding to the metallic ratios (1). □

The metallic ratios $M_{\theta\pm}$ are shown in Figure 3.

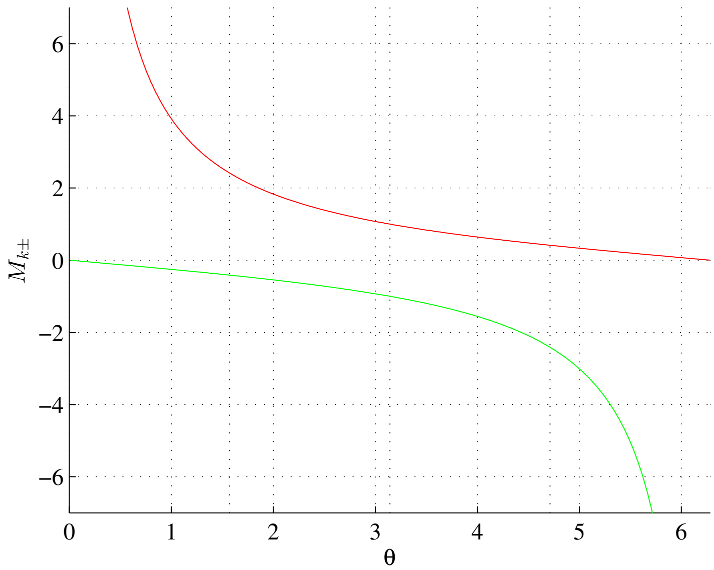

Figure 3. Metallic ratios: positive $M_{\theta+}$ (red), negative $M_{\theta-}$ (green) as a function of $0 \leq \theta \leq 2\pi$.

Equating relations (1) and (8) and solving for k gives

$$k(\theta) = \frac{2(\cos(\theta)+1)}{\sin(\theta)}, \tag{9}$$

as shown in Figure 4.

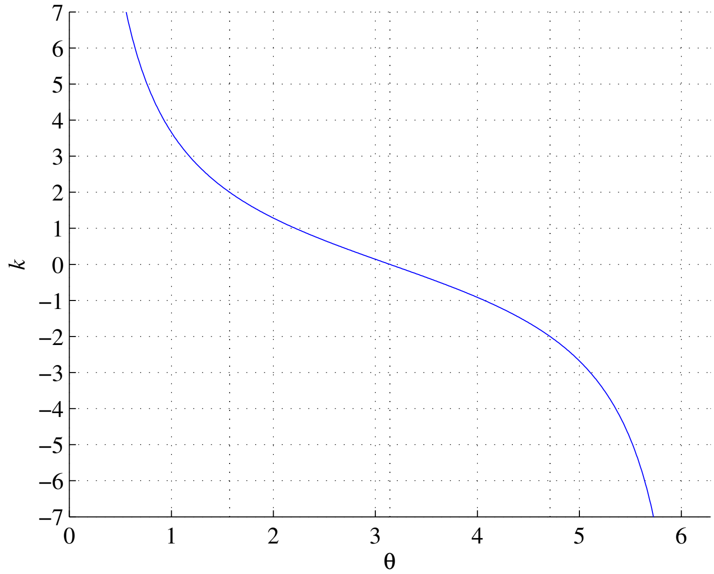

Figure 4. $k(\theta)$ for $0 \leq \theta \leq 2\pi$. $k(\pi/2) = 2$, $k(\pi) = 0$, $k(3\pi/2) = -2$.

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Conjecture 1. *For $k \neq \{0, \pm 2\}, k \in \mathbb{R}$ the angle θ_k defining the metallic ratio is the argument of a normalized complex number $z_k = e^{i\theta_k}$, wherein for $k \in \mathbb{Z}$, z_k is defined by a Pythagorean triple as*

$$z_k = \begin{cases} +\frac{a}{c} - \frac{b}{c}i & k < -\frac{\pi}{2} \\ -\frac{a}{c} - \frac{b}{c}i & -\frac{\pi}{2} < k \leq -1 \\ -\frac{b}{c} - \frac{a}{c}i & -1 < k < 0 \\ -\frac{b}{c} + \frac{a}{c}i & 0 < k < 1 \\ -\frac{a}{c} + \frac{b}{c}i & 1 \leq k < \frac{\pi}{2} \\ +\frac{a}{c} + \frac{b}{c}i & k > -\frac{\pi}{2}, \end{cases} \tag{10}$$

$a < b$ are catheti and c is hypotenuse of the right triangle shown in Figure 2, $\theta_{\pm 2} = \pm\pi/2$, and θ_0 is undefined.

Conjecture (1) has been numerically validated. This form of z_k does not hold for irrational k. For example

$$z_{\sqrt{2}} = -\frac{1}{3} + \frac{2\sqrt{2}}{3}i, \quad z_\pi = \frac{\pi + 2i}{\pi - 2i}. \tag{11}$$

Acknowledgments: I thank my wife Magdalena Bartocha for her unwavering motivation and my friend, Renata Sobajda, for her prayers.

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