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Article

Info- Geometric Analysis of the Stable $G/G/1$ Queue Manifold Dynamics with $G/G/1$ Queue Applications to E-Health

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Abstract: Information geometry is a mathematical framework that analyses the structure of statistical models using concepts from differential geometry. It treats families of probability distributions as manifolds, where the parameters of each model determine the coordinate charts. By applying info-geometric tools, we can gain insights into the characteristics of these models. The approach involves characterizing the queueing system's manifold using information geometry and presenting the exponential of the information matrix. This integration of information geometry with queueing theory provides a novel perspective for analyzing the dynamics of queueing systems, incorporating relativistic and Riemannian concepts. Some $G/G/1$ applications to E-health are highlighted. Finally, closing remarks and the next phase of research.

Keywords: stable $G/G/1$ queue; service times (ST); service utilization (SU); fisher information matrix(FIM); FIM exponential matrix of the $Stable\ G/G/1$ queue manifold ($e^{Stable\ G/G/1\ QM}$)

1. Introduction

Information geometry (IG) is a field that applies differential geometry techniques to statistics. It aims to use non-Euclidean geometry methods to analyze probability distributions and stochastic processes. IG is a cutting-edge geometric methodology that analyses models and visualizes geometry from an IG perspective. IG has wide applicability in various domains, including machine learning. This approach offers new insights and tools for understanding complex data and improving modeling techniques[1]. What's more interesting is that statistical manifolds (SMs) were studied with IG. In Figure 1, the probability $p(x|\theta)$ is coined each point in $SM(\theta)$, $\theta \in \mathbb{R}^n$[2]

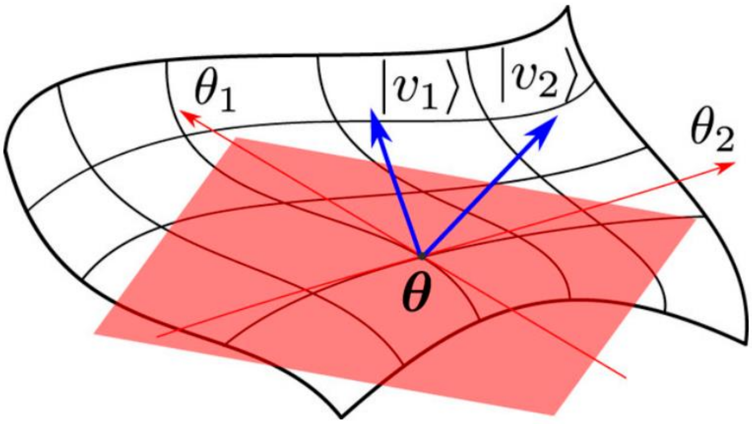

Figure 1. SM's parametrization (c.f., [2]).

A smooth statistical manifold is described mathematically by FIM from an info-geometric perspective. By quantifying the informative gap between measurements, FIM makes statistical data analysis and comparison possible.

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Amari and Dodson investigated a few exponential distributions and provided geometric structures [3]. An analogue of the regular exponential function for square matrices is the matrix exponential. Systems of linear differential equations are solved with it. Furthermore, Lie groups are matrix exponential-based [4].

The geometry of $M/D/1$ queues has only been the subject of one research work was a motivation behind starting this new to the knowledge analysis of queues, which connects information and matrix theories with differential geometry.

Following Euclidean space, geodesics are straight lines' analogues and share many of their characteristics, see Figure 2. Objects move on a geodesic in curved space-time according to General Relativity (GR), extremizing exact timing within locations. Thus, both space-time and curved spaces geometries are described by the same mathematics. Developing GR presented Einstein with its greatest mathematical challenge. In addition, a geodesic is a "straight line on a curved surface" that reduces the separation between two points.

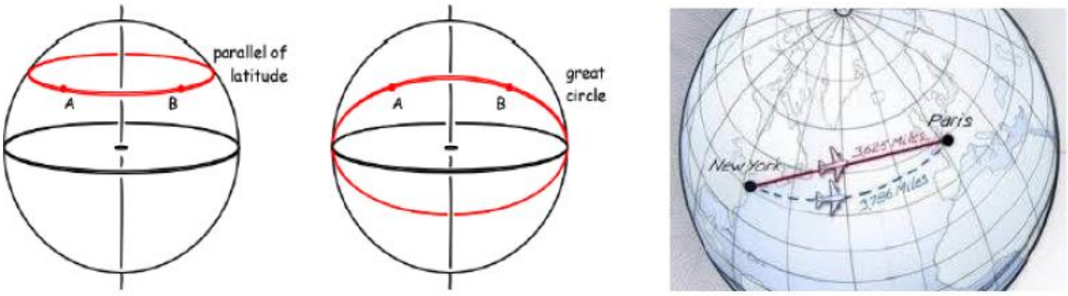

Figure 2. How curved surfaces' geodesics are geometrically represented (c.f., [6]).

In addition to measuring two distribution functions' shape similarity, FIM also relates to how much information each distribution function has regarding the parameter of the statistical manifold's probability density function.

This paper contributes to i) Obtaining the stable $G/G/1$ queue manifold FIM and its inverse. ii) $e^{Stable\ G/G/1\ QM}$ is shown to solve $\frac{dx}{dt}$ = Ax.

This is how the remainder of the paper is structured: A preliminary set of definitions related to information geometry is presented in Section 2. Section 3 introduces FIM, its inverse for a stable $G/G/1$ QM. In Section 4, one can derive$e^{Stable\ G/G/1\ QM}$. Some S $G/G/1$ applications to E-health are overviewed in section 5. Finally, Section 6 includes conclusions, some proposed open problems, and recommendations for further study.

2. IG main definitions

Definition 2.1 [7,8]

1.We call $M = \{p(x,\theta)|\theta\epsilon\Theta\}$ an SM and $p(x,\theta)$ is the probability density function.

2. The potential function Ψ is given by

$$C(x) + \sum_i F_i - \Psi(\theta) = ln\big(p(x;\theta)\big) = \mathcal{L}(x;\theta) \quad (2.1)$$

$M = \{\mathcal{L}(x;\theta)|\ \theta = (\theta_1, \theta_2, \dots, \theta_n)\epsilon\mathbb{R}^n\}$ an n-dimensional distribution manifold, with a coordinate system, namely $(\theta_1, \theta_2, \dots, \theta_n)$.

Definition 2.2 [7,8]. $\Psi(\theta)$ of (2.1) is the coordinates only part of $(-\mathcal{L}(x;\theta))$

Definition 2.3 [7,8]. FIM, namely for $i,j = 1,2,\dots,n$ $[g_{ij}]$, reads as:

$$[g_{ij}] = \left[\frac{\partial^2}{\partial\theta_i\,\partial\theta_j}\big(\Psi(\theta)\big)\right] \quad (2.2)$$

Definition 2.4[7,8]. For FIM, the inverse matrix of $[g_{ij}]$ reads as

$$[g^{ij}]= ([g_{ij}])^{-1} = \frac{adj[g_{ij}]}{\Delta}, \Delta= \det[g_{ij}] = \mathrm{I}, j = 1,2,\dots,n. \quad (2.3)$$

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The FIM's arc length representation reads as

$$(ds)^2 = \sum_{i,j=1}^{n} g_{ij}\,(d\theta^i)(d\theta^j) \tag{2.4}$$

Definition 2.6[9]

(1) Observing

$$\frac{dx}{dt} = Ax \tag{2.14}$$

The matrix exponential

$$e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!} = I + A + \frac{A^2}{2!} + \cdots + \frac{A^k}{k!} + \ldots. \tag{2.15}$$

solves (2.14).

(2) Let

$$\Phi(\delta) = \det(\mathrm{A} - \delta\mathrm{I}) \tag{2.16}$$

Assuming that

$$\Phi(\delta) = (\delta) = \det(\mathrm{A} - \delta\mathrm{I}) = 0, \tag{2.17}$$

δ of (2.17) are referred to eigen values that correspond to x, such that:

$$Ax = \delta x \tag{2.18}$$

Another way to represent e^A will be:

$$e^A = T\, e^{D_{diagonal}} T^{-1} \tag{2.19}$$

$D_{diagonal}$ defines δ −matrix diagonalization and T is matrix with only the obtained δ as columns.

3. $FIM_{G/G/1QM}$

By [17], the stable $G/G/1$ queue's Shannonian optimization reads

$$p(n) = \begin{cases} 1-\rho, & n = 0 \\ (1-\rho) g x^n, & n \geq 1 \end{cases}, \tag{3.1}$$

$g = \frac{\rho^2}{(L-\rho)(1-\rho)}, x = \frac{L-\rho}{L}$, and $L = \frac{\rho}{2}\left(1 + \frac{C_a^2 + \rho C_s^2}{1-\rho}\right)$ (Mean queue length of $G/G/1$ queue, ρ defines SU and C_s^2 denotes ST's squared coefficient of variation). $p(n)$ (c.f., (3.1)) will re-write to:

$$p(n) = \begin{cases} 1-\rho, & n = 0 \\ \frac{2\rho\left(\frac{\gamma+\rho\beta}{1-\rho} - 1\right)^{n-1}}{\left(\left(\frac{\gamma+\rho\beta}{1-\rho} + 1\right)^n\right.}, & n \geq 1 \end{cases} \tag{3.2}$$

where $\gamma = C_a^2, \beta = C_s^2$.

The reader should note that the case $\gamma = C_a^2 = 1$ in (3.2), reduces to the special case of $M/G/1$ QM [18].

Theorem 3.1. $G/G/1$ QM satisfies:

(i) $[g_{ij}]$ is

$$[g_{ij}] = \begin{bmatrix} a & b & c \\ b & e & d \\ c & d & f \end{bmatrix} \tag{3.3}$$

with

$$a = \frac{1}{\rho^2} - \frac{(\gamma+\beta)(-\gamma+\beta+2-2\rho(\beta+1))}{((1-\rho)(\gamma-1+\rho(\beta+1))^2} \tag{3.4}$$

$$b = -\frac{(\beta+1)}{(\gamma-1+\rho(\beta+1)^2} \tag{3.5}$$

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$$c = \frac{\gamma-1}{(\gamma-1+\rho(\beta+1)^2} \quad (3.6)$$

$$d = -\frac{(\rho)}{(\gamma-1+\rho(\beta+1)^2} \quad (3.7)$$

$$e = \frac{-1}{(\gamma-1+\rho(\beta+1)^2} \quad (3.8)$$

$$f = -\frac{\rho^2}{(\gamma-1+\rho(\beta+1)^2} \quad (3.9)$$

where a, b, c, d, e, f are as defined above and $\Delta = (aef + 2cbd - (ad^2 + fb^2 + ec^2)) \neq 0$ (3.10)

(ii) IFIM reads as:

$$[g^{ij}] = ([g_{ij}])^{-1} = \frac{adj[g_{ij}]}{\Delta} = \begin{bmatrix} A & B & C \\ B & E & D \\ C & D & F \end{bmatrix}, \quad (3.11)$$

with

$$A = \frac{ef-(d)^2}{\Delta} \quad (3.12)$$

$$B = \frac{cd-bf}{\Delta} \quad (3.13)$$

$$\boldsymbol{C} = \frac{bd-ec}{\Delta} \quad (3.14)$$

$$E = \frac{af-(c)^2}{\Delta} \quad (3.15)$$

$$D = \frac{bc-ad}{\Delta} \quad (3.16)$$

$$F = \frac{ae-(b)^2}{\Delta} \quad (3.17)$$

Proof

(i) By (3.2), one gets

(I) $p(0) = 1-\rho$. Thus,

$$\mathcal{L}(x;\theta) = ln(1-\rho), \theta = \theta_1 = \rho \quad (3.18)$$

$$\Psi(\theta) = -ln(1-\rho) \quad (3.19)$$

Hence,

$$\partial_1 = \frac{\partial\Psi}{\partial\rho} = \frac{1}{1-\rho} \quad (3.20)$$

$$\partial_1\partial_1 = \frac{\partial^2\Psi}{\partial\rho^2} = \frac{1}{(1-\rho)^2} \quad (3.21)$$

Therefore, FIM reads:

$$[g_{ij}] = \left[\frac{\partial^2\Psi}{\partial\rho^2}\right] = \left[\frac{1}{(1-\rho)^2}\right] \quad (3.22)$$

IFIM is:

$$[g^{ij}] = [g_{ij}]^{-1} = [(1-\rho)^2] \quad (3.23)$$

Case II: when $n \geq 1$, it follows that:

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$$\mathcal{L}(x;\theta) = (ln2 + l\,n(\rho) + (n-1)l\,n\left(\frac{\gamma+\rho\beta}{1-\rho} - 1\right) - nln\left(\frac{\gamma+\rho\beta}{1-\rho} + 1\right),$$

(3.24)

$$\theta = (\theta_1, \theta_2, \theta_3) = (\rho, \gamma, \beta)$$

$$\Psi(\theta) = l\,n\left(\frac{\gamma+\rho\beta}{1-\rho} - 1\right) - ln2 - l\,n(\rho) \quad (3.25)$$

Therefore,

$$\partial_1 = \frac{\partial\Psi}{\partial\rho} = -\frac{1}{(\rho)} + \frac{\gamma+\beta}{(1-\rho)(\gamma-1+\rho(\beta+1)} \quad (3.26)$$

$$\partial_2 = \frac{\partial\Psi}{\partial\gamma} = \frac{1}{(\gamma-1+\rho(\beta+1)} \quad (3.27)$$

$$\partial_3 = \frac{\partial\Psi}{\partial\beta} = \frac{\rho}{(\gamma-1+\rho(\beta+1)} \quad (3.28)$$

$$\partial_1\partial_2 = -\frac{(\beta+1)}{(\gamma-1+\rho(\beta+1)^2} = \partial_2\partial_1 \quad (3.29)$$

$$\partial_1\partial_3 = \frac{(\gamma-1)}{(\gamma-1+\rho(\beta+1)^2} = \partial_3\partial_1 \quad (3.30)$$

$$\partial_1\partial_1 = \frac{1}{\rho^2} - \frac{(\gamma+\beta)(-\gamma+\beta+2-2\rho(\beta+1))}{((1-\rho)(\gamma-1+\rho(\beta+1))^2} \quad (3.31)$$

$$\partial_2\partial_2 = \frac{-1}{(\gamma-1+\rho(\beta+1)^2} \quad (3.32)$$

$$\partial_2\partial_3 = -\frac{(\rho)}{(\gamma-1+\rho(\beta+1)^2} = \partial_3\partial_2 \quad (3.33)$$

$$\partial_3\partial_3 = -\frac{\rho^2}{(\gamma-1+\rho(\beta+1)^2} \quad (3.34)$$

Hence, (i) follows.

As $\gamma = 1$, we have

$$a = \frac{1}{(1-\rho)^2}, \ b = -\frac{1}{\rho^2(\beta+1}, \ c = 0, d = -\frac{1}{\rho(\beta+1)^2}, e = \frac{-1}{(\rho(\beta+1)^2}, f = -\frac{1}{(\beta+1)^2}.$$

(ii) The FIM's determinant reads

$\Delta = \det[g_{ij}] = \Delta = (aef + 2cbd - (ad^2 + fb^2 + ec^2)) \neq 0$. Hence, IFIM exists.

The corresponding IFIM reads as

$$[g^{ij}] = ([g_{ij}])^{-1} = \frac{adj[g_{ij}]}{\Delta} = = \begin{bmatrix} A & B & C \\ B & E & D \\ C & D & F \end{bmatrix}, \quad (3.35)$$

where A, B, C, D, E, F are given above. This proves (ii).

4. $e^{Stable\ G/G/1\ QM}$

Preliminary Theorem 4.1[10]

The solution of the cubic equation

$$a'w^3 + b'w^2 + c'w + d' = 0 \quad (4.1)$$

is characterized arbitrarily by

$$y = z - \frac{\sigma}{z}, \quad (4.2)$$

$$w = y - \frac{b'}{3a'}, \quad (4.3)$$

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$$z = \sqrt[3]{(-\frac{\varepsilon_1}{2}) \pm \sqrt{\varepsilon_2}}, \tag{4.4}$$

$$\varepsilon_1 = \frac{2(b')^3}{27} + \frac{d'}{a'} - \frac{b'c'}{3(a')^2}, \tag{4.5}$$

$$\varepsilon_2 = \frac{(\varepsilon_1)^2}{4} + \frac{(\varepsilon_3)^3}{27}, \tag{4.6}$$

where ε_3 is given by $\varepsilon_3 = -\frac{(b')^2}{3(a')^2} + \frac{c'}{a'}$ (4.7)

ε_2 is called the *discriminant* of the cubic equation.

Theorem 4.2. $e^{Stable\ G/G/1\ QM}$ solves $\frac{dx}{dt}$=Ax.

Proof

Recall that FIM of the stable $G/G/1$ QM $[g_{ij}], i,j = 1,2,3$ is given by

$$[g_{ij}] = \begin{bmatrix} a & b & c \\ b & e & d \\ c & d & f \end{bmatrix}, \tag{4.8}$$

where

$$a = \frac{1}{\rho^2} - \frac{(\gamma+\beta)(-\gamma+\beta+2-2\rho(\beta+1))}{((1-\rho)(\gamma-1+\rho(\beta+1))^2} \tag{4.9}$$

$$b = -\frac{(\beta+1)}{(\gamma-1+\rho(\beta+1)^2} \tag{4.10}$$

$$c = \frac{\gamma-1}{(\gamma-1+\rho(\beta+1)^2} \tag{4.11}$$

$$d = -\frac{(\rho)}{(\gamma-1+\rho(\beta+1)^2} \tag{4.12}$$

$$e = \frac{-1}{(\gamma-1+\rho(\beta+1)^2} \tag{4.13}$$

$$f = -\frac{\rho^2}{(\gamma-1+\rho(\beta+1)^2} \tag{4.14}$$

Now, we have

$\Phi(\delta) = (\delta) = \det([g_{ij}] - \delta I) = \det \begin{bmatrix} a-\delta & b & c \\ b & e-\delta & d \\ c & d & f-\delta \end{bmatrix} = 0$. Hence, it holds that:

$(a-\delta)[(e-\delta)(f-\delta) - d^2] - b[b(f-\delta) - cd] + c[bd - c(e-\delta)] = 0$ (4.15). Thus, it holds that

$$\delta^3 - (a + a(e+f) - d^2)\delta^2 + (a(e+f) - b^2 - c^2)\delta + (b^2 f + c^2 e + ad^2 - 2bcd) = 0. \tag{4.16}$$

We have

$$a' = 1, b' = -(a + a(e+f) - d^2), c' = (a(e+f) - b^2 - c^2), d' = (b^2 f + c^2 e + ad^2 - 2bcd) \tag{4.17}$$

By the preliminary theorem (4.1), it is implied that

$y = z - \frac{\varepsilon_3}{z}$, $\delta = y - \frac{b'}{3a'}$, $z = \sqrt[3]{(-\frac{\varepsilon_1}{2}) \pm \sqrt{\varepsilon_2}}, \varepsilon_1 = \frac{2(b')^3}{27} + \frac{d'}{a'} - \frac{b'c'}{3(a')^2}$, $\varepsilon_2 = \frac{(\varepsilon_1)^2}{4} + \frac{(\varepsilon_3)^3}{27}, \varepsilon_3 = -\frac{(b')^2}{3(a')^2} + \frac{c'}{a'}$

We have

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$$\varepsilon_3 = -\frac{((a+a(e+f)-d^2))^2}{3} + (a(e+f) - b^2 - c^2) \quad (4.18)$$

$$\varepsilon_1 = -\frac{2\left((a+a(e+f)-d^2)\right)^3}{27} + (b^2 f + c^2 e + ad^2 - 2bcd) + \frac{(a+a(e+f)-d^2)(a(e+f)-b^2-c^2)}{3} \quad (4.19)$$

$$\varepsilon_2 = \frac{(\varepsilon_1)^2}{4} + \frac{(\varepsilon_3)^3}{27}$$

$$= (\frac{(-\frac{2\left((a+a(e+f)-d^2)\right)^3}{27} + (b^2 f + c^2 e + ad^2 - 2bcd) + \frac{(a+a(e+f)-d^2)(a(e+f)-b^2-c^2)}{3})^2}{4}$$

$$+ \frac{(-\frac{((a+a(e+f)-d^2))^2}{3} + (a(e+f)-b^2-c^2))^3}{27}) \quad (4.20)$$

implies that we can obtain the required values eigen values $\delta_{1,2,3}$ for (4.16). Therefore, the diagonal matrix $D_{diagonal}$ is given by

$$D_{diagonal} = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{bmatrix} \quad (4.21)$$

The eigen vector $x_{\delta_1} = C$ corresponding to δ_1 satisfies

$$\begin{bmatrix} a - \delta_1 & b & c \\ b & e - \delta_1 & d \\ c & d & f - \delta_1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (4.22)$$

This implies

$$(a - \delta_1) + (b - \delta_1)x_2 + cx_3 = 0 \quad (4.23)$$

$$bx_1 + (e - \delta_1)x_2 + dx_3 = 0 \quad (4.24)$$

$$cx_1 + dx_2 + (f - \delta_1)x_3 = 0 \quad (4.25)$$

By (4.24), $x_1 = \frac{(\delta_1 - e)x_2 - dx_3}{b}$ (4.26). Consequently, by (4.25) and (4.23), it is implied that:

$c(\frac{(\delta_1 - e)x_2 - dx_3}{b}) + dx_2 + (f - \delta_1)x_3 = 0$ (4.27). Therefore, $x_2 = \frac{((\delta_1 - f) + \frac{cd}{b})x_3}{(d + \frac{c(\delta_1 - e)}{b})}$(4.28). Similarly, we have

$$x_1 = \frac{(\frac{b(f-\delta_1)-c(d+1)}{d+\frac{c}{b}(\delta_1 - f)} - c)x_3}{(a - \delta_1)} \quad (4.29)$$

Hence, it is implied that

$$x_{\delta_1} = \begin{pmatrix} \frac{(\frac{b(f-\delta_1)-c(d+1)}{d+\frac{c}{b}(\delta_1 - f)} - c)}{(a-\delta_1)} \\ \frac{((\delta_1 - f) + \frac{cd}{b})}{(d + \frac{c(\delta_1 - e)}{b})} \\ 1 \end{pmatrix} x_3 \;,\; x_3 \neq 0 \quad (4.30)$$

Also, x_{δ_2} corresponding to δ_2 reads

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$$x_{\delta_2} = \begin{pmatrix} \dfrac{\left(\dfrac{b(f-\delta_2)-c(d+1)}{d+\frac{c}{b}(\delta_2-f)}-c\right)}{(a-\delta_2)} \\ \dfrac{((\delta_2-f)+\frac{cd}{b})}{(d+\frac{c(\delta_2-e)}{b})} \\ 1 \end{pmatrix} x_3,\ x_3 \neq 0 \quad (4.31)$$

$$\text{And } x_{\delta_3} = \begin{pmatrix} \dfrac{\left(\dfrac{b(f-\delta_3)-c(d+1)}{d+\frac{c}{b}(\delta_3-f)}-c\right)}{(a-\delta_3)} \\ \dfrac{((\delta_3-f)+\frac{cd}{b})}{(d+\frac{c(\delta_3-e)}{b})} \\ 1 \end{pmatrix} x_3,\ x_3 \neq 0 \quad (4.32)$$

Hence, the matrix T is determined by

$$T = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 1 & 1 & 1 \end{bmatrix} \quad (4.33)$$

$a_1 = \dfrac{\left(\dfrac{b(f-\delta_1)-c(d+1)}{d+\frac{c}{b}(\delta_1-f)}-c\right)}{(a-\delta_1)}$, $a_2 = \dfrac{\left(\dfrac{b(f-\delta_2)-c(d+1)}{d+\frac{c}{b}(\delta_2-f)}-c\right)}{(a-\delta_2)}$, $a_3 = \dfrac{\left(\dfrac{b(f-\delta_3)-c(d+1)}{d+\frac{c}{b}(\delta_3-f)}-c\right)}{(a-\delta_3)}$, $b_1 = \dfrac{((\delta_1-f)+\frac{cd}{b})}{(d+\frac{c(\delta_1-e)}{b})}$, $b_2 = \dfrac{\left((\delta_2-f)+\frac{cd}{b}\right)}{\left(d+\frac{c(\delta_2-e)}{b}\right)}$,

$$b_3 = \frac{((\delta_3-f)+\frac{cd}{b})}{(d+\frac{c(\delta_3-e)}{b})} \quad (4.34)$$

After some manipulation, it is implied that:

$$\triangle_T = \det(T) = [a_1(b_2 - b_3) - a_2(b_1 - b_3) + a_3(b_1 - b_2)] \neq 0 \quad (4.35)$$

As a necessity that T^{-1} should exist.

After some mathematical manipulation:

$$T^{-1} = \frac{1}{\triangle_T}\begin{bmatrix} b_2 - b_3 & a_3 - a_2 & a_2 b_3 - a_3 b_2 \\ b_3 - b_1 & a_1 - a_3 & a_3 b_1 - a_1 b_3 \\ b_1 - b_2 & a_2 - a_1 & a_1 b_2 - a_2 b_1 \end{bmatrix} \quad (4.36)$$

Therefore, $e^{Stable\ G/G/1\ QM}$ reads

$$e^A = Te^D T^{-1} = T^{-1}\begin{bmatrix} e^{\delta_1} & 0 & 0 \\ 0 & e^{\delta_2} & 0 \\ 0 & 0 & e^{\delta_3} \end{bmatrix} T \quad (4.37)$$

The achieved result in (4.37) presents a novel contribution , that is $e^{Stable\ G/G/1\ QM}$ is a solution to $\frac{dx}{dt} = Ax$.

5. How can $G/G/1$ queue advance E-health.

The advancementsin e-health applications bring about new challenges for designers and maintainers. These challenges include ensuring timely transmission of health information, addressing limitations in model building and training, navigating regulatory issues related to data sharing, and safeguarding the privacy and security of medical data. Additionally, the current COVID pandemic emphasizes the need for safety in both virtual and physical environments. In [11], the authors have developed a framework for creating secure, responsive, and intelligent e-health applications. The framework focuses on three main components: Analyze, Acquire, and Authenticate, which cover the entire data lifecycle in e-health applications. The goal is to address the

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challenges related to data security, privacy, and the need for reliable medical decision support in the context of e-health. This can be visualized by Figure 4 (c.f., [11]).

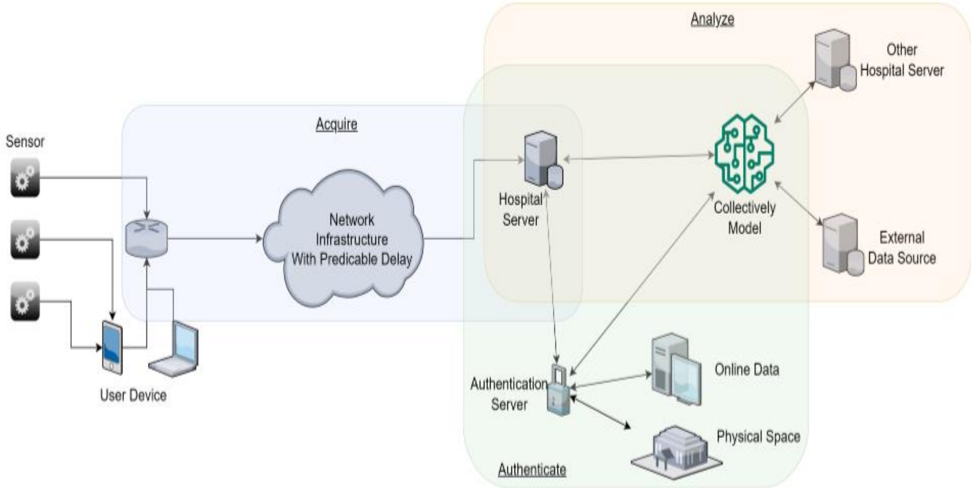

Figure 4. The framework overview refers to a comprehensive approach for developing secure, responsive, and intelligent e-health applications. It addresses challenges related to data protection, regulatory compliance, and the evolving nature of hybrid systems. The framework focuses on three key components: Analyze, Acquire, and Authenticate, covering the entire data lifecycle in e-health applications, and offers flexibility by accommodating different paradigms and stages of application development.

Regarding unrelated topics, discussed the traffic patterns of an IP network with multiple services, each of which has its distribution law. Specifically, it mentions the $G/G/1$ analysis system, which is a mathematical model used to analyze single-server systems with arbitrary inter-arrival times and unspecified service time distributions. The analysis of such systems is considered complex due to the incomplete understanding of the process dispersion. However, the standard $G/G/1$ model does not provide such solutions, except for certain distributions like the exponential distribution. The authors have also noted the necessity for additional study to look into effective systems that may be utilized to take this research a step ahead.

Systems service's arrival process descriptions are usually constrained to a small number of significant moments that define their distributions. Statistical metrics such as the coefficient of random variation or the root-mean-square deviation are used to express these moments, which are often the first two. Nonetheless, the irregular flow of incoming requests alters the working characteristics and makes it difficult to determine the precise result of the system's performance. Although the influence of higher-order moments decreases with increasing order number, the average delay is still dependent on the first two moments.The variabilityof the request service duration and the variability of the intervals between requests are represented by the squared coefficient of variation (CV). In these computations, the average values of the service rate, service time distribution mean, and request flow intensity (arrival rate) are used.

Figure 3 provides a visual representation of the relationships involved in estimating the quality and time parameters of network traffic service units. As demonstrated in [12], the variation coefficient—which is established by the root-mean-square deviation—determines the system's perceived time delay. This suggests that online traffic, like video-on-demand and Internet Protocol Television (IPTV), which have high variation coefficients, should be buffered to mitigate delays. During the transfer of data in communication networks for E-health applications, small packet losses are considered acceptable. Packet losses due to errors in communication channels with FOCL (Fibre Optic Cable Lines) deployment are possible, however they are not very significant. Figure 5's results demonstrate that the communication network data's properties are thought to be appropriate for use in e-health applications.

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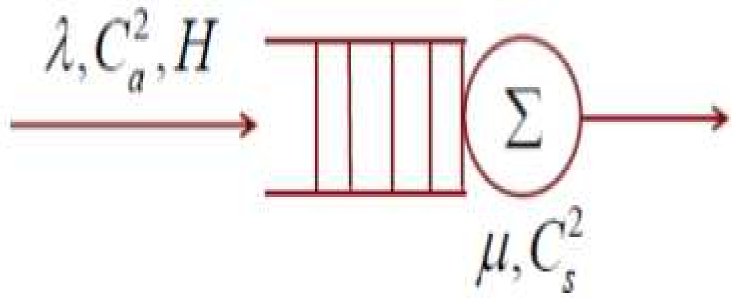

Figure 3. A Stable G/G/1 queue.

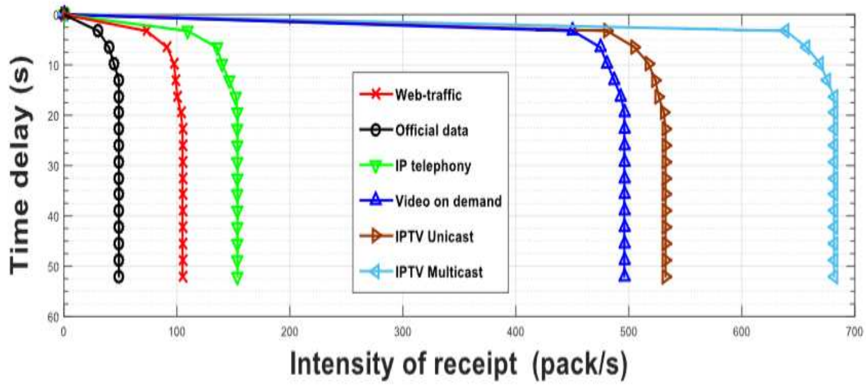

Figure 5. The correlation between packet arrival intensity and delay time in a network. It shows that the delay time grows together with the intensity of packet arrival. In terms of qualitative and temporal aspects, this information is critical for comprehending and calculating the network traffic service unit.

Figure 6 (c.f., [12]) illustrates the relationship between CPU overloading and packet loss likelihood. In E-health applications, where load capacity is usually not substantial and seldom reaches its maximum, the figure shows that lowering the load capacity improves packet loss performance.

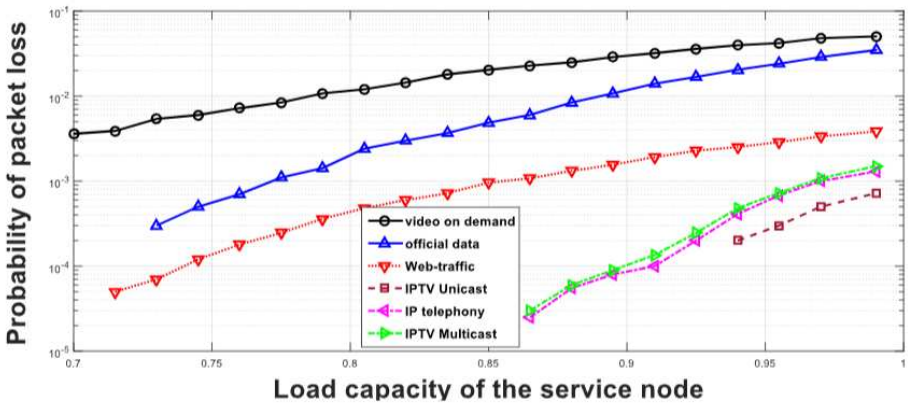

Figure 6. The relationship between packet loss probability and processor overloading in a communication network. It states that as the load capacity applied to the processor decreases, the packet loss probability improves. This information is relevant for applications like E-health, where low packet loss is desirable, as the load capacity in medical applications is typically not significant.

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The proposed work of conducted a comprehensive analysis of the MSSN (Multi-Service Switched Network) for its application in E-health. The analysis focuses on using the $G/G/1$ mathematical model to evaluate the system's capacity, considering a single-channel system with general flow and request characteristics. The goal is to understand the system's performance and suitability for E-health applications based on factors like service duration and distribution.

The authors address the likelihood of packet loss (Ploss) in a network in the context of [12], which is based on the quantity of dropped packets over a particular amount of time. The total of the waiting time in the buffer, packet processing time, and other variables is known as the packet service delay. Using the $G/G/1$ mathematical model, the authors hope to analyse a multi-service network for E-health applications while taking into account variables such as packet loss probability, latency, and jitter. In this instance, the authorsmeasured the network development qualitatively for their suggested model and contrasted it with methods that are frequently employed in E-health applications. In accordance with the Norros methodology, they also assessed the buffer space size to provide adequate service quality[13]. Additionally, they formulated a service quality algorithm for multi-service telecommunication networks, considering various mechanisms for queue formation and overloading control.

6. Conclusions and Future Work

In summary, this study reveals both FIM and IFIM for the $G/G/1$ QM. Finally, it has been proven that $FIM_{G/G/1QM}$ solves $\frac{dx}{dt}$= Ax. Notably, the strength of $G/G/1$ queue to advance E-health is highlighted. Next phase of research includes the determination of the IG structure of the Service Model manifold (c.f., [14]), the derivation of the information geometric structure of the Generalized Feller Pareto manifold(c.f., [15]), and IG powerful tools to geometrically interpret photon statistics and other unexplored phenomena in physics.

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