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Analytical Solution of a Forced Vibrating Tube Transporting Fluid

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Abstract: In this paper is proposed an analytical technique which can be used to obtain the forced response of a cantilevered tube conveying fluid. Considering the pipe subjected to an arbitrary harmonic force either distributed or concentrated, an analytical solution is found using Green's function method. The solution obtained of the closed form satisfies the differential equations in a classical sense. The presented method giving the exact solutions is more precise than the classical eigenfunction expansion or Galerkin's method with no need to the eigenfunctions or eigenvalues.

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1. Introduction

Pipes or tubes carrying fluid play a critical role in a wide range of industrial and engineering applications, including fuel pipelines, inkjet print-head nozzles, fuel-carrying tubes in space shuttle engines, and heat exchangers in nuclear power plants; thus, forced tube conveying is clearly desired [1,2]. It is also worth noting that feedback control for beams might be used for forced tube vibration [3].

From a theoretical standpoint, the fluid/structure interaction associated with the boundary moving condition is undoubtedly one of the most interesting and hardest problems in dynamical systems. For nearly 50 years, these issues have been studied both theoretically and empirically. The challenges investigated varied from changes in the natural frequencies of the pipe system caused by flowing fluid, to vibrations caused by turbulent flows, and instabilities such as flutter instability (via a Hopf bifurcation) which occurs when the flow velocity reaches a critical velocity [4]. Following [5], there are two aspects of forced vibration of pipes conveying fluid: the first is the physical aspect, which sheds more light on the dynamics of the system, and the second is related to the analytical techniques that can be used to obtain the forced response of such systems, which is the one used in this work, though the result could also be extended to the first aspect.

This paper develops an improved analytical strategy based on Green's function method to generate closed form solutions of tube carrying fluid that have continuous derivatives and satisfy the differential equations in a classical sense. Furthermore, the method does not involve solving the free vibration problem in order to obtain the eigenvalues and related eigenfunctions, which are required when utilizing classical Galerskin's method.

2. Tube Conducting Fluid General Equation of Motion

The system under consideration is a straight cantilever tube fixed to the x-axis. The fluid with an average velocity of U enters the tube at the fixed end and departs at the free end. A force is provided to the tube, allowing it to move transversely. Furthermore, the force that induces tube motion is assumed to be harmonic. Unlike the unconstrained pipe dynamical stability achieved by many others [5,6], the goal here is considerably different in the sense that we want to know the precise deflection of the tube, with the harmonic force considered either distributed or concentrated.

Considering the effects of dissipation and damping, the general equation of motion for a tube conveying fluid can be formulated based on the following assumptions: (*i*) the tube has a uniform annular cross-section; (*ii*) the length of the tube is significantly greater than its diameter; (*iii*) the effects of rotary inertia and shear deformation are neglected; (*iv*) the tube's center line is inextensible; (*v*) the tube is elastic and initially straight; and (vi) cross-sections of the tube remain plane and perpendicular to the axis of the beam, in accordance with Bernoulli-Euler beam theory. The equation can be expressed as follows [1,7]:

$$\frac{\partial^4\eta}{\partial x^4}+u^2\frac{\partial^2\eta}{\partial^2 x}+2\beta^{1/2}u\frac{\partial^2\eta}{\partial t\partial x}+\sigma\frac{\partial\eta}{\partial t}+\frac{\partial^2\eta}{\partial t^2}=F(x,t) \quad (1)$$

The resultant equation is made dimensionless by employing for *variables*,

$$x\to\frac{x}{L},\ \eta\to\frac{\eta}{L},\ t\to\left(\frac{EI}{m_t+M_f}\right)^{1/2}\frac{t}{L^2}; \quad (2)$$

and for constants,

$$u=LU\left(\frac{M_f}{EI}\right),\ \beta=\frac{M_f}{m_t+M_f},\ \sigma=\frac{cL^2}{\left[EI(m_t+M_f)\right]^{1/2}} \quad (3)$$

In the following section, we will solve the dimensionless form of the equation for a cantilever conveying fluid, which is subject to a harmonic forcing function.

3. Exact Analytical Solution

We propose a new approach based on Green's function method to solve equation (1). Assuming the applied force is harmonic, we could formulate the problem in a complex form. Let's consider a tube subjected to an arbitrary harmonic force, such that it is governed by an equation of the form,

$$F(x,t)=f(x)e^{i\omega t} \quad (4)$$

where i is the imaginary unit.

For general purpose, the force could be either distributed or concentrated at a position x_p thus,

$$f(x)=\begin{cases}F_p\delta(x-x_p)\\ f(x)\end{cases}, \quad (5)$$

therefore, the solution of the problem is sought as:

$$w(x,t)=V(x)e^{i\omega t} \quad (6)$$

where $V(x)$ is the unknown function to be determined.

As consequence of (4) and (6) in complex form, the solution of the problem is the real part of $w(x,t)$ as we will detail later. By replacing (4) in (1) we obtain,

$$V''''(x)+aV''(x)+ibV'(x)-cV(x)=f(x) \quad (7)$$

where we set $a=u^2$, $b=2\omega\beta^{1/2}$ and $c=\omega^2-1-i\sigma\omega$.

where prime (′) denotes for derivative with respect to *x*.

The solution of (7) could be expressed using green function as,

$$V(x)=\int_0^L f(x')G(x,x')dx' \quad (8)$$

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where we can determine the Green's function $G(x,x')$ by solving the following equation:

$$V''''(x)+aV''(x)+ibV'(x)-cV(x)=\delta(x-x') \tag{9}$$

We solve this equation by adopting the Laplace transform and its inverse, deducing therefore,

$$\begin{aligned} V(x,x') &= \mathrm{H}(x-x')\sum_{\alpha_i}\frac{e^{\alpha_i(x-x')}}{4\alpha_i^3+2\alpha_i a+ib} \\ &+V(0)\sum_{\alpha_i}\frac{\alpha_i^3 e^{\alpha_i x}}{4\alpha_i^3+2\alpha_i a+ib}+V'(0)\sum_{\alpha_i}\frac{\alpha_i^2 e^{\alpha_i x}}{4\alpha_i^3+2\alpha_i a+ib} \\ &+\sum_{\alpha_i}\frac{e^{\alpha_i x}}{4\alpha_i^3+2\alpha_i a+ib}(V'''(0)+aV'(0)+biV(0)) \\ &+\sum_{\alpha_i}\frac{\alpha_i e^{\alpha_i x}}{4\alpha_i^3+2\alpha_i a+ib}(V''(0)+aV(0)) \end{aligned} \tag{10}$$

where $H(x)$ is the Heaviside or step function and $\alpha_i, i=1...4$ is the solution of the following polynomial equation:

$$Z^4+aZ^2+ibZ-c=0 \tag{11}$$

This quartic equation can be analytically solved using Ferrari's method.

Let the function $\psi(x)$ be defined as:

$$\psi(x)=\sum_{\alpha_i}\frac{e^{\alpha_i x}}{4\alpha_i^3+2\alpha_i a+ib} \tag{12}$$

We then can recast (10) in the following form,

$$V(x,x')=\mathrm{H}(x-x')\psi(x-x_p)+\psi'(x)V''(0)+\psi(x)V'''(0) \tag{13}$$

where we use the boundary conditions at $x=0$ with, $V(0)=V'(0)=0$.

The last step is now to find the other boundary conditions namely $V''(0)$ and $V'''(0)$ present in (13). For this purpose, we will use the boundary conditions at the free end of the cantilevered tube; from the bending moment at $x=L$,

$$V''(L)=0=\psi''(L-x')+\psi'''(L)V''(0)+\psi''(L)V'''(0) \tag{14}$$

and from the shear force at $x=L$,

$$V'''(L)=0=\psi'''(L-x')+\psi^{(4)}(L)V''(0)+\psi'''(L)V'''(0) \tag{15}$$

Solving the previous equations (14-15) leads to determine the unknown boundary conditions at $x=0$ as:

$$V''(0)=\frac{\psi''(L)\psi'''(L-x')-\psi'''(L)\psi''(L-x')}{\psi'''(L)^2-\psi''(L)\psi^{(4)}(L)} \tag{16}$$

$$V'''(0)=\frac{\psi^{(4)}(L)\psi''(L-x')-\psi'''(L)\psi'''(L-x')}{\psi'''(L)^2-\psi''(L)\psi^{(4)}(L)} \tag{17}$$

Thus, we deduce the Green's function of the cantilevered tube conveying fluid as:

$$G(x,x') = \mathrm{H}(x-x')\psi(x-x') + \frac{\psi''(L)\psi'''(L-x') - \psi'''(L)\psi''(L-x')}{\psi'''(L)^2 - \psi''(L)\psi^{(4)}(L)}\psi''(x)$$
$$+\frac{\psi^{(4)}(L)\psi''(L-x') - \psi'''(L)\psi'''(L-x')}{\psi'''(L)^2 - \psi''(L)\psi^{(4)}(L)}\psi(x) \quad (18)$$

Finally, the dynamics response of a forced cantilevered tube conducting fluid subjected to an arbitrary harmonic force expresses as:

$$w(x) = e^{i\omega t}\int_0^L f(x')G(x,x')dx' \quad (19)$$

$f(x)$ could be taken either concentrated or distributed (5).

In the case where $f(x) = F_p\delta(x - x_p)$, we could establish the analytical solution of the tube conveying fluid submitted to a pointwise harmonic excitation as:

$$\eta(x,t) = \Re[w(x,t)]$$

where $w(x,t)$ is given by,

$$w(x,t) = F_p \exp(i\omega)\{\mathrm{H}(x-x_p)\psi(x-x_p)$$
$$+\frac{\psi''(L)\psi'''(L-x_p) - \psi'''(L)\psi''(L-x_p)}{\psi'''(L)\psi^{(4)}(L) - \psi''(L)\psi^{(5)}(L)}\psi''(x) \quad (20)$$
$$+\frac{\psi^{(5)}(L)\psi''(L-x_p) - \psi^{(4)}(L)\psi'''(L-x_p)}{\psi'''(L)\psi^{(4)}(L) - \psi''(L)\psi^{(5)}(L)}\psi(x)\}$$

Thus, the solution finally is given by:

$$v(x,t) = \Re[w(x,t)]\cos(\omega t) - \Im[w(x,t)]\sin(\omega t)$$

4. Conclusion

Considering a general equation of motion for a tube carrying fluid subjected to harmonic excitation, an analytical technique based on Green's function method has been developed to find a solution for a tube carrying fluid. The derived exact solution is more accurate than the conventional series expansion or Galerkin's method, and it does not require knowledge of the free vibration problem's eigenfunctions or eigenvalues. The proposed method might also be extended to include moment solicitation and other configurations such as multi-span tubes, boundary conditions, and even tube dynamical analysis.

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