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Article

Information Data Length Theory for the Transient $M/M/\infty$ Queueing System

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Abstract. The current paper provides a cutting-edge information data length- theoretic approach to the transient $M/M/\infty$ queueing system. This is a first time ever investigation that unifies information data length and queueing theories. The significant impact of both time and the number of states for the transient $M/M/\infty$ queue on both the upper and lower bounds of the obtained information data length is observed and noted. The paper concludes with some challenging open problems, combined with concluding remarks and future research pathways.

Keywords: information data length; transient queues; $M/M/\infty$ queueing system

1. Introduction

When clients arrive and need assistance from servers, a simple queueing structure is in place. If there are no open servers, patrons can join a queue; the queue discipline dictates the order in which patrons are served. When designing or enhancing service systems, these models aid in the computation of performance metrics.

Longer wait times are a result of higher utilisation levels, and delays get longer as utilisation rises. Variability and system size also matter; larger systems have shorter delays and more variable systems have longer delays at any given utilisation level. These ideas have effects on service system evaluation and capacity planning [1].

A typical model for arrivals in queueing systems is the Poisson process. It is assumed that consumers arrive one at a time, and that the probability of arriving at any given moment is independent of both the time and possibility of arriving prior to other customers. This model is often applicable in various contexts, such as emergency rooms and customer service call centers, and can be tested for goodness of fit using statistical measures [2].

In the context of steady (non-time dependent) queues, namely these queues with non-time dependent probability density function. For example, the $M/M/s$ or Erlang delay model is a commonly used queueing model in service systems, i.e., the assumption is that there will be one line serving the same servers, with an infinite amount of waiting room. In this model, service durations, including patient stays or provider times, are considered to have an exponential distribution, and client arrivals is Poissonian [3].

The $M/M/s$ model is still able to produce appropriate estimates of delay even in cases where the actual coefficient of variation (CV) of service time deviates somewhat from one. The $M/M/s$ model, however, may either overestimate or underestimate actual delays if the CV differs significantly from one. The average delay can be computed using the $M/G/1$ system formula, another well-known none-time dependent queue, in such scenarios involving a Poissonian arrival process with only one server [2].

1.1. $M/M/\infty$. Queueing System and the Computation of the Common Average Time for Unsaturated Site Visitors Flows Beneath Double-Parking Situations

Double parking (DP) violations of industrial trucks whilst they load and dump at transport places with inadequate curb side area may have huge poor effect on site visitors. Motivated by the

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need to examine such effect on urban streets, [4] make use of parking violation facts for New York City in conjunction with discipline facts accrued the usage of video recording and adopts a complete modelling technique that mixes to be had facts with varieties of fashions. Another implied approach was the micro-simulation version [4] advanced and calibrated to examine personal and mixed outcomes of diverse explanatory variables. The research [4] employed a macroscopic $M/M/\infty$ queueing version and micro-simulation for estimating common tour time with inside the presence of double-parking activities.

Under uncongested site visitors` situations without downstream blocking off, making use of the $M/M/\infty$ queueing version produced an amazing match with the sphere facts. Overall, the $M/M/\infty$ queueing version is a powerful technique to compute the common average time for unsaturated site visitors flows beneath double-parking situations. Micro-simulation is a greater effective device than the $M/M/\infty$ queueing version for comparing such congested situations and may be used to study person and mixed outcomes of diverse explanatory variables.

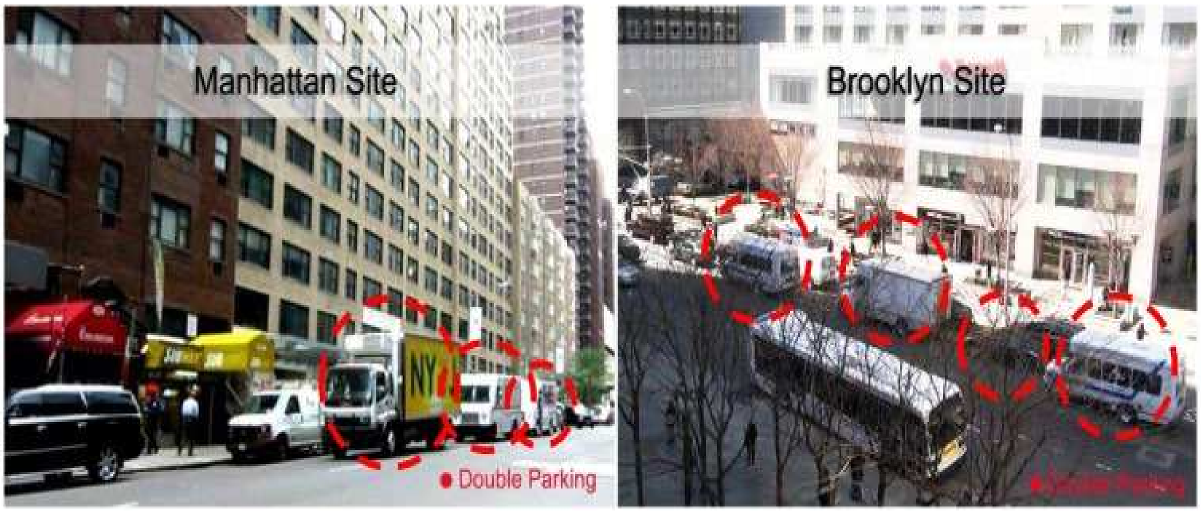

Figure 1. How DP occurs in the sites under investigation [4].

1.2. Application of $\boldsymbol{M/M/\infty}$ Birth–Death Process to Quantitively Interpret the Wavelet Dynamics in Atrial Fibrillation and Phase Singularity

It has been hypothesized that the determined range of PS or wavelets in Atrial Fibrillation (AF) might be ruled with the aid of using a not unusual set of renewal rate constants λ_f (for PS or wavelet formation) and λ_d (PS or wavelet destruction), with steady-state population dynamics modelled as an M/M/∞ birth–death manner. It has been demonstrated [5]:

(1) that λ_f and λ_d may be blended in a Markov $M/M/\infty$ manner to as it should be a version of the determined common range and population distribution of PS and wavelets in all structures at unique scales of mapping; and

(2) that slowing of the constants denoting rates, namely λ_f and λ_d is related to slower mixing rates of the $M/M/\infty$ birth–death matrix, presenting an interpretation of spontaneous AF termination.

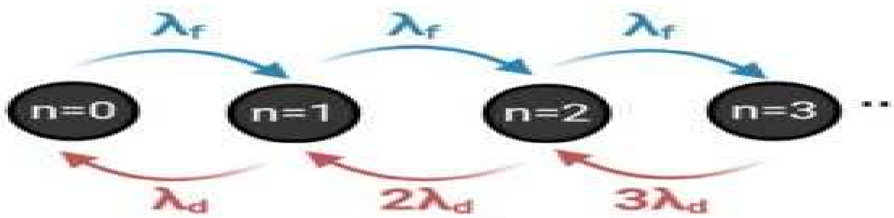

Figure 2. The birth-death process of a transient $M/M/\infty$ queueing system [5].

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It is worth mentioning that λ_f and λ_d (PS rates of formation and destruction respectively) are related by the following steady-state equation of the $M/M/\infty$ birth–death [6]

$$N = \frac{\lambda_f}{\lambda_d} \quad (1)$$

provided that N serves as the average number of PS and wavelets. Additionally, the PS and wavelet population distribution is characterized by the steady state probability p_n [6] of getting a wavelet population or a phase singularity with size n is determined by:

$$p_n = \frac{(\frac{\lambda_f}{\lambda_d})^n e^{-\frac{\lambda_f}{\lambda_d}}}{n!} \quad (2)$$

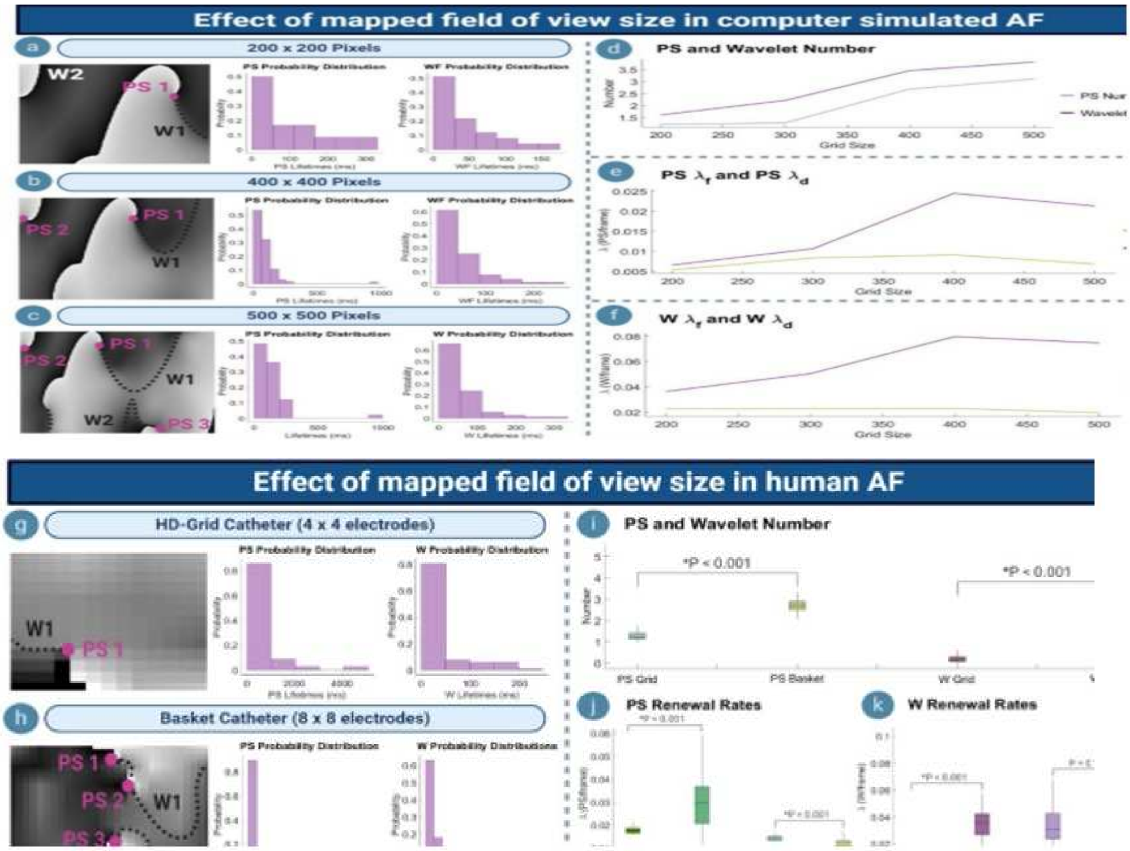

Figure 3. How a mapped field impacts view size [8].

It has been conjectured by [5] that both Equations (1) and (2) provide a strong characterization of the overall dynamics of PS and wavelet population.

To understand and identify PS and wavelet population dynamics in AF, $M/M/\infty$ birth–death approaches provide a unique quantitatively expressive architecture. There are opportunities for scientific application because this conceptual paradigm [7] has been demonstrated to work in a variety of AF studies at unique scales and mapping densities.

1.3. Information Length Theory

Shannon entropy is not the best descriptor [8] of a time series' statistical variations, which has made using other unique information theoretic notions, including Fisher Information [9–11], highly motivating. Differential entropy [12], Kullback-Leibler divergence (KD) [13], or information length

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(IL) [14,15]. By using IL, the total number of statistical variances for a specified temporal range can be found. Compared to other information metrics such as differential entropy, IL is preferred because it highlights the evolution path dependency between two states (PDFs) [16]. Time-dependent probability density functions (PDFs) provide the ability to trace time series evolution and measure variability [8], which is the basis of the IL metric's attractiveness.

Additionally, IL's formalism introduces a fascinating connection between information geometry and stochastic processes [17]. More interestingly, IL has numerous applications for quantum, fluid, and biological processes [18]. On the other hand, IL metric was the real motivation behind the provision of a novel info-geometric measure of casual information rate [19]. Its effects on the creation of entropy or free energy in the non-autonomous Ornstein-Uhlenbeck process highlight the significance of IL for stochastic thermodynamics [20]. In conclusion, [21–23] provide an examination of the interval learning (IL) computation of linear stochastic autonomous processes. This broadens the scope of application, enabling IL to be used for the abruption of event prediction and facilitating application to different engineering contexts.

1.4. IL as a Concept

Mathematically speaking, if x serves as a nth- order stochastic variable and $p(x,t)$ is a time-dependent PDF of x, then the Information Length $\mathcal{L}(t)$ corresponding to its evolution from the initial time $t_0 = 0$ to the final time $t_F = t$ is devised by:

$$\mathcal{L}(t) = \int_0^t \frac{dt_1}{\tau(t_1)} = \int_0^t \sqrt(\varepsilon(t_1))dt_1 \quad (3)$$

$$\varepsilon(t_1) = \int_{\mathbb{R}^n} \left(\frac{1}{p(x,t_1)}[\frac{\partial p(x,t_1)}{\partial t_1}]^2\right) dx \quad (4)$$

provided that $\sqrt(\varepsilon(t)$ serves as the root-mean- squared fluctuating energy rate.

Having a closer look at (3), it is essential to note that $\tau(t)$ serves as a dynamic temporal unit which provides the correlation time over which the changes of $p(x,t)$ take place [16]. Furthermore , $\tau(t)$ defines the statistical space's time unit. Having said that, $\sqrt(\varepsilon(t) = \frac{1}{\tau(t)}$, quantifies the information velocity [18].

It is preferable to compute the underlying value of the mathematical model of the related physical process to comprehend the meaning of $\mathcal{L}$. Taking the Langevin equation-described first-order stochastic process into consideration:

$$\frac{dx}{dt} = -\gamma(t)(x - f(t)) + \xi \quad (5)$$

x serves as a random variable, f defines a deterministic force, ξ represents a short-correlated random force satisfying that:

$$< \xi(t)\xi(t_1) > = 2D\delta(t - t_1) \text{ and } < \xi(t) > = 0 \quad (6)$$

where D serves as the amplitude (temperature) of the deterministic force (stochastic nose) ξ

It is to be noted that Equation (6) is so popular to be used as a descriptor of the motion of a particle under a harmonic potential in the form:

$$V(x) = \frac{1}{2}\gamma(x - f(t))^2 \quad (7)$$

Following [23,24], it is found that:

$$\varepsilon(t) = \frac{(\frac{d\beta}{dt})^2}{2\beta^2} + 2\beta(\frac{dy}{dt})^2 \quad (8)$$

$$p(x,t) = \sqrt{\frac{\beta}{\pi}e^{-(\beta(x-y))^2}}, y(t) = < x > = x(0)e^{-G(t)} + F(t)$$

$$\frac{1}{2\beta(t)} = < (x(t) - y(t))^2 > = \int_0^t 2D\, e^{-2(G(t)-G(t_1))}dt_1,$$

$$F(t) = \int_0^t e^{-(G(t)-G(t_1))}\gamma(t_1)f(t_1)dt_1, \quad (9)$$

provided that $G(t) = \int_0^t 2D\, \gamma(t')dt'$.

Considering Equation (4), it is clear that $\varepsilon(t)$ is dependent on the changes in both mean and variance defined by the corresponding dynamics of Equation (3), portraying $p(x,t)$ three-dimensional space variations, namely $(t, x, p(x,t))$ as in Figure 4, where the variation of the information velocity, $\sqrt(\varepsilon(t)$ would occur along the path starting initially from the state probability

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density function $p(x, t_0)$ to the final state at $p(x, t_F)$ as a descriptor of the speed limit from the statistical deviations of the observables [16].

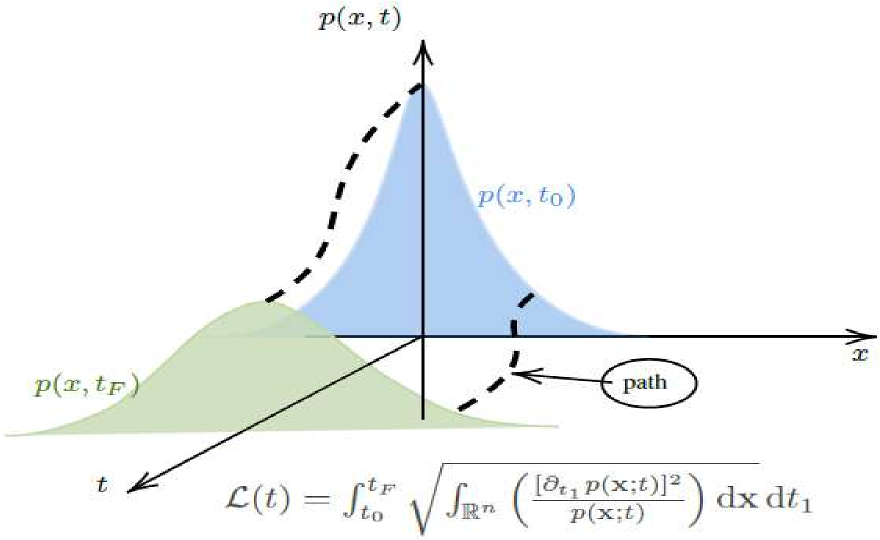

Figure 4. A graph depicting the evolution of $p(x,t)$ over time t. $\mathcal{L}(t)$ computes the total amount of statistical changes on $p(x,t)$ from t_0 to t_F [8].

Fundamentally, employing differential entropy [12], may not enable us to observe the occurring temporal statistical variations. This is a direct implication of the locality's deficiency since differential entropy is mainly for the quantification of the differences between any two given PDFs disregarding any intermediate states [17]. In a different symbolism, it only notifies us of the differences that have an influence on the underlying system's general development. IL $\mathcal{L}(t)$, on the other hand, measures any localised changes that occur along the system's course [14,17]. More crucially, IL has been touted as a cutting-edge technique for depicting an attractor structure and as a potent metric that can unite geometry and stochasticity [20,21]. From the point of stable equilibrium, the equivalent value of $lim_{t\to\infty} \mathcal{L}(t)$ would grow linearly depending on where the starting state's mean PDF $p(x,0)$ is located [15,24]. Notably, this strongly underlines that IL preserves the underlying Gaussian process' linear geometry by utilising Equation (6). More importantly, this particular property is lost when employing any other information metric [17,20]. By using definition, this emphasises that IL is a one-dimensional, model-free measure (3). Since IL is independent of data type, it may be devised by calculating the time series' time-variant PDF [8].

The road of this current paper reads: A spotlight introduction is presented in Section 1. The key findings are established in Section 2. Some challenging open problems, conclusion, and future research work are given in Section 3.

2. The Upper and Lower Bounds of The Data Information Length of Transient $M/M/\infty$ Queuing System

In this section, an exposition of a novel link between Information Length Theory (ILT) and Transient Queueing Systems (TQSs) is undertaken by deriving both upper and lower bounds of the data information length of a transient $M/M/\infty$ queuing system. In this context, it is revealed that if $\rho(t)$ serves as the time-dependent server utilization of the transient $M/M/\infty$ queuing system, then the latter obtained upper and lower bounds($UB(n,t), LB(n,t)$) respectively) are both $(n, \rho(t))$-dependent, $n = 0,1,2, \dots$.. Additionally, a typical numerical experiment is conjectured to illustrate the significant impact of time on behaviour of the devised $UB(n,t)$ and $LB(n,t)$ for different values of n.

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2.1. The Poisson Process

One of the most popular counting [25] methods is the Poisson process. It is typically employed in situations when we are counting the occurrences of specific events that seem to occur at a certain rate but are completely random (without a certain structure). For instance, let's say we know from past data that there are two earthquakes that happen in a specific region every month. The timing of earthquakes appears to be completely random except for this information. Thus, we draw the conclusion that the Poisson process may serve as a useful earthquake model. Models have

- Photons arriving on a photodiode.
- The quantity of auto accidents at a location or in a region.
- The position of users in a wireless network.
- Requests for certain publications on a web server.
- The start of conflicts.

A random process is called Poisson process [25] with the rate μ if it satisfies the following definition:

For a fixed $\boldsymbol{\mu} > 0, \boldsymbol{t} \in (\mathbf{0}, \infty)$, we define a counting process to be Poissonian with rates $\boldsymbol{\mu}$ when it satisfies the following:

1. $\boldsymbol{N}(\mathbf{0}) = \mathbf{0}$;
2. the underlying increments of $\boldsymbol{N}(\boldsymbol{t})$ are independent,
3. within any interval having a length $\boldsymbol{\vartheta} > \mathbf{0}$, the associated number of arrivals must follow a Poissonian $(\boldsymbol{\mu\vartheta})$ distribution.

Specifically, if

$$T_n = \sum_{i=1}^{n} X_i, \; T_n \sim Gamma(n, \mu) \tag{10}$$

provided that $X_i's$ serve as to be randomized independent variables satisfying $Exponential(\mu)$ variables. Then,

$$E[T_n] = \frac{1}{\mu} \tag{11}$$

$$Var[T_n] = \frac{1}{\mu^2} \tag{12}$$

This provides a simulation approach for a Poissonian process of a rate μ. We start with the generated $X_i \sim \text{Exponential}(\mu)$ to obtain the corresponding service times:

$$T_1 = X_1, \tag{13}$$

$$T_2 = X_1 + X_2, \tag{14}$$

$$T_3 = X_1 + X_2 + X_3, \tag{15}$$

Additionally, T_n must follow (11)–(12).

2.2. The Transient $\boldsymbol{M/M/\infty}$ Queue and IL

The $M/M/\infty$ queue (c., f., [26]) is a multi-server queueing model used in queueing theory, an emerging applied probabilistic discipline, with instant service-no wait arrival, as demonstrated by Figure 5. According to [27,28], the Poisson distributed service time with mean μ and mean arrival rate is λ in the $M/M/\infty$ queueing system's transient probability is given by:

$$p_n(t) = \frac{[\frac{\lambda}{\mu}(1-e^{-\mu t})]^n}{n!} \exp\{-\frac{\lambda}{\mu}(1-e^{-\mu t})\}, \; n = 0,1,2, \ldots. \tag{16}$$

Notably, as $t \to \infty$, $p_n(t)$ of Equation (16) converges to

$$p_n = \frac{\rho^n}{n!} e^{-\rho}, \; n = 0,1,2, \ldots. \tag{17}$$

ρ serves as the server utilization of the underlying queue.

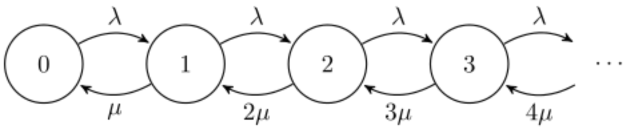

Figure 5. The state space diagram for the $M/M/\infty$ chain.

Recalling that the information length (IL) is defined to be:

$\mathcal{L}(t) = \int_0^t \frac{dt_1}{\tau(t_1)} = \int_0^t \sqrt{(\varepsilon(t_1))}dt_1$ (c.f., (3))

$\varepsilon(t_1) = \int_{\mathbb{R}^n} \left(\frac{1}{p(x,t_1)}[\frac{\partial p(x,t_1)}{\partial t_1}]^2\right) dx$ (c.f., (4))

If $\mathcal{L}_{M/M/\infty}$ serves as the IL of the transient $M/M/\infty$ queuing system, then the following theorems are devoted to the derivation of the lower and upper bounds of $\mathcal{L}_{M/M/\infty}$, namely $(LB(n,t), UB(n,t)$respectively).

Theorem 1. *The IL of the transient $M/M/\infty$ queuing system , $\mathcal{L}_{M/M/\infty}$ satisfies the following inequality:*

$\frac{2(n+1)(\rho(t))^{\frac{3}{2}}}{3\sqrt{n!}} > \mathcal{L}_{M/M/\infty}(t) > \left(\frac{\sigma\mu^n}{2^n n!}\right)\int_0^t t^n e^{-\rho(t)}\rho^{\cdot}(t)(\rho(t))^n dt$ (18)

Proof

We have

$p_n(x,t) = \frac{[\frac{\lambda}{\mu}(1-e^{-\mu t})]^n}{n!}\exp\{-\frac{\lambda}{\mu}(1-e^{-\mu t})\}$ (c.f., (16))

$= \frac{x^n}{n!}e^{-x}, x(t) = \frac{\lambda}{\mu}(1-e^{-\mu t}) = \rho(t)(1-e^{-\mu t})$ $\left(since\ \rho(t) = \frac{\lambda(t)}{\mu}\right)$ (19)

By the definition, we have $\mu > 0$, which implies that $e^{-\mu t} < 1$. Hence, $x(t) < \rho(t)$ follows. Then, we have the real number $\sigma \in (0,1)$ satisfying:

$x(t) = \sigma\rho(t)$ (20)

Consequently,

$$\frac{\partial p_n(t)}{\partial t} = \frac{\sigma x^{n-1}\rho^{\cdot}(t)}{n!}e^{-x}(n-x) \tag{21}$$

$$\frac{\left[\frac{\partial p_n(x,t)}{\partial t}\right]^2}{p_n(x,t)} = \frac{\left(\frac{\sigma x^{n-1}\rho^{\cdot}(t)}{n!}e^{-x}(n-x)\right)^2}{\frac{x^n}{n!}e^{-x}}$$

$$= \frac{\sigma^2 x^{n-2}\rho^{\cdot 2}(t)}{n!}e^{-x}(n-x)^2$$

$$= \frac{\sigma^2 x^{n-2}\rho^{\cdot 2}(t)}{n!}e^{-x}(n^2-2nx+x^2)$$

$$\leq \left(\frac{\sigma^2\rho^{\cdot 2}(t)}{n!}\right)(n^2+1) \quad (\text{Since}\quad e^{-x} \leq 1, x \in (0,1))$$

$$< \left(\frac{\rho^{\cdot 2}(t)}{n!}\right)(n^2+1)(\text{since}\quad \sigma \in (0,1)) \tag{22}$$

Hence, it follows that:

$$\mathcal{L}_{M/M/\infty} < \int_0^t \sqrt{(\int \frac{(n+1)^2\rho^{\cdot 2}(t)}{n!}dx)}\, dt$$

$$= \frac{(n+1)}{\sqrt{n!}}\int_0^t (\sqrt{\int dx}])\,\rho^{\cdot}(t)dt$$

$$= \frac{(n+1)}{\sqrt{n!}}\int_0^t (\sqrt{x}))\rho^{\cdot}(t)dt$$

$$= \frac{(n+1)}{\sqrt{n!}}\int_0^t \sqrt{\frac{\lambda}{\xi}(1-e^{-\mu t})}\rho^{\cdot}(t)dt$$

$$< \frac{(n+1)}{\sqrt{n!}}\int_0^t \sqrt{(\rho(t)}\rho^{\cdot}(t)dt \ \ (\text{Since}\ \ e^{-\mu t} \in (0,1) \quad = \frac{2(n+1)(\rho(t))^{\frac{3}{2}}}{3\sqrt{n!}} \tag{23}$$

On the other hand, by $x < \rho(t) and\ x < 1$, we have

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$$\frac{\left[\frac{\partial p_n(x,t)}{\partial t}\right]^2}{p_n(x,t)} = \frac{\sigma^2 x^{n-2} \rho^{\cdot 2}(t)}{n!} e^{-x}(n-x)^2$$
$$> e^{-2x}\left(\frac{\sigma^2 x^{n-2}\rho^{\cdot 2}(t)}{n!}\right)(n-1)^2$$
$$> e^{-2\rho(t)}\left(\frac{\sigma^2 x^{2n-1}\rho^{\cdot 2}(t)}{n!}\right)(n-1)^2$$
$$> \sigma^2 e^{-2\rho(t)}\rho^{\cdot 2}(t)\left(\frac{x^{2n-1}}{(n!)^2}\right)(n-1)^2 \tag{24}$$

Thus, it follows that:

$$\left(\sqrt{\int \frac{\left[\frac{\partial p_n(x,t)}{\partial t}\right]^2}{p_n(x,t)} dx}\right) > \left[\int \sigma^2 e^{-2\rho(t)}\rho^{\cdot 2}(t)\left(\frac{x^{2n-1}}{(n!)^2}\right)(n-1)^2\right] dx > \sigma\rho^{\cdot}(t)\ e^{-\rho(t)}\left(\frac{x^n}{n!}\right) \tag{25}$$

Hence,

$$\mathcal{L}_{M/M/\infty}(t) > \left(\frac{\sigma}{n!}\right)\int_0^t \rho^{\cdot}(t)e^{-\rho(t)}\left(\rho(t)(1-e^{-\mu t})\right)^n dt$$
$$> \left(\frac{\sigma}{n!}\right)\int_0^t e^{-\rho(t)} \quad \left(\text{Since, } (1-e^{-\mu t}) > \frac{\mu t}{2} \quad (\text{c. f., [29]})\right)$$
$$= \left(\frac{\sigma\mu^n}{2^n n!}\right)\int_0^t t^n e^{-\rho(t)}\rho^{\cdot}(t)(\rho(t))^n dt \tag{26}$$

Therefore,

$$\frac{2(n+1)(\rho(t))^{\frac{3}{2}}}{3\sqrt{n!}} > \mathcal{L}_{M/M/\infty}(t) > \left(\frac{\sigma\mu^n}{2^n n!}\right)\int_0^t t^n e^{-\rho(t)}\rho^{\cdot}(t)(\rho(t))^n dt \quad (\text{c.f., (18)})$$

In what follows, an illustration of the temporal impact as well as the potential impact of n on the behaviour of both $LB(n,t)$and $UB(n,t)$,

$$UB(n,t) = \frac{2(n+1)(\rho(t))^{\frac{3}{2}}}{3\sqrt{n!}} \quad \text{and} \quad LB(n,t) = \left(\frac{\sigma\mu^n}{2^n n!}\right)\int_0^t t^n e^{-\rho(t)}\rho^{\cdot}(t)(\rho(t))^n dt \tag{19}$$

2.3. Numerical Experiments

By choosing $\sigma = \frac{1}{2}, \rho(t) = \frac{1}{t}, \mu = 2$, it can be easily verified that these proposed choices, $UB(n,t)$ and $LB(n,t)$ will read:

$$UB(n,t) = \frac{2(n+1)}{3(t)^{\frac{3}{2}}\sqrt{(n!)}} \quad \text{and} \quad LB(n,t) = \left(-\frac{1}{2(n!)}\right)\int_0^t e^{-\frac{1}{t}}\frac{1}{t^2}dt = \left(-\frac{e^{-\frac{1}{t}}}{2(n!)}\right) \tag{20}$$

It is clear from Figure 6, that because of the progressive increase of time, $UB(n,t), n = 0,1,2$ are decreasing functions in time. Moreover, for each recorded value of time, by increasing the value of n, $UB(n,t), n = 0,1,2$ are increasing functions. More interestingly, this reveals that $UB(n,t)$ acts as decreasing function in time and an increasing function with respect to n. Fundamentally, the increase of time and $n$impacts $UB(n,t)$ to depict longer heavy tails. This translates to seeing the longest heavy tails for $UB(2,t)$, and these tails become shorter for $UB(1,t)$ and the shortest would be for $UB(0,t)$.

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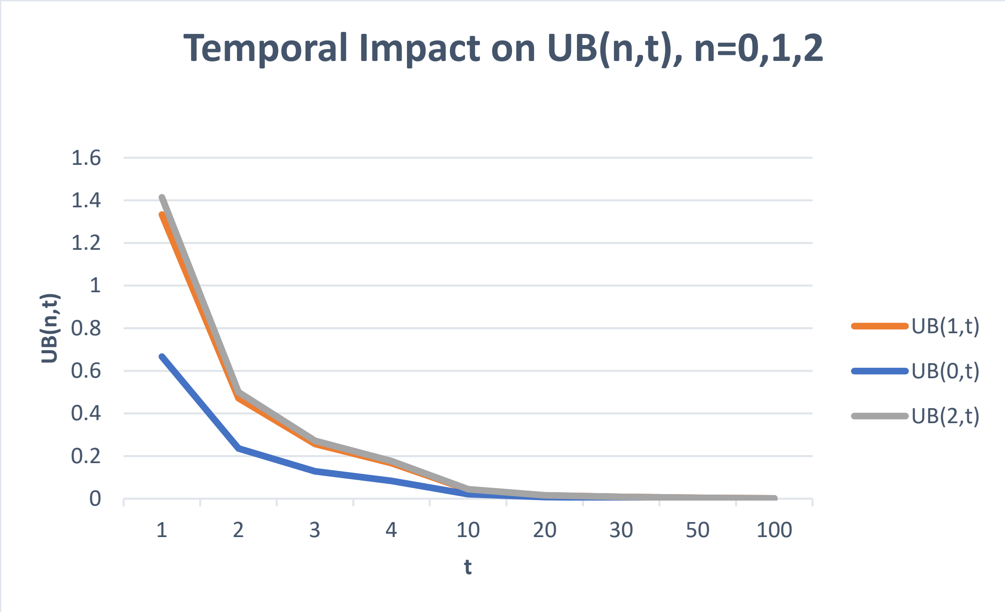

Figure 6. Significant Impact of time and n on $UB(n,t)$.

Reading Figure 7, $LB(n,t)$is decreasing in time and increasing as n increases. More potentially, the graph representation of $LB(n,t)$ produces shorter heavy tails by the increase of time and n. This shows a complete converse scenario in comparison to the recorded heavy tails phenomena in $UB(n,t)$.

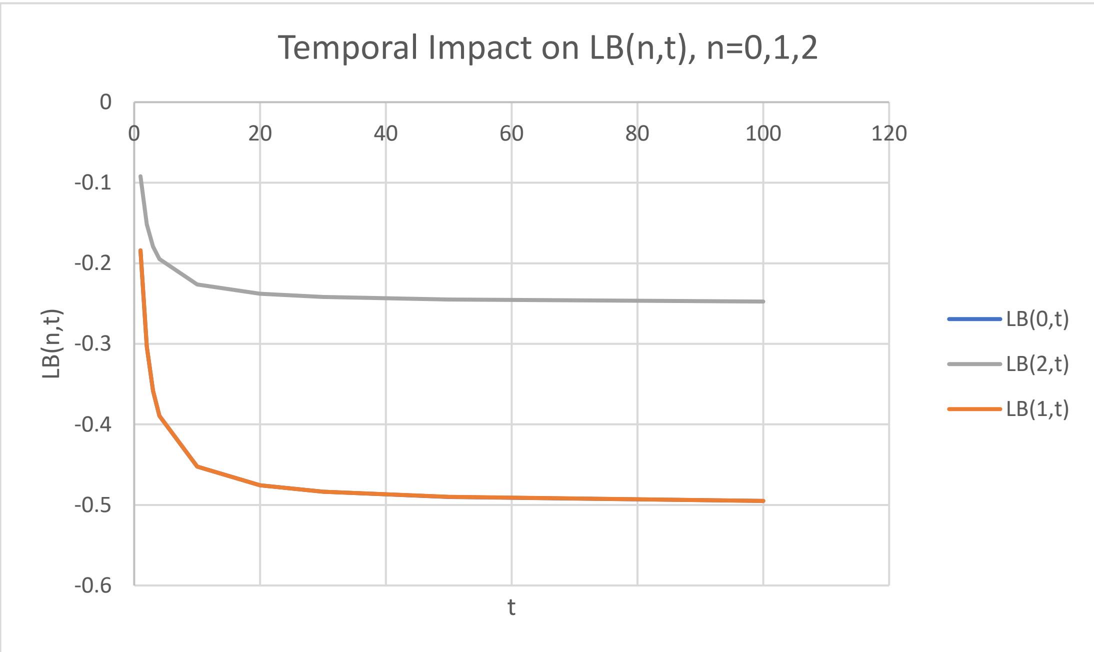

Figure 7. Significant Impact of time and n on $LB(n,t)$.

In mathematical terms, as time becomes sufficiently large($t \to \infty$), it follows that

$$lim_{t\to\infty} UB(n,t) = lim_{t\to\infty} \frac{2(n+1)}{3(t)^{\frac{3}{2}}\sqrt{n!}} = 0 = lim_{t\to\infty}\left(-\frac{e^{-\frac{1}{t}}}{2(n!)}\right) = lim_{t\to\infty}\left(-\frac{1}{2(n!)}\right)\int_0^t e^{-\frac{1}{t}}\frac{1}{t^2}dt$$
$$= lim_{t\to\infty} LB(n,t) \quad (21)$$

Engaging the findings of Equations (18) and (21), it holds that as time reaches infinity, $n = 0,1,2$

$$lim_{t\to\infty}\mathcal{L}_{M/M/\infty}(t) = 0 \quad (22)$$

Notably, it is shown that for $M/M/\infty$ transient queueing system [27,28] as $t \to \infty$, the correspondent steady state probability density function, p_n is devised by:

$$p_n = \frac{\rho^n}{n!} e^{-\rho}, \quad n = 0,1,2, \ldots. \qquad \text{(c.f., (17))}$$

Since p_n(c.f., (17)) does not depend on time, then we have by the definition of IL, (c.f., (2), (3)), that the corresponding stability phase of $M/M/\infty$ queueing system has an underlying zero information length, that is:

$$lim_{t\to\infty}\mathcal{L}_{M/M/\infty}(t) = 0, n = 0,1,2, \quad M/M/\infty \text{ is stable} \tag{23}$$

Clearly, Equations (22) and (23) are equivalent. This provides a strong validation of both obtained mathematical and numerical results. The strategy of the proof can be extended to the remaining values of $n = 3,4,5, \ldots.,$ which will show that $UB(n,t)$ will start to decrease in both (n,t), i.e, the behavioural trend will reverse in terms of the Increasability phase in n. More interestingly, it can be verified that $LB(n,t)$ will never change its behavioural trend in (n,t) by being temporarily decreasing and increasing with the respective increase of n.

In a more detailed account, communicating [30], it could be analytically demonstrated that $UB(n,t)$ and $LB(n,t)$ (c.f., (20)) are both (increasing in t)(decreasing in n) by showing that:

I) $\frac{\partial UB(n,t)}{\partial t} < 0, \frac{\partial UB(n,t)}{\partial n} > 0$ (24)

II) $\frac{\partial LB(n,t)}{\partial t} < 0, \frac{\partial LB(n,t)}{\partial n} > 0$ (25)

We have

$$\frac{\partial UB(n,t)}{\partial t} = \frac{\partial}{\partial t}\left(\frac{2(n+1)}{3(t)^{\frac{3}{2}}\sqrt{(n!)}}\right) = -\frac{(n+1)}{(t)^{\frac{5}{2}}\sqrt{n!}} < 0 \text{ for all } t > 0 \tag{26}$$

Additionally, engaging [31], the Stirling formula to compute $n!$ is written as

$$n! \sim \sqrt{(2\pi n)}\left(\frac{n}{e}\right)^n, \qquad n = 3,4, \ldots \tag{27}$$

$$\begin{aligned}
\frac{\partial UB(n,t)}{\partial n} &\sim \frac{2}{3(t)^{\frac{3}{2}}(2\pi)^{\frac{1}{4}}}\frac{\partial}{\partial n}\left(\frac{(n+1)}{\left(\frac{n}{e}\right)^{\frac{n}{2}} n^{\frac{1}{2}}}\right) = \frac{2}{3(t)^{\frac{3}{2}}(2\pi)^{\frac{1}{4}}}\frac{\partial}{\partial n}\left(\left(n^{\frac{1}{2}} + n^{-\frac{1}{2}}\right)\left(\frac{n}{e}\right)^{-\frac{n}{2}}\right) \\
&= \frac{2}{3(t)^{\frac{3}{2}}(2\pi)^{\frac{1}{4}}}\left(\left(\frac{1}{2}n^{-\frac{1}{2}} - \frac{1}{2}n^{-\frac{3}{2}}\right)\left(\frac{n}{e}\right)^{-\frac{n}{2}} + \left(n^{\frac{1}{2}} + n^{-\frac{1}{2}}\right)\left(-\frac{1}{2}(lnn - 1) - \frac{1}{2}\right)\left(\left(\frac{n}{e}\right)^{-\frac{n}{2}}\right)\right) \\
&= \frac{\left(\frac{n}{e}\right)^{-\frac{n}{2}}}{3(t)^{\frac{3}{2}}(2\pi)^{\frac{1}{4}}}\left(\left(n^{-\frac{1}{2}}(1 - lnn) - n^{-\frac{3}{2}}\right) - \left(n^{\frac{1}{2}}\right)lnn\right) < 0, n = 3,4, \ldots
\end{aligned} \tag{28}$$

This proves (I).

Additionally,

$$\frac{\partial LB(n,t)}{\partial t} = \frac{\partial}{\partial t}\left(-\frac{e^{-\frac{1}{t}}}{2(n!)}\right) = -\left(\frac{e^{-\frac{1}{t}}}{2(n!)t^2}\right) < 0 \text{ for all } t > 0, n = 3,4, \ldots \tag{29}$$

Additionally,

$$\begin{aligned}
\frac{\partial LB(n,t)}{\partial n} &\sim \left(-\frac{e^{-\frac{1}{t}}}{2\sqrt{(2\pi)}}\right)\frac{\partial}{\partial n}\left(n^{-\frac{1}{2}}\left(\frac{n}{e}\right)^{-n}\right) \\
&= \left(-\frac{e^{-\frac{1}{t}}\left(\frac{n}{e}\right)^{-\frac{n}{2}}}{2\sqrt{(2\pi)}}\right)\left(-\frac{1}{2}n^{-\frac{3}{2}} - n^{-\frac{1}{2}}lnn\right) \\
&= \left(\frac{e^{-\frac{1}{t}}n^{-\frac{3}{2}}\left(\frac{n}{e}\right)^{-\frac{n}{2}}}{4\sqrt{(2\pi)}}\right)(1 + 2nlnn) > 0 \quad \text{for all } t > 0, n = 3,4, \ldots
\end{aligned} \tag{30}$$

Hence, (II) follows.

Fundamentally, we have demonstrated with strong supporting mathematical evidence that the undertaken experiments agree with the analytic proofs.

3. Concluding Remarks, Open Problems, and Future Research

This paper contributes to the establishment of Information Length Theory of Transient Queueing Systems. The novel mathematical derivations are undertaken by finding the integral formula of the

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information length of the transient $M/M/\infty$ queuing system. Because of the complexity to derive the closed form result of the later integral formula, both the upper and the lower bounds of that integral, namely $UB(n,t), LB(n,t))$ were derived.

More interestingly, it is observed that $UB(n,t), LB(n,t))$ are both $(n,\rho(t))$-dependent, $n = 0,1,2, \ldots ..$, with $\rho(t)$ to define the time-dependent server utilization of the transient $M/M/\infty$ queuing system. Moreover, these analytic findings were validated numerically.

Here are some emerging open problems:

Open Problem One

Is it mathematically feasible to unlock the challenging problem of finding the exact analytic form of , $\mathcal{L}_{M/M/\infty}$ (c.f., (18)), rather than obtaining its upper and lower bounds? This problem is still open.

Open Problem Two

Can we extend the undertaken approach to other transient queueing systems, such as the Parthasarathian transient?

of $M/M/1$ queue [31]?

Open Problem Three

Looking at Equation (18), can we find any other strict upper and lower bounds for $\mathcal{L}_{M/M/\infty}$?

Future research pathways include the attempting to solve the proposed open problems and extending the information data length theory to explain other fields of human knowledge, such as engineering, physics and much more.

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