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## Article

# A Stochastic Sampling Approach to the Measurement Problem

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**Abstract:** The present paper reports an alternative solution of the measurement problem in quantum theory. The measurement problem can be characterized as that the U procedure verses the R procedure. The R-procedure is tested by the Yes/No measurement originally proposed by von Neuman and discussed in details by Penrose. We propose a novel stochastic sampling method to tackle the measurement paradox. Each testing sample produces a pair of Yes-number  $c$  and No-number  $d$ , which in turn generates a sample phase  $\theta_R$  with respect to the exponential form of  $(c + id)$ . All the sample phases form a group, write  $G_\psi$ . Taking  $G_\psi$  as the sampling potential, the revised Schrödinger equation has the form  $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + G_{\psi(U)} \otimes G_{\psi(R)}$ . This equation has a solution  $\psi[x, t, \theta_{U(x_i)}, \theta^{R(x^j)}]$ , where  $\theta_{U(x_i)}$  is the dynamic phase, while  $\theta^{R(x^j)}$  is the sample phase. Based on the Born rule, the probability of a sample is given by the squared magnitude of  $(c + id)$ . Some metaproperties of the new U-R model, such as natural transformation, consistency, and completeness, are presented. The present work provides a new picture of quantum mechanics and alike.

**Keywords:** U-R procedures; the Yes/No type measurement; stochastic sampling; sample phase; schrödinger equation; born probability

## 1. Introduction

Measurement problem, also called the measure paradox by Penrose [1], is a well-documented issue in quantum physics. It has caused some longstanding controversies in quantum mechanics and causes a great deal of debates. The present paper aims to provide an alternative solution for the measurement problem. In order to tackle this paradox, we have to face a challenging inquiry: shall we take the observation as part of the theory, or not. For this, we take the former approach.

The measurement paradox reflects a very deep issue about how we understand quantum physics. On one hand, by Dirac, the greater the degree of disturbance in our observation, the smaller the world we can observe. Quantum theory is about small world. Here, the disturbance in our observation should include the limitation of our experimental methods. I would add some words here that the smaller the world, the more important and more sensitive the observation. On the other hand, at the very bottom line, quantum physics is an empirical science, so it cannot stand without observations. As Penrose points out [1], "Some people might take refuge in the viewpoint that quantum states are 'not real things' being 'not measurable', or something".

Nevertheless, Penrose believes that there are powerful reasons for expecting a change. Such a change, to Penrose's view, represents a major revolution and it cannot be achieved by tinkering quantum mechanics. Yet, the necessary changes must themselves be thoroughly respectful of the central principles that lie at the heart of present-day physics, which is the same as the basic aim of this paper.

The rest of this paper contains contents as follows. Section 2 revisits the measurement paradox. We introduce two key operators, namely,  $Q$  and  $E$ . Section 3 introduces the Yes/No measurement and an application of the Dirac  $\delta$ -function. Section 4 proposes a novel approach, named stochastic sampling, step by step, in details. Section 5 formulates a revised version of the Schrödinger equation, in which the direct product of dynamic phase and sampling phase serves takes place of potential term. This vanishes the measurement paradox. Section 6 defines metaproperties such as natural

transformation, consistency, and completeness. Section 7 provides some general discussions, and outlines some topics for further research.

## 2. Revisiting measurement paradox

Let us review a few key concepts concerning the measurement paradox.

### 2.1. Observable $Q$

The measurement paradox can be simply characterized as U-procedure verses R-procedure. Here U stands for unitary, and R stands for reduction. On one hand, the U procedures work so supremely well for simple enough systems, whereas on the other, we have to give up on U and abruptly, yet stealthily, interpose the R process from time to time. The two quantum processes, U and R, are conflicts. On the one hand, U-procedure is the deterministic process of unitary evolution which can be described by Schrödinger's Equation, which controls the clear-cut temporal evolution of a definite mathematical quantity, namely the state vector  $|\psi\rangle$ . The wavefunction in the U process is single-valued, continuous, differentiable, and square-integrable. On the other hand, R-Procedure is the quantum state reduction which takes place when a 'measurement' is performed. The R process is a discontinuous random jumping of this same  $|\psi\rangle$ , where only the probabilities of the different outcomes are determined.

Observable operator  $Q$  is responsible to transfer from U-procedure to R-procedure.  $Q$  has two eigenstates, say, one is YES and the other is NO. How  $Q$  works is a mystery. Penrose [1] reviewed and discussed six approaches toward this problem. An interested reader may read his book, particularly in its §29 [1].

### 2.2. Projector $E$

The projector  $E$  was originally introduced by von Neuman [2]. Penrose [1] provides a thorough characterization of  $E$ . Indeed, his §22.5 is entitled: Yes/No measurement. It is the view of von Neuman and Penrose that all the quantum measurements are the Yes/No measurements. Once  $Q$  turns a quantum process into the R-procedure, then  $E$  projects the quantum state to Yes or else No.

## 3. The Yes/No measurement

The Yes/No (Y/N) type measurement originally proposed by von Neuman [2] and outlined in detail by Penrose [1]. Both authors stated that all the quantum theoretic experiments are Y/N type measurements, characterized by the projector  $E$ . Examples of experiments in physics, e.g., the Stern-Gerlach experiment, can be found in Sakurai [3].

E-projector: Consider an any given wavefunction  $\varphi(x)$ , where  $x$  ranges over all space points of  $X$ . For any given space point  $a$ ,  $E[\psi(x)] = a$ , is called the  $E$  operation, which stands for the any given Y/N observations.

$$\delta(x) = \begin{cases} \infty, & x=x_0 \\ 0, & x \neq x_0 \end{cases} \quad (1)$$

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} \delta(x) dx = M \quad (2)$$

In measurement theory,  $\delta(x)$  in (2) is called a testing function, and  $x_0$  in (1) is called the supporting point of  $\delta(x)$  [4]. Now we look at an important property of the  $\delta$ -function.

**Lemma 1** (Existentiality). Let  $\psi(x)$  be an any given one-dimensional wavefunction. Assume  $(x_0 + \varepsilon)$  is a R-interval. Then we have

$$\int_{x_0}^{x_0+\varepsilon} \psi(\xi) \delta(x_0 + \varepsilon) dx = \psi(x_0) \quad (3)$$

**Proof.** It is held simply by the well-known selectivity of the Dirac  $\delta$ -function [5].

To solve the measurement paradox, the existentiality is not enough. It needs to further formulate a constructive proof. The Y/N type measurement enables us to recapture paradox of U-procedure vs. R-procedure. From mathematical perspectives, it is a single-valued vs. two-valued problem. During the U-procedure, the wavefunction is single valued. While during the R-procedure, any measurement of the wavefunction becomes two-valued. However, the two procedures must share the same semantics, namely the squared modulus, i.e., the Born probability. Thus, how to generate the required Born probability from R-procedure is a sensitive issue. In Section 4, we further to formulate a constructive proof by using a novel stochastic sampling method.

Readers who are interested in a more detailed introduction to the Yes/No type measurement, I suggest to read Penrose §22.6 [1].

#### 4. Stochastic sampling

Let us consider an any given wavefunction  $\varphi(x)$ , where  $x$  ranges over all of the space points. We assume that  $\varphi(x)$  is one-dimensional without loss of generality for multi-dimensions. Thus, the corresponding Hilbert space we are currently discussing is one-dimensional, denoted by  $H$ . Hence, we may treat all the vectors in  $H$  as space points also without loss of generality. Now, it introduces an observation operator  $Q$ , which is defined below.

**Definition 1.** For any given  $a$ ,  $a \in H$ ,  $Qa = \mathbf{a}$ . We call that  $\mathbf{a}$  is the observational conjugate of  $a$ . Accordingly, we define  $H^* = \{x \mid x \text{ ranges over all possible observational } \mathbf{a}\}$ . Call  $H^*$  the observational dual space of  $H$ .

The necessity of the distinction between the space points and the observational points is analytical to the distinction between of the intuitive natural numbers and the set-theoretic enumerers in Gödel's work [6].

Consider the power set of  $H$ , write  $P(H)$ . Now, we start to select the elements from  $P(H)$ . Notice that this selection process is countable, but the cardinal number of  $P(H)$  is an uncountable infinity. We may reasonably assume this selection process is stochastic.

**Definition 2.** We introduce a new variable  $x^j$ ,  $x^j \in P(H)$ . Of course, we also have  $x^j \subset H$ , so we can introduce another variable  $x_i^j$ , where the superscript  $j$  indicates the  $j^{\text{th}}$  element stochastically selected from  $P(H)$ , the subscript  $i$  indicates that  $x$  ranges over only those space points within  $x^j$ . It is easy to see that  $x_i^j$  connects  $H$  and  $P(H)$ . Accordingly, we have

**Definition 3.** We introduce a new operator  $\hat{Q}$ , called the sample generator. For any given  $x_i^j$ ,  $\hat{Q}(x_i^j) = \mathbf{A}_i^j$ , where  $\mathbf{A}_i^j \in P(H^*)$ . Call  $\mathbf{A}_i^j$  the testing adjoint of  $x_i^j$ .

**Definition 4.** Stochastic sampling: 1. For any given finite  $x_i^j$ , once a  $x_i^j$  is stochastically selected, its adjoint  $\mathbf{A}_i^j$  becomes a testing sample. 2. For any  $x_i^j$ , if it has not been selected, then its adjoint  $\mathbf{A}_i^j$  is not a testable sample yet.

Note, this definition is analytical to the expressibility in Gödel's work [6]. (Hint, the notion of expressibility is necessary to bridge the relations in Piano arithmetic and functions in the first order theory.) While here the definition of stochastic sampling process is necessary to bridge any  $x_i^j$  and  $\mathbf{A}_i^j$  from sampling perspectives.

**Definition 5.** (R-procedure). Let  $\psi$  stand for an any given sample  $\mathbf{A}_i^j$ ,  $\varphi$  denote a YES/No type experiment, and  $q$  be a Yes/No type stimuli that  $\varphi$  can use to test  $\psi$ . By Dirac bra-ket formalism, we can write this structure as  $\langle \psi \rangle$ . When  $\varphi$  gives the stimulus  $q$  to  $\psi$ , each operational conjugate  $\mathbf{a}_i^j$  in  $\psi$  returns a Yes/No type response. Thus,  $\psi$  is a function of  $\varphi$ , write  $\psi(\varphi)$ . This is called the R-

procedure of the wavefunction. Note, this idea is from Feynman [7], who calls  $\langle\varphi|$  the final state and  $|\psi\rangle$  the initial state of a quantum theoretic experiment.

**Definition 6.** (Sample space). The sample space for the Yes/No type measurement is two-valued, i.e.,  $S = [Yes, No]$ . This means the E-projector has two and only two eigenstates, of which the eigenvalues are Yes and No.

**Definition 7.** (Sample phase). Consider projector  $E$ , for each proper sample of Yes/No type measurement,  $E(a_i^j)$  conducts a yes response or else a no response. Thus,  $E(a_i^j)$  produces a pair of the yes-number  $c$  and the no-number  $d$ , which in turn produces a sample phase  $\theta^j$  with respect to the exponential form of  $(c + id)$ . All the possible sample phases form an  $U(1)$  group, write it  $G$ . From Definitions 1 to 4, it is easy to see that  $G$  is originally generated from the wavefunction  $\psi(x)$ , so we write  $G$  as  $G_{\psi(R)}$ .

Because  $G_{\psi}$  obviously satisfies  $U(1)$  symmetry, the stochastic sampling here satisfies the required conservativeness. It is worth mentioning that, in addition to the well-documented dynamic phase and Berry phase in the literature of dynamic analysis, the sample phase introduced here is the third kind of phase. This is one significant character of the R-procedure. For the U-procedure, we have the dynamic phase potential group, write it as  $G_{\psi(U)}$ .

**Definition 8.** (Linearization). The linearization operator  $L$  is defined by  $L(c, d) = (c + id)$ .

**Definition 9.** (Sample Born probability). For any given testing sample  $A^j$ , which produces a yes-number  $c$  and a no-number  $d$ . The sample Born probability is defined by

$$P(A^j) = |(c + id)|^2. \quad (4)$$

Born probability is a kind of explanation, which serves as a semantics for the evolution of wavefunction. As Penrose points out [1], the U-procedure and R-procedure must share the same semantics, i.e., the squared magnitude of two eigenvalues.

**Theorem 1.** (Born rule) The Born probability defined by Definition 9 obeys Born rule.

**Proof.** Let  $A_i^j = \hat{Q}(x_i^j)$  be a testing sample.  $E(A_i^j)$  has two eigenstates, Yes or else No. Assume the eigenvalue for Yes is  $c$  and the eigenvalue for No is  $d$ . Then, by Definition 8, we have  $L(c, d) = (c + id)$ . Hence, by Definition 9, we have

$$p(c + id) = |\langle(c + id)|A^j\rangle|^2. \quad (5)$$

This shows that Definition 8 is conformal with respect to the Born rule (QED).

## 5. Revised Schrödinger equation

The classical Schrödinger Equation is given below

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V_{\psi}. \quad (6)$$

Now we revise Schrödinger Equation to have a new version

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + G_{\psi(U)} \otimes G_{\psi(R)}. \quad (7)$$

On the right side of revised Schrödinger Equation, the first term is responsible for the U-procedure, during which the potential is  $G_{\psi(U)}$ , and in the second term  $G_{\psi(R)}$  is responsible for R-procedure. In other words, the potential term is the direct product  $G_{\psi(U)} \otimes G_{\psi(R)}$ . Note that both terms on the right side of the revised Schrödinger Equation are originally based on the same variable  $x$ . We know that in the classical Schrödinger equation, for a single particle of mass  $m$ , moving in an external world, the

potential energy contribution is  $V = V(x, y, z)$ . In the revised Schrödinger equation, it takes not only the dynamic phase  $G_{\psi(U)}$  but also the statistical sample phase  $G_{\psi(R)}$  as the integrated potential.

The solution of the revised Schrödinger equation now has the form

$$\psi = \psi \left[ x, t, \theta_{U(x_i)}, \theta^{R(x^j)} \right] \quad (8)$$

Where  $\psi$  is the wavefunction,  $\theta_{U(x_i)}$  stands for the dynamic phase during the U-procedure and  $\theta^{R(x^j)}$  stands for the sample phase during the R-procedure. It also shows that the sample phase provides a fruitful base for Born probability and that stochastic sampling formulates the Born ensemble. It also provides a better understanding of the Born Rule.

As Penrose characterized, the measurement problem of quantum theory concerns the distinction and unification of U-procedure and R-procedure. The framework reported here postulates a unified U-R model.

**Theorem 2.** (the U-R consistency). Consider the revised Schrödinger equation (7), and let  $\psi$  be the wavefunction in the equation. As a solution, some unified U-R evolution of

$$\psi = \psi \left[ x, t, \theta_{U(x_i)}, \theta^{R(x^j)} \right] \text{ always exists, and can be mechanically constructed.}$$

**Proof.** First, the existentiality is obvious by the selectivity function shown in (3). Second, the solution of the form (8) can be constructed by following the stochastic sampling procedure explained in Section 4 (QED).

**Corollary 1.** Given *Theorem 1*, the so-called measurement paradox vanished.

**Proof.** Assumption for contradiction. Suppose there was a counter example, i.e., the measurement paradox would occur at some steps of stochastic sampling process. This would mean that the R-procedure breaks the wavefunction, which makes R-procedure is no longer  $x$  dependent. But a careful check at each step during the stochastic sampling, shows that all of the steps are  $x$  dependent. Thus, the assumption for contradiction does not hold. Hence, the measurement paradox is indeed vanished.

## 6. Metaproperties

### 6.1. Natural transformation

The contents above can be seen more intuitively in terms of category theory [8]. Both  $H$  and  $H^*$  can be treated as categories and various operators can be treated as functors between them. In Figure 1, it shows three functors from  $H$  to  $H^*$



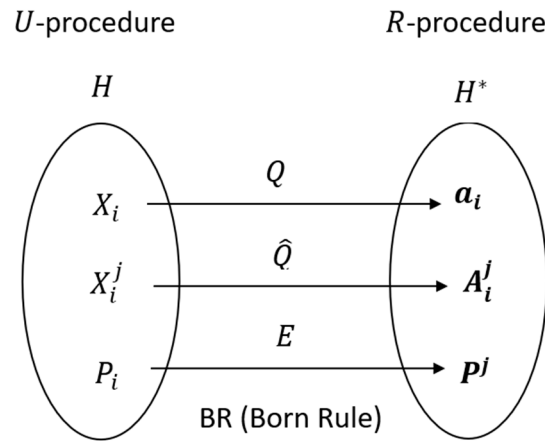


Figure 1. Three functors from  $H$  to  $H^*$ .

Further, natural transformations exist between these functors. Figure 2 depicts a natural transformation from  $Q$  to  $BR$ , where  $\hat{Q}$  stands for the functor from  $x_i^j$  to the corresponding testing sample  $A_i^j$ , and  $BR$  stands for the Born rule, which serves as the bidirectional functor connecting  $p_i$  and  $p^j$ .

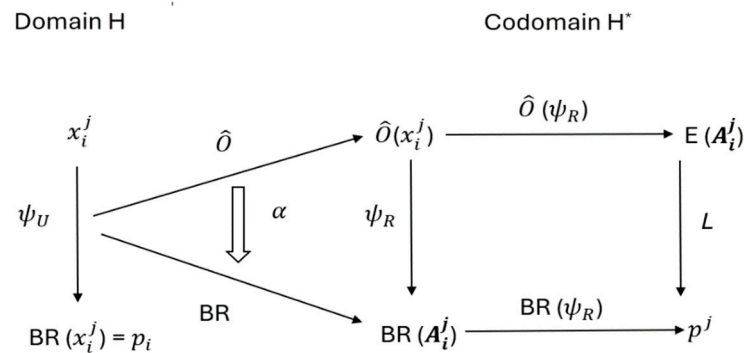


Figure 2. The natural transformation  $\alpha$  from functor  $\hat{Q}$  to functor  $BR$ .

The meaning of each symbol in the Figure 2 is defined in Section 4. The structures of the Figure 2 are explained fully in 6.2.

## 6.2. Metalogic

It is better to have a holistic picture on the work presented in this paper from metalogical perspectives. To follow the concepts in metalogic, by keeping Gödel's methods [6] in mind, we introduce the ideas about metaproperties, namely, consistency and completeness, and prove two theorems accordingly.

**Definition 10.** (consistency). Assume  $x_i^j \in P(H)$ ,  $A_i^j \in P(H^*)$ . Let  $p^j$  be the Born probability of an any given testing sample  $A_i^j$ . The U-R model is consistent if the  $p^j$  can be inferred from  $x_i^j$ .

**Theorem 3.** The new U-R model is consistent.

**Proof.**

$$BR \left[ L \left[ E \left( Q(x_i^j) \right) \right] \right]$$

$$\begin{aligned}
&\Rightarrow BR\left(L\left[\left(E(A_i^j)\right)\right]\right) \\
&\Rightarrow BR[L(c, d)] \\
&\Rightarrow BR(c + id) \\
&\Rightarrow |(c + id)|^2 \\
&\Rightarrow p^j \quad (\text{QED})
\end{aligned}$$

**Definition 11.** (inverse operators). For each operator (i.e.,  $Q, \hat{Q}, E, L, BR$ ) introduced earlier, we now introduce its corresponding inverse operator respectively:  $Q^{-1}, \hat{Q}^{-1}, E^{-1}, L^{-1}, BR^{-1}$ . As usual, for example,  $Q^{-1}Q = 1$ .

**Definition 12.** (completeness). Assume  $x_i^j \in P(H), A_i^j \in P(H^*)$ . Let  $p^j$  be the Born probability of an any given testing sample  $A_i^j$ . The U-R model is complete if  $x_i^j$  can be inferred from  $p^j$ .

**Theorem 4.** The new U-R model is complete.

**Proof.**

$$\begin{aligned}
&\hat{Q}^{-1}[E^{-1}(L^{-1}(BR^{-1}(p^j)))] \\
&\Rightarrow \hat{Q}^{-1}[E^{-1}(L^{-1}(|(c + id)|^2))] \\
&\Rightarrow \hat{Q}^{-1}[E^{-1}(c, d)] \\
&\Rightarrow \hat{Q}^{-1}(A_i^j) \\
&\Rightarrow x_i^j \quad (\text{QED})
\end{aligned}$$

From Theorems 3 and 4, we can see that the new U-R model for the evolution of the Schrödinger equation is consistent and complete. Thus, we may conclude that the open conjecture of the so-called measurement paradox, i.e., the U-procedure versus the R-procedure, of the quantum mechanics is closed.

## 7. General discussions

In its very nature, physics is an empirical science. Thus, the measurement problem must be taken seriously. In this paper, we constructed a novel stochastic sampling model, which enabled us to formulate a reasonable solution to the measurement paradox that has long been impeding the development of quantum mechanics. The revised version of Schrödinger equation should have a wide effect on the foundations of quantum theoretic modeling. Below are some further research directions.

First, to make the framework presented here relativistic, such that we may have a new picture of quantum field theory. It is also worth mentioning that sampling is actually a kind of localization.

Second, to extend the stochastic sampling approach to statistical physics, such that we may build a bridge to stochastic thermodynamics [9].

In addition, the measurement problem is a problem shared by a wide range of scientific fields. To list a few, each of the following domains commonly performs a version of the Yes/No measurement: psychology of reasoning, standard educational test, behavioral science, the Turing test, deep learning and big data models, etc. As John Wheeler once pointed out, quantum theoretic measurements are likely playing the 20-questions game with nature. It is a deep insight. As long as we are concerned with quantum observations and working on empirical issues involving micro-phenomenon, it is necessary to construe this character into our theoretical framework and corresponding models.

Theoretically, this paper contributes two key concepts, both of which generated from the stochastic sampling model: the sample phase and the phase potential. In quantum dynamic analysis of particle physics, we know that there are two kinds of phases, namely, the internal dynamic phase carried by the wavefunction and the external Berry phase for parallel transport. We add a third kind



of statistical phase, namely, the sample phase. The potential term in the classical Schrödinger equation captures the difference of space points. In the revised Schrödinger equation, not only does the potential term reflect the difference between the yes-number and the no-number within a sample, but it also provides the information on the difference from one sample to another. We are confident that these new perspectives will have many applications.

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