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[K. MAHESH KRISHNA](#)*

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Article

Nonlinear Heisenberg-Robertson-Schrodinger Uncertainty Principle

K. Mahesh Krishna

Post Doctoral Fellow, Statistics and Mathematics Unit, Indian Statistical Institute, Bangalore Centre, Karnataka 560 059, India; kmaheshak@gmail.com

Abstract: We derive an uncertainty principle for Lipschitz maps acting on subsets of Banach spaces. We show that this nonlinear uncertainty principle reduces to the Heisenberg-Robertson-Schrodinger uncertainty principle for linear operators acting on Hilbert spaces.

Keywords: uncertainty principle; lipschitz map; banach space

MSC: 26A16; 46B99

1. Introduction

Let \mathcal{H} be a complex Hilbert space and A be a possibly unbounded self-adjoint linear operator defined on domain $\mathcal{D}(A) \subseteq \mathcal{H}$. For $h \in \mathcal{D}(A)$ with $\|h\| = 1$, define the **uncertainty** of A at the point h as

$$\Delta_h(A) := \|Ah - \langle Ah, h \rangle h\| = \sqrt{\|Ah\|^2 - \langle Ah, h \rangle^2}.$$

In 1929, Robertson [1] derived the following mathematical form of the uncertainty principle (also known as uncertainty relation) of Heisenberg derived in 1927 [2] (English translation of 1927 original article by Heisenberg). Recall that for two linear operators $A : \mathcal{D}(A) \rightarrow \mathcal{H}$ and $B : \mathcal{D}(B) \rightarrow \mathcal{H}$, we define $[A, B] := AB - BA$ and $\{A, B\} := AB + BA$.

Theorem 1. [1–6] (*Heisenberg-Robertson Uncertainty Principle*) Let $A : \mathcal{D}(A) \rightarrow \mathcal{H}$ and $B : \mathcal{D}(B) \rightarrow \mathcal{H}$ be self-adjoint operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have

$$\frac{1}{2} \left(\Delta_h(A)^2 + \Delta_h(B)^2 \right) \geq \frac{1}{4} (\Delta_h(A) + \Delta_h(B))^2 \geq \Delta_h(A)\Delta_h(B) \geq \frac{1}{2} |\langle [A, B]h, h \rangle|. \quad (1)$$

In 1930, Schrodinger improved Inequality (1) [7] (English translation of 1930 original article by Schrodinger).

Theorem 2. [7] (*Heisenberg-Robertson-Schrodinger Uncertainty Principle*) Let $A : \mathcal{D}(A) \rightarrow \mathcal{H}$ and $B : \mathcal{D}(B) \rightarrow \mathcal{H}$ be self-adjoint operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have

$$\Delta_h(A)\Delta_h(B) \geq |\langle Ah, Bh \rangle - \langle Ah, h \rangle \langle Bh, h \rangle| = \frac{\sqrt{|\langle [A, B]h, h \rangle|^2 + |\langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle|^2}}{2}.$$

Theorem 2 promotes the following question.

Question 1. What is the nonlinear (even Banach space) version of Theorem 2?

In this short note, we answer Question 1 by deriving an uncertainty principle for Lipschitz maps acting on subsets of Banach spaces. We note that there is a Banach space version of uncertainty principle by Goh and Goodman [8] which differs from the results in this paper. We also note that nonlinear Maccone-Pati uncertainty principle is derived in [9].

2. Nonlinear Heisenberg-Robertson-Schrodinger Uncertainty Principle

Let \mathcal{X} be a Banach space. Recall that the collection of all Lipschitz functions $f : \mathcal{X} \rightarrow \mathbb{C}$ satisfying $f(0) = 0$, denoted by $\mathcal{X}^\#$ is a Banach space [10] w.r.t. the Lipschitz norm

$$\|f\|_{\text{Lip}_0} := \sup_{x,y \in \mathcal{X}, x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|}.$$

Let $\mathcal{M} \subseteq \mathcal{X}$ be a subset such that $0 \in \mathcal{M}$. Let $A : \mathcal{M} \rightarrow \mathcal{X}$ be a Lipschitz map such that $A(0) = 0$. Given $x \in \mathcal{M}$ and $f \in \mathcal{X}^\#$ satisfying $f(x) = 1$, we define **two uncertainties** of A at $(x, f) \in \mathcal{M} \times \mathcal{X}^\#$ as

$$\begin{aligned} \Delta(A, x, f) &:= \|Ax - f(Ax)x\|, \\ \nabla(f, A, x) &:= \|fA - f(Ax)f\|_{\text{Lip}_0}. \end{aligned}$$

Theorem 3. (Nonlinear Heisenberg-Robertson-Schrodinger Uncertainty Principle) Let \mathcal{X} be a Banach space and $\mathcal{M}, \mathcal{N} \subseteq \mathcal{X}$ be subsets such that $0 \in \mathcal{M} \cap \mathcal{N}$. Let $A : \mathcal{M} \rightarrow \mathcal{X}$, $B : \mathcal{N} \rightarrow \mathcal{X}$ be Lipschitz maps such that $A(0) = B(0) = 0$. Then for all $x \in \mathcal{M} \cap \mathcal{N}$ and $f \in \mathcal{X}^\#$ satisfying $f(x) = 1$, we have

$$\begin{aligned} \frac{1}{2} \left(\nabla(f, A, x)^2 + \Delta(B, x, f)^2 \right) &\geq \frac{1}{4} (\nabla(f, A, x) + \Delta(B, x, f))^2 \\ &\geq \nabla(f, A, x)\Delta(B, x, f) \geq |f(ABx) - f(Ax)f(Bx)|. \end{aligned}$$

Proof.

$$\begin{aligned} \nabla(f, A, x)\Delta(B, x, f) &= \|fA - f(Ax)f\|_{\text{Lip}_0} \|Bx - f(Bx)x\| \\ &\geq |[fA - f(Ax)f][Bx - f(Bx)x]| \\ &= |f(ABx) - f(Bx)f(Ax) - f(Ax)f(Bx) + f(Ax)f(Bx)f(x)| \\ &= |f(ABx) - f(Bx)f(Ax) - f(Ax)f(Bx) + f(Ax)f(Bx) \cdot 1| \\ &= |f(ABx) - f(Ax)f(Bx)|. \end{aligned}$$

□

Corollary 1. Let \mathcal{X} be a Banach space and $\mathcal{M}, \mathcal{N} \subseteq \mathcal{X}$ be subsets such that $0 \in \mathcal{M} \cap \mathcal{N}$. Let $A : \mathcal{M} \rightarrow \mathcal{X}$, $B : \mathcal{N} \rightarrow \mathcal{X}$ be Lipschitz maps such that $A(0) = B(0) = 0$. Then for all $x \in \mathcal{M} \cap \mathcal{N}$ and $f \in \mathcal{X}^\#$ satisfying $f(x) = 1$, we have

$$\nabla(f, A, x)\Delta(B, x, f) + \|fB\|_{\text{Lip}_0}\Delta(A, x, f) \geq |f([A, B]x)|.$$

Proof. Note that

$$\begin{aligned} \nabla(f, A, x)\Delta(B, x, f) &\geq |f(ABx) - f(Ax)f(Bx)| \\ &= |f(ABx - BAx) + f(BAx) - f(Ax)f(Bx)| \\ &= |f([A, B]x) + (fB)[Ax - f(Ax)x]| \\ &\geq |f([A, B]x)| - |(fB)[Ax - f(Ax)x]| \\ &\geq |f([A, B]x)| - \|fB\|_{\text{Lip}_0} \|Ax - f(Ax)x\| \\ &= |f([A, B]x)| - \|fB\|_{\text{Lip}_0} \Delta(A, x, f). \end{aligned}$$

□

Corollary 2. Let \mathcal{X} be a Banach space and $\mathcal{M}, \mathcal{N} \subseteq \mathcal{X}$ be subsets such that $0 \in \mathcal{M} \cap \mathcal{N}$. Let $A : \mathcal{M} \rightarrow \mathcal{X}$, $B : \mathcal{N} \rightarrow \mathcal{X}$ be Lipschitz maps such that $A(0) = B(0) = 0$. Then for all $x \in \mathcal{M} \cap \mathcal{N}$ and $f \in \mathcal{X}^\#$ satisfying $f(x) = 1$, we have

$$\nabla(f, A, x)\Delta(B, x, f) + \|fB\|_{\text{Lip}_0}\Delta(A, x, -f) \geq |f(\{A, B\}x)|.$$

Proof. Note that

$$\begin{aligned} \nabla(f, A, x)\Delta(B, x, f) &\geq |f(ABx) - f(Ax)f(Bx)| \\ &= |f(ABx + BAx) - f(BAx) - f(Ax)f(Bx)| \\ &= |f(\{A, B\}x) - (fB)[Ax + f(Ax)x]| \\ &\geq |f(\{A, B\}x)| - |(fB)[Ax + f(Ax)x]| \\ &\geq |f(\{A, B\}x)| - \|fB\|_{\text{Lip}_0}\|Ax + f(Ax)x\| \\ &= |f(\{A, B\}x)| - \|fB\|_{\text{Lip}_0}\Delta(A, x, -f). \end{aligned}$$

□

Corollary 3. (Functional Heisenberg-Robertson-Schrodinger Uncertainty Principle) Let \mathcal{X} be a Banach space with dual \mathcal{X}^* , $A : \mathcal{D}(A) \rightarrow \mathcal{X}$ and $B : \mathcal{D}(B) \rightarrow \mathcal{X}$ be linear operators. Then for all $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ and $f \in \mathcal{X}^*$ satisfying $f(x) = 1$, we have

$$\begin{aligned} \frac{1}{2} \left(\nabla(f, A, x)^2 + \Delta(B, x, f)^2 \right) &\geq \frac{1}{4} (\nabla(f, A, x) + \Delta(B, x, f))^2 \\ &\geq \nabla(f, A, x)\Delta(B, x, f) \geq |f(ABx) - f(Ax)f(Bx)|. \end{aligned}$$

Corollary 4. Let \mathcal{X} be a Banach space, $A : \mathcal{D}(A) \rightarrow \mathcal{X}$ and $B : \mathcal{D}(B) \rightarrow \mathcal{X}$ be linear operators. Then for all $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ and $f \in \mathcal{X}^*$ satisfying $f(x) = 1$, we have

$$\nabla(f, A, x)\Delta(B, x, f) + \|fB\|\Delta(A, x, f) \geq |f([A, B]x)|.$$

Corollary 5. Let \mathcal{X} be a Banach space, $A : \mathcal{D}(A) \rightarrow \mathcal{X}$ and $B : \mathcal{D}(B) \rightarrow \mathcal{X}$ be linear operators. Then for all $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ and $f \in \mathcal{X}^*$ satisfying $f(x) = 1$, we have

$$\nabla(f, A, x)\Delta(B, x, f) + \|fB\|\Delta(A, -x, f) = \nabla(f, A, x)\Delta(B, x, f) + \|fB\|\Delta(A, x, -f) \geq |f(\{A, B\}x)|.$$

Corollary 6. Theorem 2 follows from Theorem 3.

Proof. Let \mathcal{H} be a complex Hilbert space. $A : \mathcal{D}(A) \rightarrow \mathcal{H}$ and $B : \mathcal{D}(B) \rightarrow \mathcal{H}$ be self-adjoint operators. Let $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$. Define $\mathcal{X} := \mathcal{H}$, $\mathcal{M} := \mathcal{D}(A)$, $\mathcal{N} := \mathcal{D}(B)$, $x := h$ and

$$f : \mathcal{H} \ni u \mapsto f(u) := \langle u, h \rangle \in \mathbb{C}.$$

Then

$$\Delta(B, x, f) = \Delta(B, h, f) = \|Bh - f(Bh)h\| = \|Bh - \langle Bh, h \rangle h\| = \Delta_h(B),$$

$$\begin{aligned}
\nabla(f, A, x) &= \nabla(f, A, h) = \|fA - f(Ah)f\| = \|fA - \langle Ah, h \rangle f\| \\
&= \sup_{u \in \mathcal{H}, \|u\| \leq 1} |f(Au) - \langle Ah, h \rangle f(u)| = \sup_{u \in \mathcal{H}, \|u\| \leq 1} |\langle Au, h \rangle - \langle Ah, h \rangle \langle u, h \rangle| \\
&= \sup_{u \in \mathcal{H}, \|u\| \leq 1} |\langle u, Ah \rangle - \langle Ah, h \rangle \langle u, h \rangle| = \sup_{u \in \mathcal{H}, \|u\| \leq 1} |\langle u, Ah - \langle Ah, h \rangle h \rangle| \\
&= \|Ah - \langle Ah, h \rangle h\| = \Delta_h(A)
\end{aligned}$$

and

$$\begin{aligned}
|f(ABx) - f(Ax)f(Bx)| &= |f(ABh) - f(Ah)f(Bh)| = |\langle ABh, h \rangle - \langle Ah, h \rangle \langle Bh, h \rangle| \\
&= |\langle Bh, Ah \rangle - \langle Ah, h \rangle \langle Bh, h \rangle| = |\langle Ah, Bh \rangle - \langle Ah, h \rangle \langle Bh, h \rangle|.
\end{aligned}$$

□

References

1. Robertson, H.P. The Uncertainty Principle. *Phys. Rev.* **1929**, *34*, 163–164. doi:10.1103/PhysRev.34.163.
2. Heisenberg, W. The physical content of quantum kinematics and mechanics. In *Quantum Theory and Measurement*; Wheeler, J.A.; Zurek, W.H., Eds.; Princeton Series in Physics, Princeton University Press, Princeton, NJ, 1983; pp. 62–84.
3. Cassidy, D.C. Heisenberg, uncertainty and the quantum revolution. *Sci. Amer.* **1992**, *266*, 106–112. doi:10.1038/scientificamerican0592-106.
4. von Neumann, J. *Mathematical foundations of quantum mechanics*; Princeton University Press, Princeton, NJ, 2018; pp. xviii+304. doi:10.2307/j.ctt1wq8zhp.
5. Debnath, L.; Mikusiński, P. *Introduction to Hilbert spaces with applications*; Academic Press, Inc., San Diego, CA, 1999; pp. xx+551.
6. Ozawa, M. Heisenberg's original derivation of the uncertainty principle and its universally valid reformulations. *Current Science* **2015**, *109*, 2006–2016.
7. Schrödinger, E. About Heisenberg uncertainty relation (original annotation by A. Angelow and M.-C. Batoni). *Bulgar. J. Phys.* **1999**, *26*, 193–203 (2000). Translation of Proc. Prussian Acad. Sci. Phys. Math. Sect. **19** (1930), 296–303.
8. Goh, S.S.; Goodman, T.N.T. Uncertainty principles in Banach spaces and signal recovery. *J. Approx. Theory* **2006**, *143*, 26–35. doi:10.1016/j.jat.2006.03.009.
9. Krishna, K.M. Nonlinear Maccone-Pati Uncertainty Principle. DOI: 10.20944/preprints202401.2224.v1, 31 January 2024.
10. Weaver, N. *Lipschitz algebras*; World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018; pp. xiv+458.

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