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Article

Existence and Limit Behavior of Constraint Minimizers for a Varying Nonlocal Kirchhoff Type Energy Functional

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Abstract: In this paper, we study the constrained minimization problem for an energy functional which is related to the following Kirchhoff type equation $-\left(\eta + b\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^s\right)\Delta u + V(x)u = \mu u + \lambda|u|^p u$, where b is a positive constant, parameters $\eta \geq 0, \lambda > 0$, exponents $s > 0, 0 < p < 4$ and μ is a Lagrange multiplier. When the trapping potential $V(x)$ fulfills a polynomial function, we give a detailed limit behavior analysis of constrained minimizers for any positive sequence $\{\eta_k\}$ with $\eta_k \rightarrow 0^+$.

Keywords: Kirchhoff type energy functional; constraint minimizer; limit behavior; varying nonlocal term

2020 MSC: 32J20; 35J60; 35Q40; 46N50

1. Introduction and Main Results

We consider the following Kirchhoff type equation with a varying nonlocal term

$$-\left(\eta + b\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^s\right)\Delta u + V(x)u = \mu u + \lambda|u|^p u, \quad (1.1)$$

where $b > 0$ is a constant, parameters $\eta \geq 0, \lambda > 0$, exponents $s > 0, 0 < p < 4$ and μ is a Lagrange multiplier. The $b\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^s$ in (1.1) arises as a varying nonlocal term.

In recent years, there many articles involved in different type of varying nonlocal problems similar to (1.1) such as the model

$$\begin{cases} -C\left(\int_{\Omega} |\nabla u|^2 dx\right)^s \Delta u = h(x, u)\left(\int_{\Omega} f(x, u) dx\right)^r, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

which mainly studied the existence of solutions by using variational theory and analytical methods, (cf.[3,4,20,22]). Especially for $s = 1$ in (1.1), the Kirchhoff type constrained minimization problems are related to

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + V(x)u = \mu u + \lambda|u|^p u,$$

which have attracted a lot of mathematicians to study their existence, non-existence, uniqueness and limit behavior of constraint minimizers, etc, (cf.[7,14,15,17,18,21,25–27,29]). Coincidentally for $s = 0$ and \mathbb{R}^3 replaced by \mathbb{R}^2 , the (1.1) comes from an interesting physical context, which is associated with the well known Bose-Einstein condensates(BEC). The mathematical theory study of BEC can be described by a Gross-Pitaevskii(GP) functional, which has been associated with the elliptic equation

$$\Delta u + V(x)u = \mu u + \lambda|u|^p u,$$

see [1,5,9–12,19,23] and the related literatures. In these papers, the researchers are keen on exploring the existence, mass concentration phenomenon, uniqueness and numerical analysis of the ground state solutions for GP functional.

Inspired by the above articles, the aim of the present paper is to study the Kirchoff type equation (1.1) with a varying nonlocal term. The constrained minimization problem associated with (1.1) is defined by

$$I(\eta, s, \lambda) := \inf_{u \in \mathcal{U}} E(u), \quad (1.2)$$

where $E(u)$ fulfills

$$E(u) := \frac{\eta}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{b}{2(s+1)} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{s+1} + \frac{1}{2} \int_{\mathbb{R}^3} V(x)|u|^2 dx - \frac{\lambda}{p+2} \int_{\mathbb{R}^3} |u|^{p+2} dx. \quad (1.3)$$

The above \mathcal{U} in (1.2) is restricted to meet

$$\mathcal{U} := \left\{ u \in \mathcal{H}, \int_{\mathbb{R}^3} |u|^2 = 1 \right\}, \quad (1.4)$$

where \mathcal{H} satisfies

$$\mathcal{H} := \left\{ u \in H^1(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} V(x)|u|^2 dx < \infty \right\}$$

as well as with the norm $\|u\|_{\mathcal{H}} := \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} (1 + V(x)|u|^2) dx \right)^{\frac{1}{2}}$. To state our main results, we assume that the $V(x)$ in (1.1) satisfies

$$(V_1). V(x) \in L_{loc}^{\infty}(\mathbb{R}^3) \cap C_{loc}^{\alpha}(\mathbb{R}^3), \alpha \in (0, 1), \lim_{|x| \rightarrow \infty} V(x) = +\infty \text{ and } \min_{\mathbb{R}^3} V(x) = 0.$$

Next, we introduce an elliptic equation such as

$$-\frac{3p}{4} \Delta Q_p + \left(1 - \frac{p}{4}\right) Q_p - |Q_p|^p Q_p = 0, \quad x \in \mathbb{R}^3, \quad 0 < p < 4. \quad (1.5)$$

In fact, up to the translations, the (1.5) has a unique positive radially symmetric solution $Q_p \in H^1(\mathbb{R}^3)$ (cf.[16]). Using (1.5), we can deduce that

$$\|\nabla Q_p\|_{L^2}^2 = \|Q_p\|_{L^2}^2 = \frac{2}{p+2} \|Q_p\|_{L^{p+2}}^{p+2}, \quad 0 < p < 4. \quad (1.6)$$

Recall also from [6, Proposition 4.1] that $Q_p(x)$ has the exponential decay property

$$|\nabla Q_p(x)|, Q_p(|x|) = O(|x|^{-1} e^{-|x|}) \text{ as } |x| \rightarrow \infty. \quad (1.7)$$

At last, we give a Gagliardo-Nirenberg(G-N) type inequality (cf.[24]) such as

$$\|u\|_{L^{2+p}}^{2+p} \leq \frac{p+2}{2\|Q_p\|_{L^2}^p} \|\nabla u\|_{L^2}^{\frac{3p}{2}} \|u\|_{L^2}^{2-\frac{p}{2}}, \quad 0 < p < 4, \quad (1.8)$$

where Q_p is the unique positive solution of (1.5).

According to above results, the existence and nonexistence on constraint minimizers for $I(\eta, s, \lambda)$ are established as follows. Before this, we denote a critical constant $\lambda^* := \frac{b}{(s+1)} \|Q\|_{L^2}^{\frac{4(s+1)}{3}}$, where Q is the unique positive solution of (1.5) for $p = \frac{4(s+1)}{3}$.

Theorem 1.1. For $\eta, s > 0$, $0 < p < 4$ and (V_1) holds, then $I(\eta, s, \lambda)$ exists at least one minimizer for $p < \frac{4(s+1)}{3}$ or $p = \frac{4(s+1)}{3}$, $0 < \lambda \leq \lambda^*$. The $I(\eta, s, \lambda)$ has no minimizer for $p > \frac{4(s+1)}{3}$ or $p = \frac{4(s+1)}{3}$, $\lambda > \lambda^*$.

Theorem 1.2. For $\eta = 0$, $s > 0$, $p = \frac{4(s+1)}{3}$ and (V_1) holds, then $I(\eta, s, \lambda)$ exists at least one minimizer if $0 < \lambda < \lambda^*$. Moreover, $I(\eta, s, \lambda)$ has no minimizer for $\lambda \geq \lambda^*$

Remark that the similar conclusions appear elsewhere for studying different type of Kirchhoff equations, see [8,15,27,29]. For convenience, we give a detailed proof of Th1.1 and Th1.2 in Sec.2. In view of the above Theorems, one knows that for $\eta > 0$, $p = \frac{4(s+1)}{3}$ and $\lambda = \lambda^*$, the $I(\eta, s, \lambda)$ exists at least minimizer. However, for $\eta = 0$, $p = \frac{4(s+1)}{3}$ and $\lambda = \lambda^*$, the $I(\eta, s, \lambda^*)$ admits no minimizer. A nature question is what happen to constraint minimizers of $I(\eta, s, \lambda)$ when η tends to 0 from right?

Suppose that u_η is a minimizer for $I(\eta, s, \lambda)$, then one can restrict $u_\eta \geq 0$ due to $E(u) \geq E(|u|)$ for any $u \in \mathcal{U}$. At the same time, we always assume that $I(\eta, s, \lambda)$ admits a positive minimizer by applying the strong maximum principle to (1.1). In truth, for any positive sequence $\{\eta_k\}$ with $\eta_k \rightarrow 0^+$ as $k \rightarrow \infty$, one can verify that the positive constraint minimizers u_{η_k} satisfy $\int_{\mathbb{R}^3} |\nabla u_{\eta_k}|^2 dx \rightarrow +\infty$ as $k \rightarrow \infty$ (see Sec.3), that is, the minimizers arise blow up behavior as $\eta_k \rightarrow 0^+$. In order to get more detailed limit behavior of constraint minimizers, some appropriate assumptions on $V(x)$ are necessary. For this purpose, we assume that $V(x)$ is a form of polynomial function, and admits $n \geq 1$ isolated minima. More narrowly, there exist $n \geq 1$ distinct points $x_i \in \mathbb{R}^3$, numbers $q_i > 0$ and constant $M > 0$ fulfilling

$$(V_2). \quad V(x) = C(x) \prod_{i=1}^n |x - x_i|^{q_i} \text{ with } M < C(x) < \frac{1}{M} \text{ for all } x \in \mathbb{R}^3, \quad (1.9)$$

here $\lim_{x \rightarrow x_i} C(x)$ exists for all $1 \leq i \leq n$. For convenience, we denote

$$q = \max\{q_1, \dots, q_n\} > 0, \quad (1.10)$$

$$\theta_i = \frac{1}{\|Q\|_{L^2}^2} \lim_{x \rightarrow x_i} \frac{V(x)}{|x - x_i|^q} \int_{\mathbb{R}^3} |x|^q |Q(x)|^2 dx > 0, \quad (1.11)$$

where $Q(x)$ satisfies (1.5) for $p = \frac{4(s+1)}{3}$. Moreover, let

$$\theta = \min\{\theta_1, \dots, \theta_n\} > 0 \quad (1.12)$$

and the set of flattest global minima for $V(x)$ is denoted by

$$\mathcal{W} = \{x_i : \theta_i = \theta\}. \quad (1.13)$$

In light of Th1.1, Th1.2 and inspired by [8,15,19,29], for any positive sequence $\{\eta_k\}$ and set u_{η_k} being the positive minimizers of $I(\eta_k, s, \lambda^*)$, we next establish the following theorem on limit behavior of constraint minimizers for $I(\eta_k, s, \lambda^*)$ when $p = \frac{4(s+1)}{3}$ and $\lambda = \lambda^*$ as $\eta_k \rightarrow 0^+$.

Theorem 1.3. Assume that (V_1) and (V_2) hold. For $p = \frac{4(s+1)}{3}$, $\lambda = \lambda^*$ and any positive sequence $\{\eta_k\}$ with $\eta_k \rightarrow 0^+$ as $k \rightarrow \infty$, define $\epsilon_{\eta_k} := \left(\int_{\mathbb{R}^3} |\nabla u_{\eta_k}|^2 dx \right)^{-\frac{1}{2}}$, then the following conclusions hold.

- The u_{η_k} has a unique local maximum z_{η_k} satisfying $\lim_{k \rightarrow \infty} z_{\eta_k} = x_i$ and $x_i \in \mathcal{W}$ is a flattest global minimum of $V(x)$. Moreover, we have as $k \rightarrow \infty$

$$\epsilon_{\eta_k}^{\frac{3}{2}} u_{\eta_k}(\epsilon_{\eta_k} x + z_{\eta_k}) \rightarrow \frac{Q(|x|)}{\|Q\|_2} \text{ strongly in } H^1(\mathbb{R}^3), \quad (1.14)$$

where Q denotes the unique positive solution of (1.5) for $p = \frac{4(s+1)}{3}$.

- The ϵ_{η_k} fulfills as $k \rightarrow \infty$

$$\epsilon_{\eta_k} \approx (q\theta)^{-\frac{1}{q+2}} (\eta_k)^{\frac{1}{q+2}}. \quad (1.15)$$

- The least energy $I(\eta_k, s, \lambda^*)$ satisfies as $k \rightarrow \infty$

$$I(\eta_k, s, \lambda^*) \approx \left[\frac{1}{2} q^{\frac{2}{q+2}} + q^{\frac{-q}{q+2}} \right] \theta^{\frac{2}{q+2}} (\eta_k)^{\frac{q}{q+2}}, \quad (1.16)$$

where q, θ are stated by (1.10) and (1.12).

Notice that the $f(\eta_k) \approx g(\eta_k)$ in Th1.3 means $f/g \rightarrow 1$ as $k \rightarrow \infty$. Actually for $s = 0$ and $V(x)$ behaves the form of sinusoidal, ring-shaped, periodic and multi-well, the papers (cf.[11,12,23,28]) widely studied the mass concentration behavior of constrained minimizers. Particularly for $s = 1$, the authors in [8,15] also analyzed the limit behavior of minimizers when $\eta > 0$ as $b \rightarrow 0^+$ or $b > 0$ as $\eta \rightarrow 0^+$. As described in Th1.3, our paper gets an interesting result on this topic when there involves a varying nonlocal term, and it thus enriches the study of such issues.

The present paper is structured as follows. Sec.2 shall establish the existence and nonexistence proof of constrained minimizers for $I(\eta, s, \lambda)$ when the parameters η, λ and exponents s, p satisfy suitable range. For $p = \frac{4(s+1)}{3}$, $\lambda = \lambda^*$ and any positive sequence $\{\eta_k\}$ with $\eta_k \rightarrow 0^+$ as $k \rightarrow \infty$, in Sec.3 we plan to give the accurate energy estimation of $I(\eta_k, s, \lambda^*)$, and then analyze the detailed limit behavior of positive constrained minimizers as $\eta_k \rightarrow 0^+$.

2. Proof of Theorem 1.1 and Theorem 1.2

In this section, we shall give the proof of existence and non-existence on constraint minimizers for (1.2). Before this, one introduces the space compact embedding Theorem 2.1 in [2] such that

$$\mathcal{H} \hookrightarrow L^v(\mathbb{R}^3) (2 < v < 6). \quad (2.1)$$

For convenience, we classify the proof of Th1.1 and Th1.2 as follows two cases.

Case 1 *The existence proof of constraint minimizer.*

Proof. Under the assumption of Th1.1, for any $u \in \mathcal{U}$, we deduce from G-N inequality (1.8) that for $\eta > 0$, $p < \frac{4(s+1)}{3}$

$$\begin{aligned} E(u) &\geq \frac{\eta}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{b}{2(s+1)} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{s+1} \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^3} V(x)|u|^2 dx - \frac{\lambda}{2\|Q_p\|_{L^2}^p} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{\frac{3p}{4}}. \end{aligned} \quad (2.2)$$

For $p = \frac{4(s+1)}{3}$, $0 < \lambda \leq \lambda^*$, similar to (2.2), one also derives that

$$E(u) \geq \frac{\eta}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\lambda^* - \lambda}{2\|Q\|_{L^2}^{\frac{4(s+1)}{3}}} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{s+1} + \frac{1}{2} \int_{\mathbb{R}^3} V(x)|u|^2 dx. \quad (2.3)$$

Both $p < \frac{4(s+1)}{3}$ and $p = \frac{4(s+1)}{3}$, $0 < \lambda \leq \lambda^*$ hold, the (2.2) and (2.3) yield a fact that for any sequence $\{u_n\} \subseteq \mathcal{U}$, the $E(u_n)$ is bounded uniformly from below. Hence, there admits a minimization sequence $\{u_n\} \subseteq \mathcal{U}$ fulfilling

$$I(\eta, s, \lambda) = \lim_{n \rightarrow \infty} E(u_n). \quad (2.4)$$

In truth, one can get from (2.2) and (2.3) that $\{u_n\}$ bounded in \mathcal{H} . Applying (2.1), there exists a $\bar{u} \in \mathcal{H}$, and $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ such that as $k \rightarrow \infty$

$$u_{n_k} \rightharpoonup \bar{u} \text{ weakly in } \mathcal{H}, \quad u_{n_k} \rightarrow \bar{u} \text{ strongly in } L^\nu(\mathbb{R}^3), \quad 2 < \nu < 6. \quad (2.5)$$

Using the weak lower semi-continuity, we get

$$\liminf_{k \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla u_{n_k}|^2 dx \geq \int_{\mathbb{R}^3} |\nabla \bar{u}|^2 dx.$$

The above results give that

$$I(\eta, s, \lambda) = \liminf_{k \rightarrow \infty} E(u_{n_k}) \geq E(\bar{u}) \geq I(\eta, s, \lambda) \quad (2.6)$$

which then yields $E(\bar{u}) = I(\eta, s, \lambda)$. Hence, \bar{u} is a minimizer for $I(\eta, s, \lambda)$.

Under the assumption of Th1.2, for any $u \in \mathcal{U}$, one also derives from (1.8) that for $\eta = 0$ and $p = \frac{4(s+1)}{3}$ that

$$E(u) \geq \frac{\lambda^* - \lambda}{2\|Q\|_{L^2}^{\frac{4(s+1)}{3}}} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{s+1} + \frac{1}{2} \int_{\mathbb{R}^3} V(x)|u|^2 dx. \quad (2.7)$$

If $0 < \lambda < \lambda^*$, repeating the above procedures, one claims that $I(0, s, \lambda)$ has a minimizer. \square

Case 2 *The nonexistence proof of constraint minimizer.*

Proof. The process comes true by establishing energy estimation for $I(\eta, s, \lambda)$. To get this goal, choosing a test function such as

$$u_t(x) := \frac{\mathcal{P}_t}{\|Q\|_{L^2}} t^{\frac{3}{2}} \Phi(x - x_i) Q(t|x - x_i|) (t > 0), \quad (2.8)$$

where Q fulfills (1.5) for $p = \frac{4(s+1)}{3}$, and $x_i \in \mathcal{W}$ satisfies $V(x_i) = 0$. The function $\Phi(x) \in C_0^\infty(\mathbb{R}^3)$ in (2.8) is chosen as

$$\begin{cases} \Phi(x) = 1, & |x| \leq 1, \\ 0 \leq \Phi(x) \leq 1, & 1 < |x| < 2, \\ |\Phi(x)| = 0, & |x| \geq 2, \\ |\nabla \Phi(x)| \leq C, & x \in \mathbb{R}^3. \end{cases} \quad (2.9)$$

Notice that \mathcal{P}_t in (2.8) makes sure $\|u_t\|_{L^2}^2 = 1$. It then deduces from (1.7) and (2.8) that

$$1 \leq P_t \leq 1 + O(t^{-\infty}) \text{ and } \lim_{t \rightarrow +\infty} P_t = 1, \quad (2.10)$$

where $g(t) = O(t^{-\infty})$ means $\lim_{t \rightarrow +\infty} |g(t)|t^d = 0$ for any $d > 0$. One can attain from (1.6) that as $t \rightarrow \infty$

$$\begin{aligned} I(\eta, s, \lambda) &\leq \frac{\eta P_t^2 t^2}{2\|Q\|_{L^2}^2} \int_{\mathbb{R}^3} |\nabla Q|^2 dx + \frac{b P_t^{2(s+1)} t^{2(s+1)}}{2(s+1)\|Q\|_{L^2}^{2(s+1)}} \left(\int_{\mathbb{R}^3} |\nabla Q|^2 dx \right)^{s+1} \\ &\quad - \frac{\lambda P_t^{p+2} t^{\frac{3p}{2}}}{(p+2)\|Q\|_{L^2}^{p+2}} \int_{\mathbb{R}^3} |Q|^{p+2} dx + V(x_0) + o(1) + O(t^{-\infty}) \end{aligned} \quad (2.11)$$

which yields that for any $p > \frac{4(s+1)}{3}$, the $I(\eta, s, \lambda) \rightarrow -\infty$ as $t \rightarrow \infty$. For $\eta > 0$ and $p = \frac{4(s+1)}{3}$, we derive from (2.11) that

$$\begin{aligned} I(\eta, s, \lambda) &\leq \frac{\eta t^2}{2} + \frac{bt^{2(s+1)}}{2(s+1)} - \frac{\lambda t^{2(s+1)}}{2\|Q\|_{L^2}^{\frac{4(s+1)}{3}}} + o(1) + O(t^{-\infty}) \\ &= \frac{\eta t^2}{2} + \frac{(\lambda^* - \lambda)t^{2(s+1)}}{2\|Q\|_{L^2}^{\frac{4(s+1)}{3}}} + o(1) + O(t^{-\infty}) \rightarrow -\infty. \end{aligned} \quad (2.12)$$

which also deduces that for $\lambda > \lambda^*$, the $I(\eta, s, \lambda) \rightarrow -\infty$ as $t \rightarrow \infty$. Hence, for any $\eta > 0$, if either $p > \frac{4(s+1)}{3}$ or $p = \frac{4(s+1)}{3}$, $\lambda > \lambda^*$ holds, the $I(\eta, s, \lambda)$ has no minimizer.

For $\eta = 0$ and $p = \frac{4(s+1)}{3}$, we obtain from (2.12) that

$$I(0, s, \lambda) \leq E(u_t) = \frac{(\lambda^* - \lambda)t^{2(s+1)}}{2\|Q\|_{L^2}^{\frac{4(s+1)}{3}}} + o(1) + O(t^{-\infty}) \rightarrow -\infty. \quad (2.13)$$

One then deduces that $I(\eta, s, \lambda)$ has no minimizer due to $I(0, s, \lambda) = -\infty$ for $\lambda > \lambda^*$.

For $\eta = 0$ and $\lambda = \lambda^*$, one can get from (2.2) and (2.12) that $I(0, s, \lambda^*) = 0$. We next argue that $I(0, s, \lambda^*)$ admits no minimizer by establishing a contradiction. If this is not true, suppose that $\hat{u} \in U$ is a minimizer of $I(0, s, \lambda^*)$. As stated in Sec.1, we may assume that \hat{u} is positive. Since $V(x) \geq 0$ and $\int_{\mathbb{R}^3} |\hat{u}|^2 dx = 1$, the G-N inequality (1.8) then yields that

$$\frac{1}{(s+1)} \left(\int_{\mathbb{R}^3} |\nabla \hat{u}|^2 dx \right)^{s+1} = \frac{3\lambda^*}{2s+5} \int_{\mathbb{R}^3} |\hat{u}|^{\frac{4(s+1)}{3}+2} dx, \quad (2.14)$$

where the equality holds only for $\hat{u} = Q$, and Q is the unique positive solution of (1.5) for $p = \frac{4(s+1)}{3}$. One further obtains from (2.2) that \hat{u} satisfies

$$\int_{\mathbb{R}^3} V(x) \hat{u}^2 dx = \min_{\mathbb{R}^3} V(x) = 0. \quad (2.15)$$

However, the equalities (2.14) and (2.15) cannot be held at the same time because the first one presents a fact that \hat{u} has no compact support, and the second one needs $\hat{u} = Q$ to possess a compact support. Thus, one claims that $I(0, s, \lambda^*)$ has no minimizer. so far, the nonexistence proof of constraint minimizers is completed. \square

3. Limit Behavior Analysis of Constraint Minimizers

In this section, for $p = \frac{4(s+1)}{3}$, $\lambda = \lambda^*$ and any positive sequence $\{\eta_k\}$ with $\eta_k \rightarrow 0^+$ as $k \rightarrow \infty$, we plan to analyze the limit behavior on minimizers u_{η_k} for $I(\eta_k, s, \lambda^*)$ as $\eta_k \rightarrow 0^+$. The proof process is achieved by constructing some indispensable lemmas, which are stated as follows.

Lemma 3.1. Under the assumption of Th1.3, set $\hat{v}_{\eta_k}(x) := \epsilon_{\eta_k}^{\frac{3}{2}} u_{\eta_k}(\epsilon_{\eta_k} x)$ and $\epsilon_{\eta_k} = \left(\int_{\mathbb{R}^3} |\nabla u_{\eta_k}|^2 dx \right)^{-\frac{1}{2}} > 0$, then as $k \rightarrow \infty$, the $\epsilon_{\eta_k} \rightarrow 0$ and \hat{v}_{η_k} satisfies

$$\int_{\mathbb{R}^3} |\nabla \hat{v}_{\eta_k}|^2 dx = 1 \text{ and } \int_{\mathbb{R}^3} |\hat{v}_{\eta_k}|^{\frac{4(s+1)}{3}+2} \rightarrow \frac{b[2s+5]}{3(s+1)\lambda^*}. \quad (3.1)$$

Proof. If u_{η_k} are positive minimizers of (1.2), then u_{η_k} satisfy

$$-\left(\eta_k + b \left(\int_{\mathbb{R}^3} |\nabla u_{\eta_k}|^2 dx \right)^s \right) \Delta u_{\eta_k} + V(x) u_{\eta_k} = \mu_{\eta_k} u_{\eta_k} + \lambda^* |u_{\eta_k}|^{\frac{4(s+1)}{3}} u_{\eta_k} \quad (3.2)$$

here $\mu_{\eta_k} \in \mathbb{R}$ denote Lagrange multipliers. Set

$$\hat{v}_{\eta_k}(x) := \epsilon_{\eta_k}^{\frac{3}{2}} u_{\eta_k}(\epsilon_{\eta_k} x), \quad (3.3)$$

where $\epsilon_{\eta_k} = \left(\int_{\mathbb{R}^3} |\nabla u_{\eta_k}|^2 dx \right)^{-\frac{1}{2}} > 0$. On the contrary, we assume that $\epsilon_{\eta_k} \not\rightarrow 0$ as $\eta_k \rightarrow 0^+$, then $\{u_{\eta_k}\}$ is bounded uniformly in \mathcal{H} . Similar to the proof of Th1.1 and Th1.2 in Sec.2, one asserts that there exists a $u_0 \in \mathcal{U}$ and $\{u_{\eta_k}\}$ has a subsequence (still denoted by $\{u_{\eta_k}\}$) such that as $\eta_k \rightarrow 0^+$

$$u_{\eta_k} \rightharpoonup u_0 \text{ weakly in } \mathcal{H}, \quad u_{\eta_k} \rightarrow u_0 \text{ strongly in } L^v(\mathbb{R}^3), \quad 2 < v < 6. \quad (3.4)$$

To get our result, one needs to prove that $I(\eta_k, s, \lambda^*) \rightarrow 0$ as $\eta_k \rightarrow 0^+$. For this purpose, we choose a test function the same as (2.8). Based on (1.7) and (2.8)-(2.10), one calculates that

$$\int_{\mathbb{R}^3} |\nabla u_t|^2 dx = \frac{P_t^2 t^2}{\|Q\|_{L^2}^2} \int_{\mathbb{R}^3} |\nabla Q|^2 dx + O(t^{-\infty}) \quad (3.5)$$

and

$$\int_{\mathbb{R}^3} |u_t|^{\frac{4(s+1)}{3}+2} dx = \frac{P_t^{\frac{4(s+1)}{3}+2} t^{2(s+1)}}{\|Q\|_{L^2}^{\frac{4(s+1)}{3}+2}} \int_{\mathbb{R}^3} |Q|^{\frac{4(s+1)}{3}+2} dx + O(t^{-\infty}). \quad (3.6)$$

Since $V(x)$ satisfies (V_1) and (V_2) , one obtains that as $t \rightarrow \infty$

$$\int_{\mathbb{R}^3} V(x) |u_t|^2 dx = V(x_i) + o(1) = o(1). \quad (3.7)$$

It then follows from (1.6) and (3.5)-(3.7) that for $p = \frac{4(s+1)}{3}$ and $\lambda = \lambda^*$ as $t \rightarrow +\infty$

$$\begin{aligned} I(\eta_k, s, \lambda^*) &\leq \frac{\eta_k P_t^2 t^2}{2 \|Q\|_{L^2}^2} \int_{\mathbb{R}^3} |\nabla Q|^2 dx + \frac{b P_t^{2(s+1)} t^{2(s+1)}}{2(s+1) \|Q\|_{L^2}^{2(s+1)}} \left(\int_{\mathbb{R}^3} |\nabla Q|^2 dx \right)^{s+1} \\ &\quad - \frac{\lambda^* P_t^{p+2} t^4}{(p+2) \|Q\|_{L^2}^{p+2}} \int_{\mathbb{R}^3} |Q|^{p+2} dx + V(x_i) + o(1) + O(t^{-\infty}) \\ &= \frac{\eta_k t^2}{2} + \frac{b t^{2(s+1)}}{2(s+1)} - \frac{\lambda^* t^{2(s+1)}}{2 \|Q\|_{L^2}^{\frac{4(s+1)}{3}}} + o(1) + O(t^{-\infty}) \\ &= \frac{\eta_k t^2}{2} + o(1) + O(t^{-\infty}). \end{aligned} \quad (3.8)$$

Taking $t = (\eta_k)^{-\frac{1}{3}}$ into (3.8), it yields that as $k, t \rightarrow \infty$

$$I(\eta_k, s, \lambda^*) \leq E(u_t) = \frac{\eta_k^{\frac{1}{3}}}{2} + o(1) + O(t^{-\infty}) \rightarrow 0. \quad (3.9)$$

The (3.9) together with (1.3) and (3.4) deduces that

$$0 = I(0, s, \lambda^*) \leq E(u_0) \leq \liminf_{k \rightarrow \infty} E(u_{\eta_k}) = \lim_{k \rightarrow \infty} I(\eta_k, s, \lambda^*) = I(0, s, \lambda^*) = 0$$

which yields a fact that u_0 is a minimizer of $I(0, s, \lambda^*)$. It is a contradiction since Th1.2 shows that $I(0, s, \lambda^*)$ has no minimizer. Thus, $\epsilon_{\eta_k} \rightarrow 0$ holds as $k \rightarrow \infty$.

By (3.3), we just have $\int_{\mathbb{R}^3} |\nabla \hat{v}_{\eta_k}|^2 dx = \epsilon_{\eta_k}^{-2} \int_{\mathbb{R}^3} |\nabla u_{\eta_k}|^2 dx = 1$. Since u_{η_k} are minimizers of $I(\eta_k, s, \lambda^*)$ for any $\eta_k > 0$, one derives from (1.8) and (3.9) that as $k \rightarrow \infty$

$$0 \leq E(u_{\eta_k}) = I(\eta_k, s, \lambda^*) \leq E(u_t) \rightarrow 0 \quad (3.10)$$

which yields that as $k \rightarrow \infty$

$$\frac{b}{2(s+1)} \left(\int_{\mathbb{R}^3} |\nabla u_{\eta_k}|^2 dx \right)^{2(s+1)} - \frac{3\lambda^*}{4s+10} \int_{\mathbb{R}^3} |u_{\eta_k}|^{\frac{4(s+1)}{3}+2} dx \rightarrow 0. \quad (3.11)$$

It hence follows from (3.3) and (3.11) that as $k \rightarrow \infty$

$$\frac{b}{2(s+1)} - \frac{3\lambda^*}{4s+10} \int_{\mathbb{R}^3} |\hat{v}_{\eta_k}|^{\frac{4(s+1)}{3}+2} dx \rightarrow 0,$$

which shows as $k \rightarrow \infty$

$$\int_{\mathbb{R}^3} |\hat{v}_{\eta_k}|^{\frac{4(s+1)}{3}+2} dx \rightarrow \frac{b[2s+5]}{3(s+1)\lambda^*}.$$

□

Assume that u_{η_k} are positive minimizers of $I(\eta_k, s, \lambda^*)$ for any $\eta_k > 0$. Since $\int_{\mathbb{R}^3} |u_{\eta_k}|^2 dx = 1$, one has $u_{\eta_k}(x) \rightarrow 0$ as $|x| \rightarrow \infty$. It thus yields that $u_{\eta_k}(x)$ exists at least one local maximum, denoted by z_{η_k} . Define a function

$$v_{\eta_k}(x) := \epsilon_{\eta_k}^{\frac{3}{2}} u_{\eta_k}(\epsilon_{\eta_k} x + z_{\eta_k}), \quad (3.12)$$

where ϵ_{η_k} is given in Lem3.1. We next establish the following lemma, which is related to convergence properties of v_{η_k} and z_{η_k} .

Lemma 3.2. *Under the assumption of Th1.3, set z_{η_k} being a local maximum of u_{η_k} and v_{η_k} defined by (3.12), then we have*

(i) *There exist a finite ball $B_{2s}(0)$ and a constant $\mathcal{D} > 0$ such that*

$$\liminf_{k \rightarrow \infty} \int_{B_{2s}(0)} |v_{\eta_k}(x)|^2 dx \geq \mathcal{D} > 0. \quad (3.13)$$

(ii) *The z_{η_k} is a unique maximum of u_{η_k} and satisfies $z_{\eta_k} \rightarrow x_0$ for some $x_0 \in \mathbb{R}^3$ as $k \rightarrow \infty$. Further, the x_0 is a minimum of $V(x)$, that is, $V(x_0) = 0$.*

(iii) *The function v_{η_k} satisfies*

$$\lim_{k \rightarrow \infty} v_{\eta_k}(x) = \lim_{k \rightarrow \infty} \epsilon_{\eta_k}^{\frac{3}{2}} u_{\eta_k}(\epsilon_{\eta_k} x + z_{\eta_k}) = \frac{Q(|x|)}{\|Q\|_{L^2}} \text{ strongly in } H^1(\mathbb{R}^3), \quad (3.14)$$

where Q is the unique solution of (1.5) for $p = \frac{4(s+1)}{3}$.

Proof. (i) By (3.2), we see that v_{η_k} fulfills the elliptic equation

$$-\left(\eta_k \epsilon_{\eta_k}^{2s} + b\right) \Delta v_{\eta_k} + \epsilon_{\eta_k}^{2(s+1)} V(x) v_{\eta_k} = \epsilon_{\eta_k}^{2(s+1)} \mu_{\eta_k} v_{\eta_k} + \lambda^* |v_{\eta_k}|^{\frac{4(s+1)}{3}} v_{\eta_k}, \quad (3.15)$$

here μ_{η_k} are Lagrange multipliers. In truth, (1.2) and (3.2) give that

$$\mu_{\eta_k} = 2I(\eta_k, s, \lambda^*) + \frac{sb}{s+1} \left(\int_{\mathbb{R}^3} |\nabla u_{\eta_k}|^2 dx \right)^{s+1} - \frac{2(s+1)\lambda^*}{2s+5} \int_{\mathbb{R}^3} |u_{\eta_k}|^{\frac{4(s+1)}{3}+2} dx. \quad (3.16)$$

Repeating the proof of (3.10), one obtains that as $k \rightarrow \infty$

$$\epsilon_{\eta_k}^{2(s+1)} I(\eta_k, s, \lambda^*) \rightarrow 0 \text{ and } \int_{\mathbb{R}^3} V(\epsilon_{\eta_k} x + z_{\eta_k}) v_{\eta_k}^2(x) dx \rightarrow 0. \quad (3.17)$$

Combing (3.16), (3.17) and Lem3.1, one deduces from $0 < p = \frac{4(s+1)}{3} < 4$ that $0 < s < 2$, and then we have as $k \rightarrow \infty$

$$\mu_{\eta_k} \epsilon_{\eta_k}^{2(s+1)} = 2\epsilon_{\eta_k}^{2(s+1)} I(\eta_k, s, \lambda^*) + \frac{sb}{s+1} - \frac{2b}{3} \rightarrow \frac{(s-2)b}{3(s+1)} < 0. \quad (3.18)$$

Since u_{η_k} take local maxima at $x = z_{\eta_k}$, and v_{η_k} then get local maxima at $x = 0$. It thus yields from (3.15) and (3.18) that there exists a constant $K > 0$ satisfying as $k \rightarrow \infty$

$$v_{\eta_k}(0) \geq K > 0. \quad (3.19)$$

Furthermore, one obtains from (3.15) that

$$-\Delta v_{\eta_k} - c(x)v_{\eta_k} \leq 0, \quad x \in \mathbb{R}^3, \quad (3.20)$$

where $c(x) = \lambda^* |v_{\eta_k}|^{\frac{4(s+1)}{3}}$. In view of the De Giorgi-Nash-Moser theory (cf. Theorem 4.1 in [13]), one deduces that exist a finite ball $B_{2s}(0) \subset \mathbb{R}^3$ and constant $C > 0$ such that

$$\max_{B_s(0)} v_{\eta_k} \leq C \left(\int_{B_{2s}(0)} |v_{\eta_k}|^2 dx \right)^{\frac{1}{2}}. \quad (3.21)$$

It hence yields from (3.19) and (3.21) that there exists a constant $\mathcal{D} > 0$ satisfying

$$\liminf_{k \rightarrow \infty} \int_{B_{2s}(0)} |v_{\eta_k}|^2 dx \geq \mathcal{D} > 0, \quad (3.22)$$

which shows (3.13) holding.

(ii) On the contrary, one may assume that $|z_{\eta_k}| \rightarrow \infty$ as $k \rightarrow \infty$. By applying (3.22) and Fatou's lemma, for any large constant \mathcal{A} , one has

$$\begin{aligned} & \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^3} V(\epsilon_{\eta_k} x + z_{\eta_k}) |v_{\eta_k}(x)|^2 dx \\ & \geq \int_{B_{2s}(0)} \liminf_{k \rightarrow \infty} V(\epsilon_{\eta_k} x + z_{\eta_k}) |v_{\eta_k}(x)|^2 dx \geq \mathcal{A} > 0, \end{aligned} \quad (3.23)$$

which contradicts (3.17), and it hence shows that $|z_{\eta_k}|$ is bounded in \mathbb{R}^3 . Taking a subsequence of $\{z_{\eta_k}\}$ if necessary (still denoted by $\{z_{\eta_k}\}$), there admits a $x_0 \in \mathbb{R}^3$ such that $z_{\eta_k} \rightarrow x_0$ as $k \rightarrow \infty$. In fact, one can claim that x_0 is a minimum of $V(x)$, that is $V(x_0) = 0$. If not, repeating the proof of (3.23), it also yields a contradiction. Thus, we say that $z_{\eta_k} \rightarrow x_0$ as $k \rightarrow \infty$ and $V(x_0) = 0$.

(iii) The Lem3.1 shows that sequence $\{v_{\eta_k}\}$ is bounded in $H^1(\mathbb{R}^3)$, and under the sense of subsequence, there exists a $v_0 \in H^1(\mathbb{R}^3)$ such that $v_{\eta_k} \rightharpoonup v_0$ as $k \rightarrow \infty$. Using (3.18) and passing weak limit to (3.15), one obtains that v_0 satisfies

$$-\Delta v_0 + \frac{2-s}{3(s+1)} v_0 = \frac{\lambda^*}{b} |v_0|^{\frac{4(s+1)}{3}} v_0, \quad x \in \mathbb{R}^3. \quad (3.24)$$

where $0 < s < 2$. By (3.13) and applying the strong maximum principle to (3.24), one has $v_0 > 0$. Taking $p = \frac{4(s+1)}{3}$ in (1.5), one knows that

$$-\Delta Q + \frac{2-s}{3(s+1)} Q = \frac{1}{s+1} |Q|^{\frac{4(s+1)}{3}} Q, \quad x \in \mathbb{R}^3. \quad (3.25)$$

Because (3.25) has a unique positive radially symmetric solution $Q \in H^1(\mathbb{R}^3)$, and it hence deduces from (3.24) that

$$v_0(x) = \frac{Q(|x - y_0|)}{\|Q\|_{L^2}} \text{ for some } y_0 \in \mathbb{R}^3. \quad (3.26)$$

Similar to the procedure of Th1.1, one declares that as $k \rightarrow \infty$, $v_{\eta_k} \rightarrow v_0$ strongly in $H^1(\mathbb{R}^3)$. Using the standard elliptic regularity theory, we get from (3.15) that as $k \rightarrow \infty$

$$v_{\eta_k} \rightarrow v_0 \text{ in } C_{loc}^{2,\alpha}(\mathbb{R}^3), \alpha \in (0, 1). \quad (3.27)$$

Applying the method in [10], one knows that the $y_0 = 0$ in (3.26), and 0 is the unique global maximum of v_0 . Therefore, v_0 behaves like

$$v_0(x) = \frac{Q(|x|)}{\|Q\|_{L^2}}, \quad x \in \mathbb{R}^3. \quad (3.28)$$

By (3.27), using the technique of proving Theorem 1.2 in [11], we know that z_{η_k} is the unique global maximum of u_{η_k} . \square

To obtain more detailed description on limit behavior of constraint minimizers u_{η_k} as $\eta_k \rightarrow 0^+$, some precise energy estimation of $I(\eta_k, s, \lambda^*)$ as $\eta_k \rightarrow 0^+$ is necessary. Towards this aim, we begin with the upper bound estimation of $I(\eta_k, s, \lambda)$, which is sated as the following lemma.

Lemma 3.3. *Assume that (V_1) and (V_2) holds. If $p = \frac{4(s+1)}{3}$ and $\lambda = \lambda^*$, then for any positive sequence $\{\eta_k\}$ with $\eta_k \rightarrow 0^+$ as $k \rightarrow \infty$, the $I(\eta_k, s, \lambda)$ satisfies as $k \rightarrow \infty$*

$$I(\eta_k, s, \lambda^*) \leq \left[\frac{1}{2} q^{\frac{2}{q+2}} + q^{\frac{-q}{q+2}} \right] \theta^{\frac{2}{q+2}} (\eta_k)^{\frac{q}{q+2}} (1 + o(1)), \quad (3.29)$$

where q, θ defined by (1.10) and (1.12).

Proof. Choosing a test function the same as (2.8), it then deduces from (1.6)-(1.13) that there exist positive constants d_1, d_2 such that as $t \rightarrow +\infty$

$$\begin{aligned} & \frac{b}{2(s+1)} \left(\int_{\mathbb{R}^3} |\nabla u_t|^2 dx \right)^{s+1} - \frac{3\lambda^*}{4s+10} \int_{\mathbb{R}^3} |u_t|^{\frac{4(s+1)}{3}+2} dx \\ & \leq \frac{bt^{2(s+1)}}{2(s+1)} - \frac{\lambda^* t^{2(s+1)}}{2\|Q\|_{L^2}^{\frac{4(s+1)}{3}}} + d_1 e^{-d_2 t} = d_1 e^{-d_2 t} \end{aligned} \quad (3.30)$$

and there exist positive constants d_3, d_4 such that as $t \rightarrow +\infty$

$$\frac{\eta_k}{2} \int_{\mathbb{R}^3} |\nabla u_t|^2 dx = \frac{P_t^2 t^2}{\|Q\|_{L^2}^2} \int_{\mathbb{R}^3} |\nabla Q|^2 dx = \frac{\eta_k t^2}{2} + d_3 e^{-d_4 t}. \quad (3.31)$$

Since $V(x)$ satisfies (V_1) and (V_2) , we derive that there exist positive constants d_5, d_6 such that as $t \rightarrow +\infty$

$$\begin{aligned} & \int_{\mathbb{R}^3} V(x) u_t^2 dx \leq \frac{1}{\|Q\|_{L^2}^2} \int_{B_{\sqrt{t}}(0)} V\left(\frac{x}{t} + x_i\right) |Q|^2 dx + d_5 e^{-d_6 t} \\ & = \frac{1}{\|Q\|_{L^2}^2} \int_{B_{\sqrt{t}}(0)} C\left(\frac{x}{t} + x_i\right) \prod_{j=1}^n \left| \frac{x}{t} + x_i - x_j \right|^{q_j} |Q|^2 dx + d_5 e^{-d_6 t} \\ & = t^{-q} \frac{1}{\|Q\|_{L^2}^2} \lim_{x \rightarrow x_i} \frac{V(x)}{|x - x_i|^q} \int_{\mathbb{R}^3} |x|^q |Q(x)|^2 dx + o(t^{-q}) + d_5 e^{-d_6 t} \\ & = \theta t^{-q} + o(t^{-q}) + d_5 e^{-d_6 t}, \end{aligned} \quad (3.32)$$

where q, θ defined by (1.10) and (1.12). Combing (3.30), (3.31) and (3.32), we have

$$\begin{aligned} I(\eta_k, s, \lambda^*) &\leq \frac{\eta_k t^2}{2} + \theta t^{-q} + o(t^{-q}) + d_1 e^{-d_2 t} + d_3 e^{-d_4 t} + d_5 e^{-d_6 t} \\ &= \frac{\eta_k t^2}{2} + \theta t^{-q} (1 + o(1)). \end{aligned} \quad (3.33)$$

Taking $t = (q\theta)^{\frac{1}{q+2}} (\eta_k)^{-\frac{1}{q+2}}$, one deduces from (3.33) that as $\eta_k \rightarrow 0^+$

$$I(\eta_k, s, \lambda^*) \leq \left[\frac{1}{2} q^{\frac{2}{q+2}} + q^{\frac{-q}{q+2}} \right] \theta^{\frac{2}{q+2}} (\eta_k)^{\frac{q}{q+2}} (1 + o(1)),$$

which then gives (3.29). \square

Proof of Theorem 1.3. According to the results of Lem3.1-Lem3.3, it remains to prove (1.15) and (1.16), which can be realized by establishing the precise lower energy estimation of $I(\eta_k, s, \lambda^*)$ as $\eta_k \rightarrow 0^+$. To get this goal, we set $\{u_{\eta_k}\}$ being the positive minimizers of $I(\eta_k, s, \lambda^*)$, z_{η_k} being their unique global maxima, and we then define v_{η_k} by (3.12). Using Lem3.2, one knows that for $\{z_{\eta_k}\}$, choosing a subsequence if necessary (still stated by $\{z_{\eta_k}\}$), the $z_{\eta_k} \rightarrow x_0$ and $V(x_0) = 0$. In fact, we can go a step further, that is, we can come to the following conclusion.

$$z_{\eta_k} \rightarrow x_i \text{ and } \frac{|z_{\eta_k} - x_i|}{\epsilon_{\eta_k}} \text{ is bounded uniformly as } k \rightarrow \infty, \quad (3.34)$$

where $x_i \in \mathcal{W}$ and x_i denotes a flattest global minimum of $V(x)$. To get (3.34), we firstly claim that

$$\frac{|z_{\eta_k} - x_0|}{\epsilon_{\eta_k}} \text{ is bounded uniformly as } k \rightarrow \infty. \quad (3.35)$$

If this is false, then we assume that $\frac{|z_{\eta_k} - x_0|}{\epsilon_{\eta_k}} \rightarrow \infty$ as $k \rightarrow \infty$. It then follows from (V_2) and (3.13) that for any large positive constant \mathcal{D}

$$\begin{aligned} &\liminf_{k \rightarrow \infty} \frac{1}{\epsilon_{\eta_k}^{q_{i_0}}} \int_{\mathbb{R}^3} V(\epsilon_{\eta_k} x + z_{\eta_k}) v_{\eta_k}^2 dx \\ &\geq C \int_{B_{2s}(0)} \liminf_{k \rightarrow \infty} \left| x + \frac{z_{\eta_k} - x_0}{\epsilon_{\eta_k}} \right|^{q_{i_0}} \cdot \prod_{j=1, j \neq i_0}^n |\epsilon_{\eta_k} x + z_{\eta_k} - x_j|^{q_j} v_{\eta_k}^2 dx \geq \mathcal{D}. \end{aligned} \quad (3.36)$$

Recall from G-N inequality (1.8), we also have for $p = \frac{4(s+1)}{3}$ and $\lambda = \lambda^*$

$$\liminf_{k \rightarrow \infty} \left(\frac{b}{2(s+1)} \left(\int_{\mathbb{R}^3} |\nabla u_{\eta_k}|^2 dx \right)^{s+1} - \frac{\lambda^*}{p+2} \int_{\mathbb{R}^3} |u_{\eta_k}|^{p+2} dx \right) \geq 0 \quad (3.37)$$

which together with (3.36) then gives

$$\liminf_{k \rightarrow \infty} I(\eta_k, s, \lambda^*) = \liminf_{k \rightarrow \infty} E(\eta_k) \geq \frac{\eta_k \epsilon_{\eta_k}^{-2}}{2} + \mathcal{D} \epsilon_{\eta_k}^{q_{i_0}} \geq \mathcal{E} \eta_k^{\frac{q_{i_0}}{q_{i_0}+2}}, \quad (3.38)$$

where \mathcal{E} is a arbitrarily large constant. However, this is a contradiction with the upper energy in Lem3.3. Hence, the (3.35) is holding. In truth, the upper energy of $I(\eta_k, s, \lambda^*)$ also compels that $x_0 = x_i \in \mathcal{W}$. If not, by repeating the proof process from (3.35) to (3.37), one still derives a contradiction. Thus, we complete the proof of (3.34).

Using (3.34) and similar to estimation of (3.36), one can deduce that there admits a $\hat{x} \in \mathbb{R}^3$ such that

$$\begin{aligned} & \liminf_{k \rightarrow \infty} \frac{1}{\epsilon_{\eta_k}^q} \int_{\mathbb{R}^3} V(\epsilon_{\eta_k} x + z_{\eta_k}) v_{\eta_k}^2 dx \\ &= \lim_{x \rightarrow x_i} \frac{V(x)}{|x - x_i|^q} \int_{\mathbb{R}^3} |x + \hat{x}|^q v_0^2 dx \\ &\geq \lim_{x \rightarrow x_i} \frac{V(x)}{|x - x_i|^q} \int_{\mathbb{R}^3} |x|^q v_0^2 dx = \theta, \end{aligned} \quad (3.39)$$

where θ, q given by (1.10) and (1.12). As a fact, the equality in (3.39) holds only for $\bar{x} = 0$. One then calculates from (3.38) and (3.39) that

$$\liminf_{k \rightarrow \infty} I(\eta_k, s, \lambda^*) = \liminf_{k \rightarrow \infty} E(\eta_k) \geq \frac{\eta_k \epsilon_{\eta_k}^{-2}}{2} + \theta \epsilon_{\eta_k}^q. \quad (3.40)$$

Due to the restriction of energy upper bound in Lem3.3, it yields that ϵ_{λ_k} be the form of

$$\epsilon_{\eta_k} = (q\theta)^{-\frac{1}{q+2}} (\eta_k)^{\frac{1}{q+2}} \quad (3.41)$$

which gives (1.15). Taking (3.41) into (3.40), one then derives that

$$\liminf_{k \rightarrow \infty} I(\eta_k, s, \lambda^*) \geq \left[\frac{1}{2} q^{\frac{2}{q+2}} + q^{\frac{-q}{q+2}} \right] \theta^{\frac{2}{q+2}} (\eta_k)^{\frac{q}{q+2}}. \quad (3.42)$$

which together with Lem3.3 yields that as $k \rightarrow \infty$

$$I(\eta_k, s, \lambda^*) \approx \left[\frac{1}{2} q^{\frac{2}{q+2}} + q^{\frac{-q}{q+2}} \right] \theta^{\frac{2}{q+2}} (\eta_k)^{\frac{q}{q+2}}. \quad (3.43)$$

So far, we have finished the proof of Th1.3. \square

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