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On Induced Topologies by Ideal, Primal, Filter and Grill

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Abstract: In this paper the one-to-one correspondences and equivalences between ideals, primals, filters and grills are introduced. It is shown the local functions and the topological spaces induced by them are the same. From this point of view, the topological properties with respect to one topology can be derived from topological properties valid in the corresponding topology.

Keywords: ideal; prima; grill; filter; local function; finer topology; compatibility; codense system

MSC: 54A05; 54A10

1. Introduction

Many topologies with important applications in mathematics have been defined using some additional mathematical structures. These structures include grills, ideals, filters, and primals. These classical structures are undoubtedly some of the most important objects of topology.

A very interesting area of application of additional structures is the technique of induced topologies. Namely, the creation of a new topology from a given topology derived from ideals, primals, filters and grills.

The notion of ideal topological spaces was studied by Kuratowski [18] and Vaidyanathaswamy [34], and it is the famous method of creating a new topology derived from ideal. Roughly speaking, if an ideal is given on a topological space, a new topology (called an ideal topology) can be obtained by an ideal-associated local function or its dual operator. This concept was further investigated in various directions by Jankovic and Hamlett [11–14], Dontchev et al. [10], Mandal and Mukherjee [20], and Mukherjee et al. [24]. Properties such as decomposition of continuity, separation axioms, connectedness, compactness, resolvability, and compatibility of topology with an ideal are central research topics of ideal topological spaces. The classical result is that if a given set is locally of the first category at every point, then it is of the first category proven by Banach [7] for metric spaces and extended to arbitrary spaces by Kuratowski [18]. Oxtoby [25] proved the union of open sets of the first category is of the first category. Similar results were studied for an ideal topological space by Kaniewski at al. [17].

The other classical structure of general topology in the literature is the notion of grills. It was introduced by Choquet [9] and further developed by Mandal [19], Roy and Mukherjee [28–30], Roy and et al [31], Thorn [33], and many others. The main idea is based on a similar course to the grill-associated topology. A few operators were introduced, basic properties of these operators and induced topologies were studied. Also, the suitability of topology for a given grill was defined. Grill topological spaces are still the focus of research, see Azzam et al. [6], Kalaivani et al.[15], Rajasekaran et al.[26].

Recently, Acharjee et al. [1] introduced a new structure called primal. They define the notion of primal topological space by utilizing two new operators and investigate many fundamental properties of this new structure and these two operators. Moreover, the notion of primal is the dual structure of grills. Furthermore, some new studies have been developed regarding primal topological spaces since the introduction of primal; for more details see [2–5].

Filters that are used in general topology to characterize such important concepts as continuity, initial and final structures, compactness, etc., were introduced by Henri Cartan [8]. They allow for a

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general theory of convergence in topological spaces when sequences do not suffice. For the purposes of our article, it is sufficient to state that ideals and filters are mutually dual concepts. Similarly, grills and primals are dual. This fact is stated in the literature, but the equality of local functions induced by primals, grills, filters, and ideals has rarely been explicitly proven. The exception to this is Kandil et al. [16] and Renukadevi [27], where the equality of local functions generated by ideals and grills is proven. Also, a new topology can be induced from a grill and a filter on the same set, see Modak [21,22]. At the end of the article, we will draw some other applications where topological properties characterized by filters can be replaced by other structures, see Modak at al. [23], Selim et al. [32].

In general, we can consider four systems, namely an ideal $\mathcal{I}$, a primal $\mathcal{P}$, a filter $\mathcal{F}$ and a grill $\mathcal{G}$ on a topological space (X,τ), see Definition 1. A derivation of a new topology that is finer than the original topology τ is as follows: The local function $A^*_{\mathcal{I}}$, $A^*_{\mathcal{P}}$, $A^*_{\mathcal{F}}$, $A^*_{\mathcal{G}}$, see Definition 1, derived from $\mathcal{I},\mathcal{P},\mathcal{F},\mathcal{G}$ and τ defines the Kuratowski closure operator $cl_{\mathcal{I}}$, $cl_{\mathcal{P}}$, $cl_{\mathcal{F}}$, $cl_{\mathcal{G}}$, see Definition 3. In the final step, a new topology on X is defined, denoted by $\tau_{\mathcal{I}}$, $\tau_{\mathcal{P}}$, $\tau_{\mathcal{F}}$, $\tau_{\mathcal{G}}$, respectively, see Theorem 9. If we look at the achieved results, we can see a striking similarity. In fact, the local functions and topologies generated by this way are equivalent.

The main concept of the article is as follows: Using correspondence between two systems (Theorem 1–6), it is possible to define their equivalence, see Definition 2. Two equivalent systems generate the same topology (Theorem 10), and the results achieved in one topology can be used in a topology determined by an equivalent system. In the last part, we will show the application of this equivalence on examples of compatibility and codense topologies.

Definition 1. *Let X be a nonempty set. A nonempty system $\mathcal{I}$, $\mathcal{P}$, $\mathcal{F}$, $\mathcal{G}$ of subsets of X is said be an ideal, a primal, a filter, a grill on X if it satisfies the following conditions*

(1) $X \notin \mathcal{I}$	(1) $X \notin \mathcal{P}$
(2) $A \in \mathcal{I}$ and $B \subset A \Rightarrow B \in \mathcal{I}$	(2) $A \in \mathcal{P}$ and $B \subset A \Rightarrow B \in \mathcal{P}$
(3) $A \in \mathcal{I}$ and $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$	(3) $A \cap B \in \mathcal{P} \Rightarrow A \in \mathcal{P}$ or $B \in \mathcal{P}$
(1) $\emptyset \notin \mathcal{F}$	(1) $\emptyset \notin \mathcal{G}$
(2) $A \in \mathcal{F}$ and $A \subset B \Rightarrow B \in \mathcal{F}$	(2) $A \in \mathcal{G}$ and $A \subset B \Rightarrow B \in \mathcal{G}$
(3) $A \in \mathcal{F}$ and $B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$	(3) $A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$

respectively. Furthermore, if τ is a topology on X, for $A \subset X$ we define four local functions

$$A^*_{\mathcal{I}} = \{x \in X : A \cap U \notin \mathcal{I} \text{ for any } U \in \tau, x \in U\},$$
$$A^*_{\mathcal{P}} = \{x \in X : (X \setminus A) \cup (X \setminus U) \in \mathcal{P} \text{ for any } U \in \tau, x \in U\},$$
$$A^*_{\mathcal{F}} = \{x \in X : (X \setminus A) \cup (X \setminus U) \notin \mathcal{F} \text{ for any } U \in \tau, x \in U\},$$
$$A^*_{\mathcal{G}} = \{x \in X : A \cap U \in \mathcal{G} \text{ for any } U \in \tau, x \in U\}.$$

An ideal topological space, a primal topological space, a filter topological space, a grill topological space is a topological space (X,τ) with an ideal $\mathcal{I}$, a primal $\mathcal{P}$, a filter $\mathcal{F}$, a grill $\mathcal{G}$ and it is denoted by $(X,\tau,\mathcal{I})$, $(X,\tau,\mathcal{P})$, $(X,\tau,\mathcal{F})$, $(X,\tau,\mathcal{G})$, respectively. Let $\mathbb{I},\mathbb{P},\mathbb{F},\mathbb{G}$ be a family of all ideals, primals, filters, grills on X, respectively. Put $\mathbb{X} = \mathbb{I} \cup \mathbb{P} \cup \mathbb{F} \cup \mathbb{G}$. If $\mathcal{Z} \in \mathbb{X}$, then $(X,\tau,\mathcal{Z})$ is called a $\mathcal{Z}$-topological space.

2. Main results

In the following two parts we present sixteen operators $\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2}$ (Theorem 1–6) between pairs of systems $\mathbb{Z}_1, \mathbb{Z}_2 \in \{\mathbb{I},\mathbb{P},\mathbb{F},\mathbb{G}\}$ and their properties and compositions are studied. Next, the equivalence between $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathbb{X}$ is defined (Definotion 2). The equality of local functions and the equality of generated topologies are proved, provided that $\mathcal{Z}_1$ and $\mathcal{Z}_2$ and equivalent.

Define four identity operators

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$\mathsf{H}_{\mathbb{I}}^{\mathbb{I}} : \mathbb{I} \to \mathbb{I}$ by $\mathsf{H}_{\mathbb{I}}^{\mathbb{I}}(\mathcal{I}) = \mathcal{I}$, $\mathsf{H}_{\mathbb{P}}^{\mathbb{P}} : \mathbb{P} \to \mathbb{P}$ by $\mathsf{H}_{\mathbb{P}}^{\mathbb{P}}(\mathcal{P}) = \mathcal{P}$,
$\mathsf{H}_{\mathbb{F}}^{\mathbb{F}} : \mathbb{F} \to \mathbb{F}$ by $\mathsf{H}_{\mathbb{F}}^{\mathbb{F}}(\mathcal{F}) = \mathcal{F}$, $\mathsf{H}_{\mathbb{G}}^{\mathbb{G}} : \mathbb{G} \to \mathbb{G}$ by $\mathsf{H}_{\mathbb{G}}^{\mathbb{G}}(\mathcal{G}) = \mathcal{G}$.

In the next six theorems the proofs of items (1)–(10) are left to the reader. The rest will be proven.

Theorem 1. *Let $\mathcal{I}, \mathcal{P}$ be an ideal, a primal on X, respectively. Define*

$\mathsf{H}_{\mathbb{P}}^{\mathbb{I}} : \mathbb{I} \to \mathbb{P}$ *by* $\mathsf{H}_{\mathbb{P}}^{\mathbb{I}}(\mathcal{I}) = \{A \subset X : X \setminus A \notin \mathcal{I}\}$,

$\mathsf{H}_{\mathbb{I}}^{\mathbb{P}} : \mathbb{P} \to \mathbb{I}$ *by* $\mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P}) = \{A \subset X : X \setminus A \notin \mathcal{P}\}$.

Then $\mathsf{H}_{\mathbb{P}}^{\mathbb{I}}(\mathcal{I})$, $\mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P})$ is a primal, an ideal on X, respectively and

(1) $A \in \mathsf{H}_{\mathbb{P}}^{\mathbb{I}}(\mathcal{I}) \Leftrightarrow X \setminus A \notin \mathcal{I}$
(2) $A \notin \mathsf{H}_{\mathbb{P}}^{\mathbb{I}}(\mathcal{I}) \Leftrightarrow X \setminus A \in \mathcal{I}$
(3) $X \setminus A \in \mathsf{H}_{\mathbb{P}}^{\mathbb{I}}(\mathcal{I}) \Leftrightarrow A \notin \mathcal{I}$
(4) $X \setminus A \notin \mathsf{H}_{\mathbb{P}}^{\mathbb{I}}(\mathcal{I}) \Leftrightarrow A \in \mathcal{I}$
(5) $A \in \mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P}) \Leftrightarrow X \setminus A \notin \mathcal{P}$
(6) $A \notin \mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P}) \Leftrightarrow X \setminus A \in \mathcal{P}$
(7) $X \setminus A \in \mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P}) \Leftrightarrow A \notin \mathcal{P}$
(8) $X \setminus A \notin \mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P}) \Leftrightarrow A \in \mathcal{P}$

(9) $\mathsf{H}_{\mathbb{P}}^{\mathbb{I}}(\mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P})) = \mathsf{H}_{\mathbb{P}}^{\mathbb{P}}(\mathcal{P}) = \mathcal{P}$
(10) $\mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathsf{H}_{\mathbb{P}}^{\mathbb{I}}(\mathcal{I})) = \mathsf{H}_{\mathbb{I}}^{\mathbb{I}}(\mathcal{I}) = \mathcal{I}$
(11) $A^*_{\mathsf{H}_{\mathbb{P}}^{\mathbb{I}}(\mathcal{I})} = A^*_{\mathcal{I}}$
(12) $A^*_{\mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P})} = A^*_{\mathcal{P}}$

Proof. We prove $\mathsf{H}_{\mathbb{P}}^{\mathbb{I}}(\mathcal{I})$ is a primal. Since $\emptyset \in \mathcal{I}$, $X \notin \mathsf{H}_{\mathbb{P}}^{\mathbb{I}}(\mathcal{I})$.

Let $A \in \mathsf{H}_{\mathbb{P}}^{\mathbb{I}}(\mathcal{I})$ and $B \subset A$. Then $X \setminus A \notin \mathcal{I}$. Since $X \setminus A \subset X \setminus B$ and $\mathcal{I}$ is an ideal, $X \setminus B \notin \mathcal{I}$, so $B \in \mathsf{H}_{\mathbb{P}}^{\mathbb{I}}(\mathcal{I})$. Let $A \cap B \in \mathsf{H}_{\mathbb{P}}^{\mathbb{I}}(\mathcal{I})$. Then $(X \setminus A) \cup (X \setminus B) = X \setminus A \cap B \notin \mathcal{I}$. Since $\mathcal{I}$ is an ideal, $X \setminus A$ or $X \setminus B$ is not from $\mathcal{I}$. So $A \in \mathsf{H}_{\mathbb{P}}^{\mathbb{I}}(\mathcal{I})$ or $B \in \mathsf{H}_{\mathbb{P}}^{\mathbb{I}}(\mathcal{I})$. That means $\mathsf{H}_{\mathbb{P}}^{\mathbb{I}}(\mathcal{I})$ is a primal.

We prove $\mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P})$ is an ideal. Since $\emptyset \in \mathcal{P}$, $X \notin \mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P})$.

Let $A \in \mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P})$ and $B \subset A$. Then $X \setminus A \notin \mathcal{P}$. Since $X \setminus A \subset X \setminus B$ and $\mathcal{P}$ is a primal, $X \setminus B \notin \mathcal{P}$, so $B \in \mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P})$. Let $A, B \in \mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P})$. Then $X \setminus A \notin \mathcal{P}$, $X \setminus B \notin \mathcal{P}$ and $(X \setminus A) \cap (X \setminus B) = X \setminus A \cup B \notin \mathcal{P}$. So $A \cup B \in \mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P})$. That means $\mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P})$ is an ideal.

(11): $x \in A^*_{\mathsf{H}_{\mathbb{P}}^{\mathbb{I}}(\mathcal{I})}$ if and only if $(X \setminus A) \cup (X \setminus U) = X \setminus A \cap U \in \mathsf{H}_{\mathbb{P}}^{\mathbb{I}}(\mathcal{I})$ for any nbhd U of x if and only if $A \cap U \notin \mathcal{I}$ if and only if $x \in A^*_{\mathcal{I}}$.

(12): $x \in A^*_{\mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P})}$ if and only if $A \cap U \notin \mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P})$ for any nbhd U of x if and only if $X \setminus A \cap U = (X \setminus A) \cup (X \setminus U) \in \mathcal{P}$ if and only if $x \in A^*_{\mathcal{P}}$. □

Theorem 2. *Let $\mathcal{I}, \mathcal{G}$ be an ideal, a grill on X, respectively. Define*

$\mathsf{H}_{\mathbb{G}}^{\mathbb{I}} : \mathbb{I} \to \mathbb{G}$ *by* $\mathsf{H}_{\mathbb{G}}^{\mathbb{I}}(\mathcal{I}) = \{A \subset X : A \notin \mathcal{I}\}$,

$\mathsf{H}_{\mathbb{I}}^{\mathbb{G}} : \mathbb{G} \to \mathbb{I}$ *by* $\mathsf{H}_{\mathbb{I}}^{\mathbb{G}}(\mathcal{G}) = \{A \subset X : A \notin \mathcal{G}\}$.

Then $\mathsf{H}_{\mathbb{G}}^{\mathbb{I}}(\mathcal{I})$, $\mathsf{H}_{\mathbb{I}}^{\mathbb{G}}(\mathcal{G})$ is a grill, an ideal on X, respectively and

(1) $A \in \mathsf{H}_{\mathbb{G}}^{\mathbb{I}}(\mathcal{I}) \Leftrightarrow A \notin \mathcal{I}$
(2) $A \notin \mathsf{H}_{\mathbb{G}}^{\mathbb{I}}(\mathcal{I}) \Leftrightarrow A \in \mathcal{I}$
(3) $X \setminus A \in \mathsf{H}_{\mathbb{G}}^{\mathbb{I}}(\mathcal{I}) \Leftrightarrow X \setminus A \notin \mathcal{I}$
(4) $X \setminus A \notin \mathsf{H}_{\mathbb{G}}^{\mathbb{I}}(\mathcal{I}) \Leftrightarrow X \setminus A \in \mathcal{I}$
(5) $A \in \mathsf{H}_{\mathbb{I}}^{\mathbb{G}}(\mathcal{G}) \Leftrightarrow A \notin \mathcal{G}$
(6) $A \notin \mathsf{H}_{\mathbb{I}}^{\mathbb{G}}(\mathcal{G}) \Leftrightarrow A \in \mathcal{G}$
(7) $X \setminus A \in \mathsf{H}_{\mathbb{I}}^{\mathbb{G}}(\mathcal{G}) \Leftrightarrow X \setminus A \notin \mathcal{G}$
(8) $X \setminus A \notin \mathsf{H}_{\mathbb{I}}^{\mathbb{G}}(\mathcal{G}) \Leftrightarrow X \setminus A \in \mathcal{G}$

(9) $\mathsf{H}_{\mathbb{G}}^{\mathbb{I}}(\mathsf{H}_{\mathbb{I}}^{\mathbb{G}}(\mathcal{G})) = \mathsf{H}_{\mathbb{G}}^{\mathbb{G}}(\mathcal{G}) = \mathcal{G}$
(10) $\mathsf{H}_{\mathbb{I}}^{\mathbb{G}}(\mathsf{H}_{\mathbb{G}}^{\mathbb{I}}(\mathcal{I})) = \mathsf{H}_{\mathbb{I}}^{\mathbb{I}}(\mathcal{I}) = \mathcal{I}$
(11) $A^*_{\mathsf{H}_{\mathbb{G}}^{\mathbb{I}}(\mathcal{I})} = A^*_{\mathcal{I}}$
(12) $A^*_{\mathsf{H}_{\mathbb{I}}^{\mathbb{G}}(\mathcal{G})} = A^*_{\mathcal{G}}$

Proof. We prove $\mathsf{H}_{\mathbb{G}}^{\mathbb{I}}(\mathcal{I})$ is a grill. Since $\emptyset \in \mathcal{I}, \emptyset \notin \mathsf{H}_{\mathbb{G}}^{\mathbb{I}}(\mathcal{I})$.

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Let $A \in \mathsf{H}^{\mathbb{I}}_{\mathbb{G}}(\mathcal{I})$ and $A \subset B$. Then $A \notin \mathcal{I}$. Since $\mathcal{I}$ is an ideal, $B \notin \mathcal{I}$, so $B \in \mathsf{H}^{\mathbb{I}}_{\mathbb{G}}(\mathcal{I})$. Let $A \cup B \in \mathsf{H}^{\mathbb{I}}_{\mathbb{G}}(\mathcal{I})$. Then $A \cup B \notin \mathcal{I}$. Since $\mathcal{I}$ is an ideal, A or B is not from $\mathcal{I}$. So $A \in \mathsf{H}^{\mathbb{I}}_{\mathbb{G}}(\mathcal{I})$ or $B \in \mathsf{H}^{\mathbb{I}}_{\mathbb{G}}(\mathcal{I})$. That means $\mathsf{H}^{\mathbb{I}}_{\mathbb{G}}(\mathcal{I})$ is a grill.

We prove $\mathsf{H}^{\mathbb{G}}_{\mathbb{I}}(\mathcal{G})$ is an ideal. Since $X \in \mathcal{G}$, $X \notin \mathsf{H}^{\mathbb{G}}_{\mathbb{I}}(\mathcal{G})$.

Let $A \in \mathsf{H}^{\mathbb{G}}_{\mathbb{I}}(\mathcal{G})$ and $B \subset A$. Then $A \notin \mathcal{G}$. Since $\mathcal{G}$ is a grill, $B \notin \mathcal{G}$, so $B \in \mathsf{H}^{\mathbb{G}}_{\mathbb{I}}(\mathcal{G})$. Let $A, B \in \mathsf{H}^{\mathbb{G}}_{\mathbb{I}}(\mathcal{G})$. Then $A, B \notin \mathcal{G}$. Since $\mathcal{G}$ is a grill, $A \cup B \notin \mathcal{G}$. So $A \cup B \in \mathsf{H}^{\mathbb{G}}_{\mathbb{I}}(\mathcal{G})$. That means $\mathsf{H}^{\mathbb{G}}_{\mathbb{I}}(\mathcal{G})$ is an ideal.

(11): $x \in A^*_{\mathsf{H}^{\mathbb{I}}_{\mathbb{G}}(\mathcal{I})}$ if and only if $A \cap U \in \mathsf{H}^{\mathbb{I}}_{\mathbb{G}}(\mathcal{I})$ for any nbhd U of x if and only if $A \cap U \notin \mathcal{I}$ if and only if $x \in A^*_{\mathcal{I}}$.

(12): $x \in A^*_{\mathsf{H}^{\mathbb{G}}_{\mathbb{I}}(\mathcal{G})}$ if and only if $A \cap U \notin \mathsf{H}^{\mathbb{G}}_{\mathbb{I}}(\mathcal{G})$ for any nbhd U of x if and only if $A \cap U \in \mathcal{G}$ if and only if $x \in A^*_{\mathcal{G}}$. $\square$

Theorem 3. *Let $\mathcal{F}, \mathcal{G}$ be a filter, a grill on X, respectively. Define*

$\mathsf{H}^{\mathbb{F}}_{\mathbb{G}} : \mathbb{F} \to \mathbb{G}$ *by* $\mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathcal{F}) = \{A \subset X : X \setminus A \notin \mathcal{F}\}$,

$\mathsf{H}^{\mathbb{G}}_{\mathbb{F}} : \mathbb{G} \to \mathbb{F}$ *by* $\mathsf{H}^{\mathbb{G}}_{\mathbb{F}}(\mathcal{G}) = \{A \subset X : X \setminus A \notin \mathcal{G}\}$.

Then $\mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathcal{F})$, $\mathsf{H}^{\mathbb{G}}_{\mathbb{F}}(\mathcal{G})$ is a grill, a filter on X, respectively and

(1) $A \in \mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathcal{F}) \Leftrightarrow X \setminus A \notin \mathcal{F}$	(5) $A \in \mathsf{H}^{\mathbb{G}}_{\mathbb{F}}(\mathcal{G}) \Leftrightarrow X \setminus A \notin \mathcal{G}$
(2) $A \notin \mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathcal{F}) \Leftrightarrow X \setminus A \in \mathcal{F}$	(6) $A \notin \mathsf{H}^{\mathbb{G}}_{\mathbb{F}}(\mathcal{G}) \Leftrightarrow X \setminus A \in \mathcal{G}$
(3) $X \setminus A \in \mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathcal{F}) \Leftrightarrow A \notin \mathcal{F}$	(7) $X \setminus A \in \mathsf{H}^{\mathbb{G}}_{\mathbb{F}}(\mathcal{G}) \Leftrightarrow A \notin \mathcal{G}$
(4) $X \setminus A \notin \mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathcal{F}) \Leftrightarrow A \in \mathcal{F}$	(8) $X \setminus A \notin \mathsf{H}^{\mathbb{G}}_{\mathbb{F}}(\mathcal{G}) \Leftrightarrow A \in \mathcal{G}$
(9) $\mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathsf{H}^{\mathbb{G}}_{\mathbb{F}}(\mathcal{G})) = \mathsf{H}^{\mathbb{G}}_{\mathbb{G}}(\mathcal{G}) = \mathcal{G}$	(10) $\mathsf{H}^{\mathbb{G}}_{\mathbb{F}}(\mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathcal{F})) = \mathsf{H}^{\mathbb{F}}_{\mathbb{F}}(\mathcal{F}) = \mathcal{F}$
(11) $A^*_{\mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathcal{F})} = A^*_{\mathcal{F}}$	(12) $A^*_{\mathsf{H}^{\mathbb{G}}_{\mathbb{F}}(\mathcal{G})} = A^*_{\mathcal{G}}$

Proof. We prove $\mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathcal{F})$ is a grill. Since $X \in \mathcal{F}, \emptyset \notin \mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathcal{F})$.

Let $A \in \mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathcal{F})$ and $A \subset B$. Then $X \setminus A \notin \mathcal{F}$. Since $\mathcal{F}$ is a filter and $X \setminus B \subset X \setminus A$, $X \setminus B \notin \mathcal{F}$, so $B \in \mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathcal{F})$. Let $A \cup B \in \mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathcal{F})$. Then $X \setminus A \cup B = (X \setminus A) \cap (X \setminus B) \notin \mathcal{F}$. Since $\mathcal{F}$ is an filter, $X \setminus A$ or $X \setminus B$ is not from $\mathcal{F}$. So $A \in \mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathcal{I})$ or $B \in \mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathcal{I})$. That means $\mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathcal{I})$ is a grill.

We prove $\mathsf{H}^{\mathbb{G}}_{\mathbb{F}}(\mathcal{G})$ is a filter. Since $X \in \mathcal{G}, \emptyset \notin \mathsf{H}^{\mathbb{G}}_{\mathbb{F}}(\mathcal{G})$.

Let $A \in \mathsf{H}^{\mathbb{G}}_{\mathbb{F}}(\mathcal{G})$ and $A \subset B$. Then $X \setminus A \notin \mathcal{G}$. Since $\mathcal{G}$ is a grill and $X \setminus B \subset X \setminus A$, $X \setminus B \notin \mathcal{G}$, so $B \in \mathsf{H}^{\mathbb{G}}_{\mathbb{F}}(\mathcal{G})$. Let $A, B \in \mathsf{H}^{\mathbb{G}}_{\mathbb{F}}(\mathcal{G})$. Then $X \setminus A \notin \mathcal{G}, X \setminus B \notin \mathcal{G}$. Then $(X \setminus A) \cup (X \setminus B) = X \setminus A \cap B \notin \mathcal{G}$. So, $A \cap B \in \mathsf{H}^{\mathbb{G}}_{\mathbb{F}}(\mathcal{G})$. That means $\mathsf{H}^{\mathbb{G}}_{\mathbb{F}}(\mathcal{G})$ is a filter.

(11): $x \in A^*_{\mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathcal{F})}$ if and only if $A \cap U \in \mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathcal{F})$ for any nbhd U of x if and only if $X \setminus A \cap U \notin \mathcal{F}$ if and only if $x \in A^*_{\mathcal{F}}$.

(12): $x \in A^*_{\mathsf{H}^{\mathbb{G}}_{\mathbb{F}}(\mathcal{G})}$ if and only if $X \setminus A \cap U \notin \mathsf{H}^{\mathbb{G}}_{\mathbb{F}}(\mathcal{G})$ for any nbhd U of x if and only if $A \cap U \in \mathcal{G}$ if and only if $x \in A^*_{\mathcal{G}}$. $\square$

Theorem 4. *Let $\mathcal{F}, \mathcal{P}$ be a filter, a primal on X, respectively. Define*

$\mathsf{H}^{\mathbb{F}}_{\mathbb{P}} : \mathbb{F} \to \mathbb{P}$ *by* $\mathsf{H}^{\mathbb{F}}_{\mathbb{P}}(\mathcal{F}) = \{A \subset X : A \notin \mathcal{F}\}$,

$\mathsf{H}^{\mathbb{P}}_{\mathbb{F}} : \mathbb{P} \to \mathbb{F}$ *by* $\mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathcal{P}) = \{A \subset X : A \notin \mathcal{P}\}$.

Then $\mathsf{H}^{\mathbb{F}}_{\mathbb{P}}(\mathcal{F})$, $\mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathcal{P})$ is a primal, a filter on X, respectively and

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(1) $A \in \mathsf{H}^{\mathbb{F}}_{\mathbb{P}}(\mathcal{F}) \Leftrightarrow A \notin \mathcal{F}$ (5) $A \in \mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathcal{P}) \Leftrightarrow A \notin \mathcal{P}$
(2) $A \notin \mathsf{H}^{\mathbb{F}}_{\mathbb{P}}(\mathcal{F}) \Leftrightarrow A \in \mathcal{F}$ (6) $A \notin \mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathcal{P}) \Leftrightarrow A \in \mathcal{P}$
(3) $X \setminus A \in \mathsf{H}^{\mathbb{F}}_{\mathbb{P}}(\mathcal{F}) \Leftrightarrow X \setminus A \notin \mathcal{F}$ (7) $X \setminus A \in \mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathcal{P}) \Leftrightarrow X \setminus A \notin \mathcal{P}$
(4) $X \setminus A \notin \mathsf{H}^{\mathbb{F}}_{\mathbb{P}}(\mathcal{F}) \Leftrightarrow X \setminus A \in \mathcal{F}$ (8) $X \setminus A \notin \mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathcal{P}) \Leftrightarrow X \setminus A \in \mathcal{P}$

(9) $\mathsf{H}^{\mathbb{F}}_{\mathbb{P}}(\mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathcal{P})) = \mathsf{H}^{\mathbb{P}}_{\mathbb{P}}(\mathcal{P}) = \mathcal{P}$ (10) $\mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathsf{H}^{\mathbb{F}}_{\mathbb{P}}(\mathcal{F})) = \mathsf{H}^{\mathbb{F}}_{\mathbb{F}}(\mathcal{F}) = \mathcal{F}$
(11) $A^*_{\mathsf{H}^{\mathbb{F}}_{\mathbb{P}}(\mathcal{F})} = A^*_{\mathcal{F}}$ (12) $A^*_{\mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathcal{P})} = A^*_{\mathcal{P}}$

Proof. We prove $\mathsf{H}^{\mathbb{F}}_{\mathbb{P}}(\mathcal{F})$ is a primal. Since $X \in \mathcal{F}$, $X \notin \mathsf{H}^{\mathbb{F}}_{\mathbb{P}}(\mathcal{F})$.

Let $A \in \mathsf{H}^{\mathbb{F}}_{\mathbb{P}}(\mathcal{F})$ and $B \subset A$. Then $A \notin \mathcal{F}$. Since $\mathcal{F}$ is a filter, $B \notin \mathcal{F}$, so $B \in \mathsf{H}^{\mathbb{F}}_{\mathbb{P}}(\mathcal{F})$. Let $A \cap B \in \mathsf{H}^{\mathbb{F}}_{\mathbb{P}}(\mathcal{F})$. Then $A \cap B \notin \mathcal{F}$. Since $\mathcal{F}$ is a filter, A or B is not from $\mathcal{F}$. So $A \in \mathsf{H}^{\mathbb{F}}_{\mathbb{P}}(\mathcal{F})$ or $B \in \mathsf{H}^{\mathbb{F}}_{\mathbb{P}}(\mathcal{F})$. That means $\mathsf{H}^{\mathbb{F}}_{\mathbb{P}}(\mathcal{F})$ is a primal.

We prove $\mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathcal{P})$ is a filter. Since $\emptyset \in \mathcal{P}$, $\emptyset \notin \mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathcal{P})$.

Let $A \in \mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathcal{P})$ and $A \subset B$. Then $A \notin \mathcal{P}$. Since $\mathcal{P}$ is a primal, $B \notin \mathcal{P}$, so $B \in \mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathcal{P})$. Let $A, B \in \mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathcal{P})$. Then $A, B \notin \mathcal{P}$. Since $\mathcal{P}$ is a primal, $A \cap B \notin \mathcal{P}$. So $A \cap B \in \mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathcal{P})$. That means $\mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathcal{P})$ is a filter.

(11): $x \in A^*_{\mathsf{H}^{\mathbb{F}}_{\mathbb{P}}(\mathcal{F})}$ if and only if $(X \setminus A) \cup (X \setminus U) = X \setminus A \cap U \in \mathsf{H}^{\mathbb{F}}_{\mathbb{P}}(\mathcal{F})$ for any nbhd U of x if and only if $X \setminus A \cap U \notin \mathcal{F}$ if and only if $x \in A^*_{\mathcal{F}}$.

(12): $x \in A^*_{\mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathcal{P})}$ if and only if $X \setminus A \cap U \notin \mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathcal{P})$ for any nbhd U of x if and only if $X \setminus A \cap U = (X \setminus A) \cup (X \setminus U) \in \mathcal{P}$ if and only if $x \in A^*_{\mathcal{P}}$. □

Theorem 5. *Let $\mathcal{G}$, $\mathcal{P}$ be a grill, a primal on X, respectively. Define*

$\mathsf{H}^{\mathbb{G}}_{\mathbb{P}} : \mathbb{G} \to \mathbb{P}$ *by* $\mathsf{F}^{\mathbb{G}}_{\mathbb{P}}(\mathcal{G}) = \{A \subset X : X \setminus A \in \mathcal{G}\}$,

$\mathsf{H}^{\mathbb{P}}_{\mathbb{G}} : \mathbb{P} \to \mathbb{G}$ *by* $\mathsf{H}^{\mathbb{P}}_{\mathbb{G}}(\mathcal{P}) = \{A \subset X : X \setminus A \in \mathcal{P}\}$.

Then $\mathsf{H}^{\mathbb{G}}_{\mathbb{P}}(\mathcal{G})$, $\mathsf{H}^{\mathbb{P}}_{\mathbb{G}}(\mathcal{P})$ is a primal, a grill on X, respectively and

(1) $A \in \mathsf{H}^{\mathbb{G}}_{\mathbb{P}}(\mathcal{G}) \Leftrightarrow X \setminus A \in \mathcal{G}$ (5) $A \in \mathsf{H}^{\mathbb{P}}_{\mathbb{G}}(\mathcal{P}) \Leftrightarrow X \setminus A \in \mathcal{P}$
(2) $A \notin \mathsf{H}^{\mathbb{G}}_{\mathbb{P}}(\mathcal{G}) \Leftrightarrow X \setminus A \notin \mathcal{G}$ (6) $A \notin \mathsf{H}^{\mathbb{P}}_{\mathbb{G}}(\mathcal{P}) \Leftrightarrow X \setminus A \notin \mathcal{P}$
(3) $X \setminus A \in \mathsf{H}^{\mathbb{G}}_{\mathbb{P}}(\mathcal{G}) \Leftrightarrow A \in \mathcal{G}$ (7) $X \setminus A \in \mathsf{H}^{\mathbb{P}}_{\mathbb{G}}(\mathcal{P}) \Leftrightarrow A \in \mathcal{P}$
(4) $X \setminus A \notin \mathsf{H}^{\mathbb{G}}_{\mathbb{P}}(\mathcal{G}) \Leftrightarrow A \notin \mathcal{G}$ (8) $X \setminus A \notin \mathsf{H}^{\mathbb{P}}_{\mathbb{G}}(\mathcal{P}) \Leftrightarrow A \notin \mathcal{P}$

(9) $\mathsf{H}^{\mathbb{G}}_{\mathbb{P}}(\mathsf{H}^{\mathbb{P}}_{\mathbb{G}}(\mathcal{P})) = \mathsf{H}^{\mathbb{P}}_{\mathbb{P}}(\mathcal{P}) = \mathcal{P}$ (10) $\mathsf{H}^{\mathbb{P}}_{\mathbb{G}}(\mathsf{H}^{\mathbb{G}}_{\mathbb{P}}(\mathcal{G})) = \mathsf{H}^{\mathbb{G}}_{\mathbb{G}}(\mathcal{G}) = \mathcal{G}$
(11) $A^*_{\mathsf{H}^{\mathbb{G}}_{\mathbb{P}}(\mathcal{G})} = A^*_{\mathcal{G}}$ (12) $A^*_{\mathsf{H}^{\mathbb{P}}_{\mathbb{G}}(\mathcal{P})} = A^*_{\mathcal{P}}$

Proof. We prove $\mathsf{H}^{\mathbb{G}}_{\mathbb{P}}(\mathcal{G})$ is a primal. Since $\emptyset \notin \mathcal{G}$, $X \notin \mathsf{H}^{\mathbb{G}}_{\mathbb{P}}(\mathcal{G})$.

Let $A \in \mathsf{H}^{\mathbb{G}}_{\mathbb{P}}(\mathcal{G})$ and $B \subset A$. Then $X \setminus A \in \mathcal{G}$. Since $\mathcal{G}$ is a grill and $X \setminus A \subset X \setminus B$, $X \setminus B \in \mathcal{G}$, so $B \in \mathsf{H}^{\mathbb{G}}_{\mathbb{P}}(\mathcal{G})$. Let $A \cap B \in \mathsf{H}^{\mathbb{G}}_{\mathbb{P}}(\mathcal{G})$. Then $X \setminus A \cap B = (X \setminus A) \cup (X \setminus B) \in \mathcal{G}$. Since $\mathcal{G}$ is a grill, $X \setminus A$ or $X \setminus B$ is from $\mathcal{G}$. So $A \in \mathsf{H}^{\mathbb{G}}_{\mathbb{P}}(\mathcal{G})$ or $B \in \mathsf{H}^{\mathbb{G}}_{\mathbb{P}}(\mathcal{G})$. That means $\mathsf{H}^{\mathbb{G}}_{\mathbb{P}}(\mathcal{G})$ is a primal.

We prove $\mathsf{H}^{\mathbb{P}}_{\mathbb{G}}(\mathcal{P})$ is a grill. Since $X \notin \mathcal{P}$, $\emptyset \notin \mathsf{H}^{\mathbb{P}}_{\mathbb{G}}(\mathcal{P})$.

Let $A \in \mathsf{H}^{\mathbb{P}}_{\mathbb{G}}(\mathcal{P})$ and $A \subset B$. Then $X \setminus A \in \mathcal{P}$. Since $\mathcal{P}$ is a primal and $X \setminus B \subset X \setminus A$, $X \setminus B \in \mathcal{P}$, so $B \in \mathsf{H}^{\mathbb{P}}_{\mathbb{G}}(\mathcal{P})$. Let $A \cup B \in \mathsf{H}^{\mathbb{P}}_{\mathbb{G}}(\mathcal{P})$. Then $X \setminus A \cup B = (X \setminus A) \cap (X \setminus B) \in \mathcal{P}$. Since $\mathcal{P}$ is a primal, $X \setminus A$ or $X \setminus B$ is from $\mathcal{P}$. So $A \in \mathsf{H}^{\mathbb{P}}_{\mathbb{G}}(\mathcal{P})$ or $B \in \mathsf{H}^{\mathbb{P}}_{\mathbb{G}}(\mathcal{P})$. That means $\mathsf{H}^{\mathbb{P}}_{\mathbb{G}}(\mathcal{P})$ is a grill.

(11): $x \in A^*_{\mathsf{H}^{\mathbb{G}}_{\mathbb{P}}(\mathcal{G})}$ if and only if $(X \setminus A) \cup (X \setminus U) = X \setminus A \cap U \in \mathsf{H}^{\mathbb{G}}_{\mathbb{P}}(\mathcal{G})$ for any nbhd U of x if and only if $A \cap U \in \mathcal{G}$ if and only if $x \in A^*_{\mathcal{G}}$.

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(12): $x \in A^*_{\mathsf{H}^{\mathbb{P}}_{\mathbb{G}}(\mathcal{P})}$ if and only if $A \cap U \in \mathsf{H}^{\mathbb{P}}_{\mathbb{G}}(\mathcal{P})$ for any nbhd U of x if and only if $X \setminus A \cap U = (X \setminus A) \cup (X \setminus U) \in \mathcal{P}$ if and only if $x \in A^*_{\mathcal{P}}$. □

Theorem 6. *Let $\mathcal{F}, \mathcal{I}$ be a filter, a ideal on X, respectively. Define*

$\mathsf{H}^{\mathbb{F}}_{\mathbb{I}} : \mathbb{F} \to \mathbb{I}$ *by* $\mathsf{H}^{\mathbb{F}}_{\mathbb{I}}(\mathcal{F}) = \{A \subset X : X \setminus A \in \mathcal{F}\}$,

$\mathsf{H}^{\mathbb{I}}_{\mathbb{F}} : \mathbb{I} \to \mathbb{F}$ *by* $\mathsf{H}^{\mathbb{I}}_{\mathbb{F}}(\mathcal{I}) = \{A \subset X : X \setminus A \in \mathcal{I}\}$.

Then $\mathsf{H}^{\mathbb{F}}_{\mathbb{I}}(\mathcal{F})$, $\mathsf{H}^{\mathbb{I}}_{\mathbb{F}}(\mathcal{I})$ is an ideal, a filter on X, respectively and

(1) $A \in \mathsf{H}^{\mathbb{F}}_{\mathbb{I}}(\mathcal{F}) \Leftrightarrow X \setminus A \in \mathcal{F}$ (5) $A \in \mathsf{H}^{\mathbb{I}}_{\mathbb{F}}(\mathcal{I}) \Leftrightarrow X \setminus A \in \mathcal{I}$
(2) $A \notin \mathsf{H}^{\mathbb{F}}_{\mathbb{I}}(\mathcal{F}) \Leftrightarrow X \setminus A \notin \mathcal{F}$ (6) $A \notin \mathsf{H}^{\mathbb{I}}_{\mathbb{F}}(\mathcal{I}) \Leftrightarrow X \setminus A \notin \mathcal{I}$
(3) $X \setminus A \in \mathsf{H}^{\mathbb{F}}_{\mathbb{I}}(\mathcal{F}) \Leftrightarrow A \in \mathcal{F}$ (7) $X \setminus A \in \mathsf{H}^{\mathbb{I}}_{\mathbb{F}}(\mathcal{I}) \Leftrightarrow A \in \mathcal{I}$
(4) $X \setminus A \notin \mathsf{H}^{\mathbb{F}}_{\mathbb{I}}(\mathcal{F}) \Leftrightarrow A \notin \mathcal{F}$ (8) $X \setminus A \notin \mathsf{H}^{\mathbb{I}}_{\mathbb{F}}(\mathcal{I}) \Leftrightarrow A \notin \mathcal{I}$

(9) $\mathsf{H}^{\mathbb{F}}_{\mathbb{I}}(\mathsf{H}^{\mathbb{I}}_{\mathbb{F}}(\mathcal{I})) = \mathsf{H}^{\mathbb{I}}_{\mathbb{I}}(\mathcal{I}) = \mathcal{I}$ (10) $\mathsf{H}^{\mathbb{I}}_{\mathbb{F}}(\mathsf{H}^{\mathbb{F}}_{\mathbb{I}}(\mathcal{F})) = \mathsf{H}^{\mathbb{F}}_{\mathbb{F}}(\mathcal{F}) = \mathcal{F}$
(11) $A^*_{\mathsf{H}^{\mathbb{F}}_{\mathbb{I}}(\mathcal{F})} = A^*_{\mathcal{F}}$ (12) $A^*_{\mathsf{H}^{\mathbb{I}}_{\mathbb{F}}(\mathcal{I})} = A^*_{\mathcal{I}}$

Proof. We prove $\mathsf{H}^{\mathbb{F}}_{\mathbb{I}}(\mathcal{F})$ is an ideal. Since $\emptyset \notin \mathcal{F}$, $X \notin \mathsf{H}^{\mathbb{F}}_{\mathbb{I}}(\mathcal{F})$.

Let $A \in \mathsf{H}^{\mathbb{F}}_{\mathbb{I}}(\mathcal{F})$ and $B \subset A$. Then $X \setminus A \in \mathcal{F}$. Since $X \setminus A \subset X \setminus B$ and $\mathcal{F}$ is a filter, $X \setminus B \in \mathcal{F}$, so $B \in \mathsf{H}^{\mathbb{F}}_{\mathbb{I}}(\mathcal{F})$. Let $A, B \in \mathsf{H}^{\mathbb{F}}_{\mathbb{I}}(\mathcal{F})$. Then $X \setminus A, X \setminus B \in \mathcal{F}$. Since $\mathcal{F}$ is a filter, $(X \setminus A) \cap (X \setminus B) = X \setminus A \cup B \in \mathcal{F}$. So $A \cup B \in \mathsf{H}^{\mathbb{F}}_{\mathbb{I}}(\mathcal{F})$. That means $\mathsf{H}^{\mathbb{F}}_{\mathbb{I}}(\mathcal{F})$ is an ideal.

We prove $\mathsf{H}^{\mathbb{I}}_{\mathbb{F}}(\mathcal{I})$ is a filter. Since $X \notin \mathcal{I}, \emptyset \notin \mathsf{H}^{\mathbb{I}}_{\mathbb{F}}(\mathcal{I})$.

Let $A \in \mathsf{H}^{\mathbb{I}}_{\mathbb{F}}(\mathcal{I})$ and $A \subset B$. Then $X \setminus A \in \mathcal{I}$. Since $X \setminus B \subset X \setminus A$ and $\mathcal{I}$ is an ideal, $X \setminus B \in \mathcal{I}$, so $B \in \mathsf{H}^{\mathbb{I}}_{\mathbb{F}}(\mathcal{I})$. Let $A, B \in \mathsf{H}^{\mathbb{I}}_{\mathbb{F}}(\mathcal{I})$. Then $X \setminus A, X \setminus B \in \mathcal{I}$. Since $\mathcal{I}$ is an ideal, $(X \setminus A) \cup (X \setminus B) = X \setminus A \cap B \in \mathcal{I}$. So $A \cap B \in \mathsf{H}^{\mathbb{I}}_{\mathbb{F}}(\mathcal{I})$. That means $\mathsf{H}^{\mathbb{I}}_{\mathbb{F}}(\mathcal{I})$ is a filter.

(11): $x \in A^*_{\mathsf{H}^{\mathbb{F}}_{\mathbb{I}}(\mathcal{F})}$ if and only if $A \cap U \notin \mathsf{H}^{\mathbb{F}}_{\mathbb{I}}(\mathcal{F})$ for any nbhd U of x if and only if $X \setminus A \cap U \notin \mathcal{F}$ if and only if $x \in A^*_{\mathcal{F}}$.

(12): $x \in A^*_{\mathsf{H}^{\mathbb{I}}_{\mathbb{F}}(\mathcal{I})}$ if and only if $X \setminus A \cap U \notin \mathsf{H}^{\mathbb{I}}_{\mathbb{F}}(\mathcal{I})$ for any nbhd U of x if and only if $A \cap U \notin \mathcal{I}$ if and only if $x \in A^*_{\mathcal{I}}$. □

Proposition 1. *Let $\mathbb{Z}_1, \mathbb{Z}_2 \in \{\mathbb{I}, \mathbb{P}, \mathbb{F}, \mathbb{G}\}$. If $\mathbb{Z}_1 \neq \mathbb{Z}_2$, then each of the twelve operators $\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2}$ from Theorem 1-6 is bijective and $(\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2})^{-1} = \mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}_1}$, consequently $\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2}$ and $\mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}_1}$ are mutually inverse, so $\mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}_1} \circ \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2} = \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_1}$.*

Proof. It follows from Theorem 1-6 items (9) and (10). □

Proposition 2. $\mathsf{H}^{\mathbb{P}}_{\mathbb{F}} \circ \mathsf{H}^{\mathbb{I}}_{\mathbb{P}} = \mathsf{H}^{\mathbb{I}}_{\mathbb{F}}$, $\mathsf{H}^{\mathbb{G}}_{\mathbb{I}} \circ \mathsf{H}^{\mathbb{F}}_{\mathbb{G}} = \mathsf{H}^{\mathbb{F}}_{\mathbb{I}}$, $\mathsf{H}^{\mathbb{F}}_{\mathbb{G}} \circ \mathsf{H}^{\mathbb{P}}_{\mathbb{F}} = \mathsf{H}^{\mathbb{P}}_{\mathbb{G}}$, $\mathsf{H}^{\mathbb{I}}_{\mathbb{P}} \circ \mathsf{H}^{\mathbb{G}}_{\mathbb{I}} = \mathsf{H}^{\mathbb{G}}_{\mathbb{P}}$.

Proof. $A \in \mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathsf{H}^{\mathbb{I}}_{\mathbb{P}}(\mathcal{I}))$, $\mathsf{H}^{\mathbb{G}}_{\mathbb{I}}(\mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathcal{F})) \Leftrightarrow A \notin \mathsf{H}^{\mathbb{I}}_{\mathbb{P}}(\mathcal{I}), \mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathcal{F}) \Leftrightarrow X \setminus A \in \mathcal{I}, \mathcal{F} \Leftrightarrow A \in \mathsf{H}^{\mathbb{I}}_{\mathbb{F}}(\mathcal{I}), \mathsf{H}^{\mathbb{F}}_{\mathbb{I}}(\mathcal{F})$, respectively. So, $\mathsf{H}^{\mathbb{P}}_{\mathbb{F}} \circ \mathsf{H}^{\mathbb{I}}_{\mathbb{P}} = \mathsf{H}^{\mathbb{I}}_{\mathbb{F}}$, $\mathsf{H}^{\mathbb{G}}_{\mathbb{I}} \circ \mathsf{H}^{\mathbb{F}}_{\mathbb{G}} = \mathsf{H}^{\mathbb{F}}_{\mathbb{I}}$.

$A \in \mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathcal{P}))$, $\mathsf{H}^{\mathbb{I}}_{\mathbb{P}}(\mathsf{H}^{\mathbb{G}}_{\mathbb{I}}(\mathcal{G})) \Leftrightarrow X \setminus A \notin \mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathcal{P}), \mathsf{H}^{\mathbb{G}}_{\mathbb{I}}(\mathcal{G}) \Leftrightarrow X \setminus A \in \mathcal{P}, \mathcal{G} \Leftrightarrow A \in \mathsf{H}^{\mathbb{P}}_{\mathbb{G}}(\mathcal{P}), \mathsf{H}^{\mathbb{G}}_{\mathbb{P}}(\mathcal{G})$, respectively. So, $\mathsf{H}^{\mathbb{F}}_{\mathbb{G}} \circ \mathsf{H}^{\mathbb{P}}_{\mathbb{F}} = \mathsf{H}^{\mathbb{P}}_{\mathbb{G}}$, $\mathsf{H}^{\mathbb{I}}_{\mathbb{P}} \circ \mathsf{H}^{\mathbb{G}}_{\mathbb{I}} = \mathsf{H}^{\mathbb{G}}_{\mathbb{P}}$. □

Proposition 3. *Let $\mathbb{Z}, \mathbb{Z}_1, \mathbb{Z}_2 \in \{\mathbb{I}, \mathbb{P}, \mathbb{F}, \mathbb{G}\}$. The next conditions are equivalent*

(1) $\mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_2} \circ \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}} = \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2}$ *(2)* $\mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_1} \circ \mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}} = \mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}_1}$
(3) $\mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}} \circ \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2} = \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}}$ *(4)* $\mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}_1} \circ \mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_2} = \mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_1}$
(5) $\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2} \circ \mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_1} = \mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_2}$ *(6)* $\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}} \circ \mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}_1} = \mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}}$

Proof. (1) $\Leftrightarrow$ (2):

$\mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_2} \circ \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}} = \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2} \Leftrightarrow (\mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_2} \circ \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}})^{-1} = (\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2})^{-1} \Leftrightarrow \mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_1} \circ \mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}} = \mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}_1}$

(1) $\Leftrightarrow$ (3):

$\mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_2} \circ \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}} = \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2} \Leftrightarrow \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}} = (\mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_2})^{-1} \circ \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2} \Leftrightarrow \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}} = \mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}} \circ \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2}$

(3) $\Leftrightarrow$ (4):

$\mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}} \circ \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2} = \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}} \Leftrightarrow (\mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}} \circ \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2})^{-1} = (\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}})^{-1} \Leftrightarrow \mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}_1} \circ \mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_2} = \mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_1}$

(1) $\Leftrightarrow$ (5):

$\mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_2} \circ \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}} = \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2} \Leftrightarrow \mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_2} = \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2} \circ (\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}})^{-1} \Leftrightarrow \mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_2} = \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2} \circ \mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_1}$

(5) $\Leftrightarrow$ (6):

$\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2} \circ \mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_1} = \mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_2} \Leftrightarrow (\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2} \circ \mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_1})^{-1} = (\mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_2})^{-1} \Leftrightarrow \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}} \circ \mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}_1} = \mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}}$ □

Proposition 4. *Let $\mathbb{Z}, \mathbb{Z}_1, \mathbb{Z}_2 \in \{\mathbb{I}, \mathbb{P}, \mathbb{F}, \mathbb{G}\}$. Then for 64 possibilities the equation $\mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_2} \circ \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}} = \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2}$ holds.*

Proof. By Proposition 2 and Proposition 3, the equation $\mathsf{H}^{\mathbb{Z}}_{\mathbb{Z}_2} \circ \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}} = \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2}$ holds for 24 possibilities ($\mathbb{Z}, \mathbb{Z}_1, \mathbb{Z}_2$ are mutually different):

$$
\begin{array}{llll}
\mathsf{H}^{\mathbb{P}}_{\mathbb{F}} \circ \mathsf{H}^{\mathbb{I}}_{\mathbb{P}} = \mathsf{H}^{\mathbb{I}}_{\mathbb{F}} & \mathsf{H}^{\mathbb{G}}_{\mathbb{I}} \circ \mathsf{H}^{\mathbb{F}}_{\mathbb{G}} = \mathsf{H}^{\mathbb{F}}_{\mathbb{I}} & \mathsf{H}^{\mathbb{F}}_{\mathbb{G}} \circ \mathsf{H}^{\mathbb{P}}_{\mathbb{F}} = \mathsf{H}^{\mathbb{P}}_{\mathbb{G}} & \mathsf{H}^{\mathbb{I}}_{\mathbb{P}} \circ \mathsf{H}^{\mathbb{G}}_{\mathbb{I}} = \mathsf{H}^{\mathbb{G}}_{\mathbb{P}} \\
\mathsf{H}^{\mathbb{P}}_{\mathbb{I}} \circ \mathsf{H}^{\mathbb{F}}_{\mathbb{P}} = \mathsf{H}^{\mathbb{F}}_{\mathbb{I}} & \mathsf{H}^{\mathbb{G}}_{\mathbb{F}} \circ \mathsf{H}^{\mathbb{I}}_{\mathbb{G}} = \mathsf{H}^{\mathbb{I}}_{\mathbb{F}} & \mathsf{H}^{\mathbb{F}}_{\mathbb{P}} \circ \mathsf{H}^{\mathbb{G}}_{\mathbb{F}} = \mathsf{H}^{\mathbb{G}}_{\mathbb{P}} & \mathsf{H}^{\mathbb{I}}_{\mathbb{G}} \circ \mathsf{H}^{\mathbb{P}}_{\mathbb{I}} = \mathsf{H}^{\mathbb{P}}_{\mathbb{G}} \\
\mathsf{H}^{\mathbb{F}}_{\mathbb{P}} \circ \mathsf{H}^{\mathbb{I}}_{\mathbb{F}} = \mathsf{H}^{\mathbb{I}}_{\mathbb{P}} & \mathsf{H}^{\mathbb{I}}_{\mathbb{G}} \circ \mathsf{H}^{\mathbb{F}}_{\mathbb{I}} = \mathsf{H}^{\mathbb{F}}_{\mathbb{G}} & \mathsf{H}^{\mathbb{G}}_{\mathbb{F}} \circ \mathsf{H}^{\mathbb{P}}_{\mathbb{G}} = \mathsf{H}^{\mathbb{P}}_{\mathbb{F}} & \mathsf{H}^{\mathbb{P}}_{\mathbb{I}} \circ \mathsf{H}^{\mathbb{G}}_{\mathbb{P}} = \mathsf{H}^{\mathbb{G}}_{\mathbb{I}} \\
\mathsf{H}^{\mathbb{F}}_{\mathbb{I}} \circ \mathsf{H}^{\mathbb{P}}_{\mathbb{F}} = \mathsf{H}^{\mathbb{P}}_{\mathbb{I}} & \mathsf{H}^{\mathbb{I}}_{\mathbb{F}} \circ \mathsf{H}^{\mathbb{G}}_{\mathbb{I}} = \mathsf{H}^{\mathbb{G}}_{\mathbb{F}} & \mathsf{H}^{\mathbb{G}}_{\mathbb{P}} \circ \mathsf{H}^{\mathbb{F}}_{\mathbb{G}} = \mathsf{H}^{\mathbb{F}}_{\mathbb{P}} & \mathsf{H}^{\mathbb{P}}_{\mathbb{G}} \circ \mathsf{H}^{\mathbb{I}}_{\mathbb{P}} = \mathsf{H}^{\mathbb{I}}_{\mathbb{G}} \\
\mathsf{H}^{\mathbb{I}}_{\mathbb{F}} \circ \mathsf{H}^{\mathbb{P}}_{\mathbb{I}} = \mathsf{H}^{\mathbb{P}}_{\mathbb{F}} & \mathsf{H}^{\mathbb{F}}_{\mathbb{I}} \circ \mathsf{H}^{\mathbb{G}}_{\mathbb{F}} = \mathsf{H}^{\mathbb{G}}_{\mathbb{I}} & \mathsf{H}^{\mathbb{P}}_{\mathbb{G}} \circ \mathsf{H}^{\mathbb{F}}_{\mathbb{P}} = \mathsf{H}^{\mathbb{F}}_{\mathbb{G}} & \mathsf{H}^{\mathbb{G}}_{\mathbb{P}} \circ \mathsf{H}^{\mathbb{I}}_{\mathbb{G}} = \mathsf{H}^{\mathbb{I}}_{\mathbb{P}} \\
\mathsf{H}^{\mathbb{I}}_{\mathbb{P}} \circ \mathsf{H}^{\mathbb{F}}_{\mathbb{I}} = \mathsf{H}^{\mathbb{F}}_{\mathbb{P}} & \mathsf{H}^{\mathbb{F}}_{\mathbb{G}} \circ \mathsf{H}^{\mathbb{I}}_{\mathbb{F}} = \mathsf{H}^{\mathbb{I}}_{\mathbb{G}} & \mathsf{H}^{\mathbb{P}}_{\mathbb{F}} \circ \mathsf{H}^{\mathbb{G}}_{\mathbb{P}} = \mathsf{H}^{\mathbb{G}}_{\mathbb{F}} & \mathsf{H}^{\mathbb{G}}_{\mathbb{I}} \circ \mathsf{H}^{\mathbb{P}}_{\mathbb{G}} = \mathsf{H}^{\mathbb{P}}_{\mathbb{I}}
\end{array}
$$

Other cases for $\mathbb{Z}_1 \neq \mathbb{Z}_2$ are trivial:

$\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_1} \circ \mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}_1} = \mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}_1}$ (12 possibilities), $\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2} \circ \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_1} = \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2}$ (12 possibilities),
$\mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}_1} \circ \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2} = \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_1}$ (12 possibilities), $\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_1} \circ \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_1} = \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_1}$ (4 possibilities).
□

For a composition of finitely many operators, the domain (the codomain) of the resulting operator is equal to the domain (codomain) of the first (last) operator and the value of local function is independent on the operators as the following theorem states.

Theorem 7. *Let $\mathbb{Z}_k \in \{\mathbb{I}, \mathbb{P}, \mathbb{F}, \mathbb{G}\}$, $k = 1, \ldots, n$ and $\mathcal{Z} \in \mathbb{Z}_1$. Then*

$$\mathsf{H}^{\mathbb{Z}_{n-1}}_{\mathbb{Z}_n} \circ \mathsf{H}^{\mathbb{Z}_{n-2}}_{\mathbb{Z}_{n-1}} \circ \cdots \circ \mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}_3} \circ \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2} = \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_n}$$
$$A^*_{\left(\mathsf{H}^{\mathbb{Z}_{n-1}}_{\mathbb{Z}_n} \circ \mathsf{H}^{\mathbb{Z}_{n-2}}_{\mathbb{Z}_{n-1}} \circ \cdots \circ \mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}_3} \circ \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2}\right)(\mathcal{Z})} = A^*_{\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_n}(\mathcal{Z})} = A^*_{\mathcal{Z}}$$

Consequently,

(1) $A^*_{\mathcal{I}} = A^*_{\mathsf{H}^{\mathbb{I}}_{\mathbb{I}}(\mathcal{I})} = A^*_{\mathsf{H}^{\mathbb{I}}_{\mathbb{P}}(\mathcal{I})} = A^*_{\mathsf{H}^{\mathbb{I}}_{\mathbb{G}}(\mathcal{I})} = A^*_{\mathsf{H}^{\mathbb{I}}_{\mathbb{F}}(\mathcal{I})}$
(2) $A^*_{\mathcal{P}} = A^*_{\mathsf{H}^{\mathbb{P}}_{\mathbb{P}}(\mathcal{P})} = A^*_{\mathsf{H}^{\mathbb{P}}_{\mathbb{I}}(\mathcal{P})} = A^*_{\mathsf{H}^{\mathbb{P}}_{\mathbb{F}}(\mathcal{P})} = A^*_{\mathsf{H}^{\mathbb{P}}_{\mathbb{G}}(\mathcal{P})}$
(3) $A^*_{\mathcal{F}} = A^*_{\mathsf{H}^{\mathbb{F}}_{\mathbb{F}}(\mathcal{F})} = A^*_{\mathsf{H}^{\mathbb{F}}_{\mathbb{G}}(\mathcal{F})} = A^*_{\mathsf{H}^{\mathbb{F}}_{\mathbb{P}}(\mathcal{F})} = A^*_{\mathsf{H}^{\mathbb{F}}_{\mathbb{I}}(\mathcal{F})}$
(4) $A^*_{\mathcal{G}} = A^*_{\mathsf{H}^{\mathbb{G}}_{\mathbb{G}}(\mathcal{G})} = A^*_{\mathsf{H}^{\mathbb{G}}_{\mathbb{I}}(\mathcal{G})} = A^*_{\mathsf{H}^{\mathbb{G}}_{\mathbb{F}}(\mathcal{G})} = A^*_{\mathsf{H}^{\mathbb{G}}_{\mathbb{P}}(\mathcal{G})}$

Proof. For the first equation we use the mathematical induction. If $n = 3$, it follows from Proposition 4. Suppose the equation holds for $n > 3$. Then
$\mathsf{H}^{\mathbb{Z}_n}_{\mathbb{Z}_{n+1}} \circ \mathsf{H}^{\mathbb{Z}_{n-1}}_{\mathbb{Z}_n} \circ \mathsf{H}^{\mathbb{Z}_{n-2}}_{\mathbb{Z}_{n-1}} \circ \cdots \circ \mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}_3} \circ \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2} = \mathsf{H}^{\mathbb{Z}_n}_{\mathbb{Z}_{n+1}} \circ \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_n} = \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_{n+1}}$, by Proposition 4. The second equation follows from the first one and from Theorem 1–6. □

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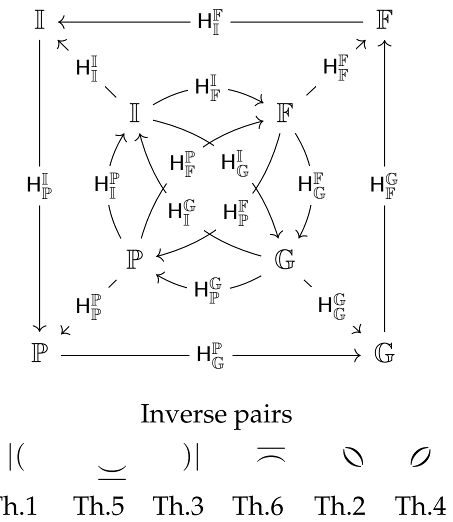
The set $\{\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2} : \mathbb{Z}_1, \mathbb{Z}_2 \in \{\mathbb{I}, \mathbb{P}, \mathbb{F}, \mathbb{G}\}\}$ consisting of 16 operators is enclosed under composition. The results can be interpreted by the next diagram.



3. Applications

We have defined 16 operators, which can be expressed by one notation $\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2}$ where $\mathbb{Z}_1, \mathbb{Z}_2 \in \{\mathbb{I}, \mathbb{P}, \mathbb{F}, \mathbb{G}\}$. Between the members of $\mathbb{Z}_1$ and the members of $\mathbb{Z}_2$ we can define an equivalence as the next definition states. Note if $\mathcal{Z} \in \mathbb{X}$, then $\mathcal{Z} \in \mathbb{Z}$ for some $\mathbb{Z} \in \{\mathbb{I}, \mathbb{P}, \mathbb{F}, \mathbb{G}\}$.

Definition 2. *Let $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathbb{X}$. $\mathcal{Z}_1$ is equivalent to $\mathcal{Z}_2$ if $\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2}(\mathcal{Z}_1) = \mathcal{Z}_2$ and $\mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}_1}(\mathcal{Z}_2) = \mathcal{Z}_1$ where $\mathcal{Z}_1 \in \mathbb{Z}_1$, $\mathcal{Z}_2 \in \mathbb{Z}_2$ and $\mathbb{Z}_1, \mathbb{Z}_2 \in \{\mathbb{I}, \mathbb{P}, \mathbb{F}, \mathbb{G}\}$. This relation is denoted by $\mathcal{Z}_1 \sim \mathcal{Z}_2$.*

Lemma 1. *For any $\mathcal{Z} \in \mathbb{X}$, $\mathcal{Z} \sim \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2}(\mathcal{Z})$ where $\mathcal{Z} \in \mathbb{Z}_1$ and $\mathbb{Z}_1, \mathbb{Z}_2 \in \{\mathbb{I}, \mathbb{P}, \mathbb{F}, \mathbb{G}\}$. Moreover, $\sim$ is an equivalence relation.*

Proof. Since $\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2}(\mathcal{Z}) = \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2}(\mathcal{Z})$ and $\mathcal{Z} = \mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}_1}(\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2}(\mathcal{Z}))$, $\mathcal{Z} \sim \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2}(\mathcal{Z})$. It is clear $\sim$ is reflexive and symmetric. Let $\mathcal{Z}_1 \sim \mathcal{Z}_2 \sim \mathcal{Z}_3$. Then $\mathcal{Z}_2 = \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2}(\mathcal{Z}_1)$ and $\mathcal{Z}_3 = \mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}_3}(\mathcal{Z}_2) = \mathsf{H}^{\mathbb{Z}_2}_{\mathbb{Z}_3}(\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2}(\mathcal{Z}_1)) = \mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_3}(\mathcal{Z}_1)$ and $\mathcal{Z}_1 = \mathsf{H}^{\mathbb{Z}_3}_{\mathbb{Z}_1}(\mathcal{Z}_3)$, so $\mathcal{Z}_1 \sim \mathcal{Z}_3$. $\square$

In the next definition we define a dual operator $A^{\triangleright}_{\mathcal{Z}}$ (see Hamlett et al. [12]) to the operator $A^*_{\mathcal{Z}}$ and a closure $cl_{\mathcal{Z}}(\cdot)$ and an interior $int_{\mathcal{Z}}(\cdot)$ operator.

Definition 3. *Let $\mathcal{Z} \in \mathbb{X}$. Define the next operators*

$A^{\triangleright}_{\mathcal{Z}} = X \setminus (X \setminus A)^*_{\mathcal{Z}}$,
$cl_{\mathcal{Z}}(A) = A \cup A^*_{\mathcal{Z}}$,
$int_{\mathcal{Z}}(A) = A \cap A^{\triangleright}_{\mathcal{Z}}$.

Lemma 2. *Let $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathbb{X}$. If $\mathcal{Z}_1 \sim \mathcal{Z}_2$, then $A^*_{\mathcal{Z}_2} = A^*_{\mathcal{Z}_1}$, $A^{\triangleright}_{\mathcal{Z}_2} = A^{\triangleright}_{\mathcal{Z}_1}$, $cl_{\mathcal{Z}_2}(A) = cl_{\mathcal{Z}_1}(A)$, $int_{\mathcal{Z}_2}(A) = int_{\mathcal{Z}_1}(A)$. Consequently, if $\mathcal{I} \sim \mathcal{P} \sim \mathcal{F} \sim \mathcal{G}$, then $A^*_{\mathcal{I}} = A^*_{\mathcal{P}} = A^*_{\mathcal{F}} = A^*_{\mathcal{G}}$.*

Proof. Suppose $\mathcal{Z}_1 \sim \mathcal{Z}_2$. Then $\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2}(\mathcal{Z}_1) = \mathcal{Z}_2$. By Theorem 7, $A^*_{\mathcal{Z}_2} = A^*_{\mathsf{H}^{\mathbb{Z}_1}_{\mathbb{Z}_2}(\mathcal{Z}_1)} = A^*_{\mathcal{Z}_1}$. Other equalities are obvious. $\square$

Remark 1. *It is well known (see, for example, Jankovic and Hamlett [13]) that an ideal topology $\tau_{\mathcal{I}}$ derived from a topology τ on a set X and an ideal $\mathcal{I}$ on X is defined by a Kuratowski closure operator $cl_{\mathcal{I}}(A) = A \cup A^*_{\mathcal{I}}$*

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and $\tau_{\mathcal{I}}$ is finer then τ. A base for $\tau_{\mathcal{I}}$ is described as $\beta(\tau,\mathcal{I}) = \{G \setminus A : G \in \tau, A \in \mathcal{I}\}$ and $\tau_{\mathcal{I}} = \{A \subset X : cl_{\mathcal{I}}(X \setminus A) = X \setminus A\}$.

In the literature we can find many properties of local functions. Designation of operators is different. For example, $A^*_{\mathcal{P}} = A^{\diamond}_{\mathcal{P}}$, $A^{\triangleright}_{\mathcal{P}} = \Psi(A)$ in Al-Omari et al. [2] or $A^*_{\mathcal{G}} = \Phi_{\mathcal{G}}(A)$, $A^{\triangleright}_{\mathcal{G}} = \Gamma_{\mathcal{G}}(A)$ in Roy et al. [31]. We will list some of them below regardless of what system $\mathcal{Z} \in \mathbb{Z}$ they apply to.

Theorem 8. *Let $\mathcal{Z} \in \mathbb{X}$. Then*

(1) $\emptyset^*_{\mathcal{Z}} = \emptyset$,
(2) If $A \subset B$, then $A^*_{\mathcal{Z}} \subset B^*_{\mathcal{Z}}$,
(3) $(A \cup B)^*_{\mathcal{Z}} = A^*_{\mathcal{Z}} \cup B^*_{\mathcal{Z}}$,
(4) $A^*_{\mathcal{Z}} \setminus B^*_{\mathcal{Z}} = (A \setminus B)^*_{\mathcal{Z}} \setminus B^*_{\mathcal{Z}}$,
(5) $(A^*_{\mathcal{Z}})^*_{\mathcal{Z}} \subset A^*_{\mathcal{Z}}$,
(6) If $A \subset B$, then $A^{\triangleright}_{\mathcal{Z}} \subset B^{\triangleright}_{\mathcal{Z}}$,
(7) $A^{\triangleright}_{\mathcal{Z}} \subset (A^{\triangleright}_{\mathcal{Z}})^{\triangleright}_{\mathcal{Z}}$,
(8) $(A \cap B)^{\triangleright}_{\mathcal{Z}} = A^{\triangleright}_{\mathcal{Z}} \cap B^{\triangleright}_{\mathcal{Z}}$.

Proof. Let $\mathcal{I} := \mathsf{H}^{\mathbb{Z}}_{\mathbb{I}}(\mathcal{Z})$. Then $A^*_{\mathcal{I}} = A^*_{\mathsf{H}^{\mathbb{Z}}_{\mathbb{I}}(\mathcal{Z})} = A^*_{\mathcal{Z}}$, by Theorem 7. Since all items hold for $\mathcal{I} \in \mathbb{I}$ (see, for example, Jankovic and Hamlett [13]), they hold for $\mathcal{Z} \in \mathbb{P} \cup \mathbb{F} \cup \mathbb{G}$. □

Similarly, the next theorem holds for $\mathcal{I} \in \mathbb{I}$, so it holds for any $\mathcal{Z} \in \mathbb{X}$.

Theorem 9. *Let $\mathcal{Z} \in \mathbb{X}$. A family $\tau_{\mathcal{Z}} = \{A \subset X : cl_{\mathcal{Z}}(X \setminus A) = X \setminus A\}$ is a topology finer then τ and the next conditions are equivalent.*

(1) $A \in \tau_{\mathcal{Z}}$,
(2) $int_{\mathcal{Z}}(A) = A$,
(3) $A \subset int_{\mathcal{Z}}(A)$,
(4) $A \subset A^{\triangleright}_{\mathcal{Z}}(A)$,
(5) $cl_{\mathcal{Z}}(X \setminus A) = X \setminus A$,
(6) $cl_{\mathcal{Z}}(X \setminus A) \subset X \setminus A$,
(7) $(X \setminus A)^*_{\mathcal{Z}} \subset X \setminus A$.

Theorem 10. *Let $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathbb{X}$. If $\mathcal{Z}_1 \sim \mathcal{Z}_2$, then $\tau_{\mathcal{Z}_1} = \tau_{\mathcal{Z}_2}$. Consequently, if $\mathcal{I} \sim \mathcal{P} \sim \mathcal{F} \sim \mathcal{G}$, then $\tau_{\mathcal{I}} = \tau_{\mathcal{P}} = \tau_{\mathcal{F}} = \tau_{\mathcal{G}}$. A simple base for the open sets of $\tau_{\mathcal{I}}, \tau_{\mathcal{P}}, \tau_{\mathcal{F}}, \tau_{\mathcal{G}}$ is described as follows:*

$\beta(\tau,\mathcal{I}) = \{G \setminus A : G \in \tau, A \in \mathcal{I}\}$,
$\beta(\tau,\mathcal{P}) = \{G \cap A : G \in \tau, A \notin \mathcal{P}\}$,
$\beta(\tau,\mathcal{F}) = \{G \cap A : G \in \tau, A \in \mathcal{F}\}$,
$\beta(\tau,\mathcal{G}) = \{G \setminus A : G \in \tau, A \notin \mathcal{G}\}$, *respectively.*

Proof. The equality $\tau_{\mathcal{Z}_1} = \tau_{\mathcal{Z}_2}$ follows from Lemma 2.

By Remark 1, $\beta(\tau,\mathcal{I}) = \{G \setminus A : G \in \tau, A \in \mathcal{I}\}$ is a base for $\tau_{\mathcal{I}}$. $H \in \beta(\tau,\mathcal{I})$ if and only if $H = G \setminus A = G \cap (X \setminus A)$ where $G \in \tau$ and $A \in \mathcal{I}$ if and only if $H = G \cap B$ where $G \in \tau$ and $B := X \setminus A \notin \mathcal{P}$ (by Theorem 1 (4)), so $H \in \beta(\tau,\mathcal{P})$ if and only if $H = G \cap B$ where $G \in \tau$ and $B \in \mathcal{F}$ (by Theorem 4 (5)), so $H \in \beta(\tau,\mathcal{F})$ if and only if $H = G \setminus A = G \setminus (X \setminus B)$ where $G \in \tau$ and $X \setminus B \notin \mathcal{G}$ (by Theorem 3 (4)), so $H \in \beta(\tau,\mathcal{G})$. □

Definition 4. *A set A is $\mathcal{I}$-small ($\mathcal{P}$-small, $\mathcal{F}$-small, $\mathcal{G}$-small) if $A \in \mathcal{I}$ ($X \setminus A \notin \mathcal{P}$, $X \setminus A \in \mathcal{F}$, $A \notin \mathcal{G}$) and A is locally $\mathcal{I}$-small ($\mathcal{P}$-small, $\mathcal{F}$-small, $\mathcal{G}$-small) if for any $a \in A$ there is a set $U \in \tau$ containing a such that $A \cap U$ is $\mathcal{I}$-small ($\mathcal{P}$-small, $\mathcal{F}$-small, $\mathcal{G}$-small). Let $\mathcal{Z} \in \mathbb{X}$. $\mathcal{Z}$ is said to be compatible with τ if any locally $\mathcal{Z}$-small set is $\mathcal{Z}$-small, denoted by $\mathcal{Z} \sim \tau$.*

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Remark 2. *Let $\mathcal{I} \sim \mathcal{P} \sim \mathcal{F} \sim \mathcal{G}$. Then $\mathcal{I} = \mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P})$, $\mathcal{F} = \mathsf{H}_{\mathbb{F}}^{\mathbb{P}}(\mathcal{P})$ $\mathcal{G} = \mathsf{H}_{\mathbb{G}}^{\mathbb{F}}(\mathcal{F})$. So, $A \in \mathcal{I} \Leftrightarrow X \setminus A \notin \mathcal{P} \Leftrightarrow X \setminus A \in \mathcal{F} \Leftrightarrow A \notin \mathcal{G}$. That means if $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathbb{X}$ and $\mathcal{Z}_1 \sim \mathcal{Z}_2$, a set A is $\mathcal{Z}_1$-small (locally $\mathcal{Z}_1$-small) if and only if A is $\mathcal{Z}_2$-small (locally $\mathcal{Z}_2$-small) and $\mathcal{Z}_1 \sim \tau$ if and only if $\mathcal{Z}_2 \sim \tau$. Consequently,*

(1) *If $\mathcal{I} \sim \tau$, then $\mathsf{H}_{\mathbb{P}}^{\mathbb{I}}(\mathcal{I}) \sim \tau$, $\mathsf{H}_{\mathbb{F}}^{\mathbb{I}}(\mathcal{I}) \sim \tau$, $\mathsf{H}_{\mathbb{G}}^{\mathbb{I}}(\mathcal{I}) \sim \tau$.*
(2) *If $\mathcal{P} \sim \tau$, then $\mathsf{H}_{\mathbb{F}}^{\mathbb{P}}(\mathcal{P}) \sim \tau$, $\mathsf{H}_{\mathbb{G}}^{\mathbb{P}}(\mathcal{P}) \sim \tau$, $\mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P}) \sim \tau$.*
(3) *If $\mathcal{F} \sim \tau$, then $\mathsf{H}_{\mathbb{G}}^{\mathbb{F}}(\mathcal{F}) \sim \tau$, $\mathsf{H}_{\mathbb{I}}^{\mathbb{F}}(\mathcal{F}) \sim \tau$, $\mathsf{H}_{\mathbb{P}}^{\mathbb{F}}(\mathcal{F}) \sim \tau$.*
(4) *If $\mathcal{G} \sim \tau$, then $\mathsf{H}_{\mathbb{I}}^{\mathbb{G}}(\mathcal{G}) \sim \tau$, $\mathsf{H}_{\mathbb{P}}^{\mathbb{G}}(\mathcal{G}) \sim \tau$, $\mathsf{H}_{\mathbb{F}}^{\mathbb{G}}(\mathcal{G}) \sim \tau$.*

In the theory of ideal topological spaces there are several characterizations of compatibility. For $(X, \tau, \mathcal{I})$ the most common equivalent conditions are as follows (see, for example, Hamlett and Jankovic [12], Jankovic and Hamlett [13]).

Theorem 11. *The next are equivalent*

(1) *$\mathcal{I} \sim \tau$,*
(2) *for every $A \subset X$, if $A \cap A^*_{\mathcal{I}} = \emptyset$, then $A \in \mathcal{I}$,*
(3) *for every $A \subset X$, $A \setminus A^*_{\mathcal{I}} \in \mathcal{I}$,*
(4) *for every $A \subset X$, if $(X \setminus A) \cup (X \setminus A)^{\triangleright}_{\mathcal{I}} = X$, then $A \in \mathcal{I}$,*
(5) *for every $A \subset X$, $A^{\triangleright}_{\mathcal{I}} \setminus A \in \mathcal{I}$.*

Regardless of a concept we work in, a compatibility of $\mathcal{Z}_1$ can be characterized by another equivalent system $\mathcal{Z}_2$ and by operators $(\cdot)^*_{\mathcal{Z}_2}$ and $(\cdot)^{\triangleright}_{\mathcal{Z}_2}$.

Theorem 12. *Let $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathbb{X}$. If $\mathcal{Z}_1 \sim \mathcal{Z}_2$, then the next are equivalent*

(1) *$\mathcal{Z}_1 \sim \tau$,*
(2) *for every $A \subset X$, if $A \cap A^*_{\mathcal{Z}_2} = \emptyset$, then A is $\mathcal{Z}_2$-small,*
(3) *for every $A \subset X$, $A \setminus A^*_{\mathcal{Z}_2}$ is $\mathcal{Z}_2$-small,*
(4) *for every $A \subset X$, if $(X \setminus A) \cup (X \setminus A)^{\triangleright}_{\mathcal{Z}_2} = X$, then A is $\mathcal{Z}_2$-small,*
(5) *for every $A \subset X$, $A^{\triangleright}_{\mathcal{Z}_2} \setminus A$ is $\mathcal{Z}_2$-small.*

Proof. Let $\mathcal{I}_1 := \mathsf{H}_{\mathbb{I}}^{\mathbb{Z}_1}(\mathcal{Z}_1)$ where $\mathcal{Z}_1 \in \mathbb{Z}_1 \in \{\mathbb{P}, \mathbb{F}, \mathbb{G}\}$. Since $\mathcal{I}_1 \sim \mathcal{Z}_1 \sim \mathcal{Z}_2$, $\mathcal{Z}_1 \sim \tau$ if and only if $\mathcal{I}_1 \sim \tau$ (by Remark 2), A is $\mathcal{I}_1$-small if and only if A is $\mathcal{Z}_2$-small (by Remark 2), $A^*_{\mathcal{I}_1} = A^*_{\mathcal{Z}_1} = A^*_{\mathcal{Z}_2}$, $A^{\triangleright}_{\mathcal{I}_1} = A^{\triangleright}_{\mathcal{Z}_1} = A^{\triangleright}_{\mathcal{Z}_2}$ (by Lemma 2). Then the next are equivalent (note (ii) $\Leftrightarrow$ (2), (iii) $\Leftrightarrow$ (3), (iv) $\Leftrightarrow$ (4), (v) $\Leftrightarrow$ (5))

(1) $\mathcal{Z}_1 \sim \tau$,
(i) $\mathcal{I}_1 \sim \tau$,
(ii) for every $A \subset X$, if $A \cap A^*_{\mathcal{I}_1} = \emptyset$, then A is $\mathcal{I}_1$-small,
(iii) for every $A \subset X$, $A \setminus A^*_{\mathcal{I}_1}$ is $\mathcal{I}_1$-small,
(iv) for every $A \subset X$, if $(X \setminus A) \cup (X \setminus A)^{\triangleright}_{\mathcal{I}_1} = X$, then A is $\mathcal{I}_1$-small,
(v) for every $A \subset X$, $A^{\triangleright}_{\mathcal{I}_1} \setminus A$ is $\mathcal{I}_1$-small.
(2) for every $A \subset X$, if $A \cap A^*_{\mathcal{Z}_2} = \emptyset$, then A is $\mathcal{Z}_2$-small,
(3) for every $A \subset X$, $A \setminus A^*_{\mathcal{Z}_2}$ is $\mathcal{Z}_2$-small,
(4) for every $A \subset X$, if $(X \setminus A) \cup (X \setminus A)^{\triangleright}_{\mathcal{Z}_2} = X$, then A is $\mathcal{Z}_2$-small,
(5) for every $A \subset X$, $A^{\triangleright}_{\mathcal{Z}_2} \setminus A$ is $\mathcal{Z}_2$-small.

□

Remark 3. *Note, in a primal case (see Al-Omari et al. [2]) a compatibility is called "a topology suitable for a primal". More precisely, we prove τ is suitable for a primal $\mathcal{P}$ if and only if $\mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P})$ is compatible with τ. Proof: τ is suitable for a primal $\mathcal{P}$ if and only if $X \setminus A \notin \mathcal{P}$ whenever $A \cap A^{\diamond}_{\mathcal{P}} = \emptyset$ (see Al-Omatri et al. [2, Theorem 4.2]) if and only if $X \setminus A \notin \mathcal{P}$ whenever $A \cap A^*_{\mathcal{P}} = \emptyset$ if and only if A is $\mathcal{P}$-small whenever $A \cap A^*_{\mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P})} = \emptyset$ if and only if A is $\mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P})$-small whenever $A \cap A^*_{\mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P})} = \emptyset$ if and only if $\mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P})$ is compatible with τ.*

Definition 5. *A set A is $\mathcal{I}$-big ($\mathcal{P}$-big, $\mathcal{F}$-big, $\mathcal{G}$-big) if A is not $\mathcal{I}$-small ($\mathcal{P}$-small, $\mathcal{F}$-small, $\mathcal{G}$-small), equivalently $A \notin \mathcal{I}$ ($X \setminus A \in \mathcal{P}$, $X \setminus A \notin \mathcal{F}$, $A \in \mathcal{G}$). Let $\mathcal{Z} \in \mathbb{Z}$. A topology τ is $\mathcal{Z}$-codense, if any nonempty open set is $\mathcal{Z}$-big.*

Remark 4. *Let $\mathcal{I} \sim \mathcal{P} \sim \mathcal{F} \sim \mathcal{G}$. Then $\mathcal{I} = \mathsf{H}_{\mathbb{I}}^{\mathbb{P}}(\mathcal{P})$, $\mathcal{F} = \mathsf{H}_{\mathbb{F}}^{\mathbb{P}}(\mathcal{P})$, $\mathcal{G} = \mathsf{H}_{\mathbb{G}}^{\mathbb{F}}(\mathcal{F})$. So, $A \notin \mathcal{I} \Leftrightarrow X \setminus A \in \mathcal{P} \Leftrightarrow X \setminus A \notin \mathcal{F} \Leftrightarrow A \in \mathcal{G}$. That means if $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathbb{X}$ and $\mathcal{Z}_1 \sim \mathcal{Z}_2$, a set A is $\mathcal{Z}_1$-big if and only if A is $\mathcal{Z}_2$-big and τ is $\mathcal{Z}_1$-codense if and only if τ is $\mathcal{Z}_2$-codense.*

From the theory of ideal topological spaces we have the next characterization.

Theorem 13. *τ is $\mathcal{I}$-codense if and only if $X_{\mathcal{I}}^* = X$.*

The property of being $\mathcal{Z}_1$-codense can be characterized by the operators $(\cdot)_{\mathcal{Z}_2}^*$ and $(\cdot)_{\mathcal{Z}_2}^{\triangleright}$ with respect to another equivalent system $\mathcal{Z}_2$.

Theorem 14. *Let $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathbb{X}$. If $\mathcal{Z}_1 \sim \mathcal{Z}_2$, then the next are equivalent*

(1) τ *is* $\mathcal{Z}_1$*-codense,*
(2) $X_{\mathcal{Z}_2}^* = X$,
(3) $\emptyset_{\mathcal{Z}_2}^{\triangleright} = \emptyset$.

Proof. Let $\mathcal{I} := \mathsf{H}_{\mathbb{I}}^{\mathbb{Z}_2}(\mathcal{Z}_2)$ where $\mathcal{Z}_2 \in \mathbb{Z}_2 \in \{\mathbb{P}, \mathbb{F}, \mathbb{G}\}$. Then $\mathcal{Z}_2 \sim \mathcal{I}$. By Remark 4 and Theorem 13, τ is $\mathcal{Z}_1$-codense if and only if τ is $\mathcal{Z}_2$-codense if and only if τ is $\mathcal{I}$-codense if and only if $X = X_{\mathcal{I}}^* = X^*_{\mathsf{H}_{\mathbb{I}}^{\mathbb{Z}_2}(\mathcal{Z}_2)} = X_{\mathcal{Z}_2}^*$ (by Theorem 7), so (1) $\Leftrightarrow$ (2). The equivalence (2) $\Leftrightarrow$ (3) follows from equation $\emptyset_{\mathcal{Z}_2}^{\triangleright} = X \setminus X_{\mathcal{Z}_2}^*$. □

Finally, one more application in topology. Many topological properties can be characterized by filters. Equivalent characterization can be done by grills, primals and ideals by this way: a primal $\mathcal{P}$, a grill $\mathcal{G}$, an ideal $\mathcal{I}$ converges to a point x (is an ultraprimal, an ultragrill, an ultraideal) if the corresponding filter $\mathsf{H}_{\mathbb{F}}^{\mathbb{P}}(\mathcal{P})$, $\mathsf{H}_{\mathbb{F}}^{\mathbb{G}}(\mathcal{G})$, $\mathsf{H}_{\mathbb{F}}^{\mathbb{I}}(\mathcal{I})$ converges to x (is ultrafilter), respectively. For example, a topological space is Hausdorff (compact) if and only if each primal, grill, ideal has at most one limit (each ultraprimal, ultragrill, ultraideal converges to at least one point), see Modak at al. [23], Selim et al. [32].

4. Conclusions

The main result of the work is based on the unification of approaches to the creation of new topologies derived from ideals, primals, filters and grills. Each approach has its equivalent counterparts in other approaches, and results valid in one approach can be used in others according to the following scheme: Definitions, theorems and proof methods in a space $(X, \tau, \mathcal{Z})$, $\mathcal{Z} \in \mathbb{X}$, are valid and usable in the other three spaces $(X, \tau, \mathsf{H}_{\mathbb{Z}_1}^{\mathbb{Z}_2}(\mathcal{Z}))$ where $\mathcal{Z} \in \mathbb{Z}_2 \in \{\mathbb{I}, \mathbb{P}, \mathbb{F}, \mathbb{G}\}$ and $\mathbb{Z}_1 \in \{\mathbb{I}, \mathbb{P}, \mathbb{F}, \mathbb{G}\}$, $\mathbb{Z}_1 \neq \mathbb{Z}_2$. From this point of view, many proofs do not need to be done, and it is enough to do them only once in one space. Due to the fact that ideals are the most widespread set systems we can say that the results established in topological spaces with grills, primals and filters are nothing but the results established in ideal topological spaces. On the other hand, such diversity, even if equivalent, can be a stimulus for further research and applications.

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