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Finding All Efficient Solutions of an Interval Multiple Objective Linear Programming Problem

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Abstract: In this paper, we propose a method to find all efficient solutions of an interval multiple objective linear programming (IMOLP) problem with interval coefficients in the objective functions, the constraint matrix and the right-hand side vector. The set of all efficient solutions of an IMOLP problem can completely supply information to the decision makers in finding their most preferred solutions but, up to now, not all efficient solutions of an IMOLP problem can be found by known methods. A numerical example is given to illustrate the working of the proposed methods.

Keywords: Interval multiple objective linear programming; the efficient basic set; the efficient set

MSC: 90C29; 90C50; 90C90

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1. Introduction

Multiple objective linear programming (MOLP) models play an important role in solving and investigating real-life practical problems. There is a practical fact that the exact values of the coefficients of an MOLP model are very difficult to determine but intervals containing them can be easily determined. Thus, practical problems can be described more correctly and more easily by using interval multiple objective linear programming (IMOLP) models than using MOLP models. For brevity of presentation, we shall use the following notation: For two matrices A and B of the same size, $A \le B$ if and only if $a_{ij} \le b_{ij}$, where a_{ij} and b_{ij} are elements of A and B, respectively.

An interval multiple objective linear programming (IMOLP) problem, denoted by $P(IA, IC, ib)$, can be stated as follows:

"maximize" Cx, (1)

$Ax \le b$, (2)

$C \in IC$, $A \in IA$, $b \in ib$, (3)

where $IA = \{A | \underline{A} \le A \le \overline{A}\}$ is an $m \times n$ interval matrix, $IC = \{C | \underline{C} \le C \le \overline{C}\}$ is a $k \times n$ interval matrix, $ib = \{b | \underline{b} \le b \le \overline{b}\}$ is an m interval vector, $\underline{A}$, $\overline{A}$, $\underline{C}$, $\overline{C}$, $\underline{b}$ and $\overline{b}$ are determined. For every $A \in IA$, $C \in IC$ and $b \in ib$ a multiple objective linear programming (MOLP) problem, denoted by $P(A, C, b)$, is obtained from problem (1)-(3). Let $L(A, b)$ and $L(IA, ib)$ be the feasible set of $P(A, C, b)$ and problem (1)-(3), respectively. A point $x \in L(A, b)$ is called *efficient* for $P(A, C, b)$ if there is no $y \in L(A, b)$ such that $Cx \le Cy$ and $Cx \ne Cy$. A feasible point of problem (1)-(3) is called *efficient* for it if there are $A \in IA$, $C \in IC$ and $b \in ib$ such that it is efficient for $P(A, C, b)$. The set of all efficient solutions of a problem $P(A, C, b)$ (of a problem $P(IA, IC, ib)$) is called an *efficient set* of it. Let $E(A, C, b)$ and $E(IA, IC, ib)$ be the efficient sets of $P(A, C, b)$ and $P(IA, IC, ib)$, respectively. It is easily seen that

$$L(IA, ib) = \left\{ x \in R^m \middle| \begin{array}{l} Ax \le b, \\ A \in IA, b \in ib \end{array} \right\}, \quad (4)$$

$$L(IA, ib) = \cup \{ L(A, b) | A \in IA, b \in ib \}, \quad (5)$$

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$E(IA,IC,ib)=\cup\{E(A,C,b)|A\in IA,C\in IC,b\in ib\}$. (6)

The notion of efficient solutions of IMOLP problem (1)-(3) can be found in, for example, Tu [34], Allahdadi and Batamiz [1]. Another name of an efficient solution of IMOLP problem (1)-(3) is *possibly efficient* used in, for example, Bitran [6], Inuiguchi and Kume [17], Oliveira and Antunes [24]. There are many known methods for finding the efficient set of an MOLP problem, see, for example, Yu and Zeleny [40], Isermann [19], Ecker et al. [10], Dauer and Liu [9], Armand and Malivert [3], Armand [2], Sayin [32], Dauer and Gallagher [8], Benson [5], Tu ([34–38]), Yan et al. [39], Foroughi and Jafari [12], Pourkarimi et al. [25], Krichen et al. [21], Tohidi and Hassasi [33], Rudloff et at. [31]. It can be easily seen that all the known methods for finding the efficient set of an MOLP problem, in general, must recompute the efficient set when the data of the MOLP problem is changed. Therefore, they cannot find the efficient set of an IMOLP problem in a general case. Solving an IMOLP problem is much more difficult than solving an MOLP problem. Methods to solve and investigate an IMOLP problem are increasingly developed. Chanas and Kuchta [7], Ishibuchi and Tanaka [20] study interval coefficients only in objective functions, Tu [34] investigates interval coefficients only in the right-hand side vector. Some efficient solutions can be found by methods given by Urli and Nadeau [41] based on an interactive method, by Inuiguchi and Kume [18] based on a goal programming method, by Rivaz and Yaghoobi [27] based on a weighted sum of maximum regrets, Hajiagha et al. [14], Rivaza and Saeidib [26] based on fuzzy programming methods, etc. Efficient extreme points of an IMOLP problem are dealt with in Inuiguchi and Kume [17]. If the coefficients of an IMOLP problem are satisfied probability distributions, then stochastic programming methods can be used to study this problem, see, for example, Batamiz and Allahdadi [4]. Theoretically, an IMOLP problem can be stated based on fuzzy numbers and can be solved by fuzzy programming methods. However, huge difficulties in this way lie in constructing adequate membership functions and finding all efficient solutions of an IMOLP problem.

In this paper, we propose a method to determine all efficient solutions of an IMOLP problem. This method is developed based on the methods of Tu [34,35] and can be easily implemented.

The paper is organized as follows: A method for determining the efficient set of an IMOLP problem is presented in Section 2. An example is given in Section 3 to illustrate the performance of the proposed methods.

2. A Method for Determining the Efficient Set of an IMOLP Problem

2.1. A General Case

Corresponding to IMOLP problem (1)-(3), we consider the following set:

$$\tilde{L}(IA,IC)=\left\{p=(y,z)^T\in R^{m+k}\left| \begin{array}{l} y^TA-z^TC=e^TC, y\ge 0, z\ge 0,\\ A\in IA,\ C\in IC\end{array}\right.\right\}, \quad (7)$$

where $e=(1,...,1)^T\in R^k$. For every $A\in IA$ and $C\in IC$, the following convex polyhedron, denoted by $\tilde{L}(A,C)$, is directly obtained from (7):

$$\tilde{L}(A,C)=\left\{p=(y,z)^T\in R^{m+k}\left|y^TA-z^TC=e^TC, y\ge 0, z\ge 0\right.\right\}.$$

Thus, it is clear that

$\tilde{L}(IA,IC)=\cup\{\tilde{L}(A,C)|A\in IA,C\in IC\}$. (8)

A formula presented in the following property to describe the solution set of an interval linear equation is given by Oettli-Prager [23]:

Property 2.1. $X=\{x\,|\,|A_cx-b_c|\le\Delta|x|+\delta\}$,

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where $X=\left\{x\in R^n \,|\, Ax=b, A\in IA, b\in ib\right\}$, $A_c=(\underline{A}+\overline{A})/2$, $b_c=(\underline{b}+\overline{b})/2$, $\Delta=(\overline{A}-\underline{A})/2$, $\delta=(\overline{b}-\underline{b})/2$, $|x|=(|x_1|,\ldots,|x_n|)^T$, IA *and* ib *are defined in problem* (1)-(3).

This property is proven by Rohn ([29], Theorem 2.1, p.43) in a special case when $n=m$ but his proof can be easily modified to prove Property 2.1. In the case when $x\ge 0$, we get the following property:

Property 2.2. *If* $x\ge 0$, *then* $X=\left\{x\in R^n \,\middle|\, \begin{array}{l}\underline{A}x\le\overline{b},\\ -\overline{A}x\le-\underline{b}\end{array}\right\}$.

Proof. Since $x\ge 0$, $|A_c x-b_c|\le\Delta|x|+\delta$ $\Leftrightarrow |(\overline{A}x-\underline{b})-(\overline{b}-\underline{A}x)|\le(\overline{A}x-\underline{b})+(\overline{b}-\underline{A}x)$ $\Leftrightarrow \begin{cases}\overline{A}x\ge\underline{b}\\ \overline{b}\ge\underline{A}x\end{cases}$.

Therefore, based on Property 2.1 we have $X=\left\{x\in R^n \,\middle|\, \begin{array}{l}\underline{A}x\le\overline{b},\\ -\overline{A}x\le-\underline{b}\end{array}\right\}$. The proof is complete. □

The solution set of an interval linear inequality with non-negative variables is given in the following property:

Property 2.3. $IS(IA,ib)=\left\{x\in R^n \,\middle|\, \begin{array}{l}\underline{A}x+I_m y\le\overline{b},\\ -\overline{A}x-I_m y\le-\underline{b},\\ x\ge 0, y\ge 0\end{array}\right\}$,

where

$$IS(IA,ib)=\left\{x\in R^n \,\middle|\, \begin{array}{l}Ax\le b, x\ge 0,\\ A\in IA, b\in ib\end{array}\right\}$$ *and* I_m *is the unit matrix in* R^m.

Proof. $IS(IA,ib)=\left\{x\in R^n \,\middle|\, \begin{array}{l}Ax\le b, x\ge 0,\\ A\in IA, b\in ib\end{array}\right\}=\left\{x\in R^n \,\middle|\, \begin{array}{l}Ax+I_m y=b,\\ A\in IA, b\in ib,\\ x\ge 0, y\ge 0\end{array}\right\}=\left\{x\in R^n \,\middle|\, \begin{array}{l}Dz=b, z=(x,y)^T\ge 0,\\ D\in ID, b\in ib\end{array}\right\}$

, where $D=(A,I_m)$, $\overline{D}=(\overline{A},I_m)$, $\underline{D}=(\underline{A},I_m)$ and $ID=\left\{D \,|\, \underline{D}\le D\le\overline{D}\right\}$ is an $m\times(m+n)$ interval matrix. Based on Property 2.2, we have $IS(IA,ib)=\left\{x\in R^n \,\middle|\, \begin{array}{l}\underline{A}x+I_m y\le\overline{b},\\ -\overline{A}x-I_m y\le-\underline{b},\\ x\ge 0, y\ge 0\end{array}\right\}$. The proof is complete.

□

Noting that the variables in (7) are non-negative, based on Property 2.2 the following property can be easily obtained:

Property 2.4. $\tilde{L}(IA,IC)=\overline{L}$,

where

$$\overline{L}=\left\{p=(y,z)^T\in R^{m+k} \,\middle|\, \begin{array}{l}y^T\underline{A}-z^T\overline{C}\le e^T\overline{C},\\ -y^T\overline{A}+z^T\underline{C}\le-e^T\underline{C}\\ y\ge 0, z\ge 0\end{array}\right\}.$$

This property is also presented in Li et al. ([22], Theorem 2.5), Rohn [30].

Remark 2.1. Since $\overline{L}$ is a convex polyhedron described by a system of linear inequalities with non-negative variables, $\overline{L}$ has an extreme point if and only if it is not empty.

In order to find extreme points of $\overline{L}$ based on the simplex method, $\overline{L}$ is stated in the following form:

$$\overline{\overline{L}} = \left\{ p = \left(y, y^1, y^2, z\right)^T \in R^{m+2n+k} \middle| \begin{array}{l} y^T \underline{A} + I_n y^1 - z^T \overline{C} = e^T \overline{C}, \\ -y^T \overline{A} + I_n y^2 + z^T \underline{C} = -e^T \underline{C}, \\ y \geq 0, y^1 \geq 0, y^2 \geq 0, z \geq 0 \end{array} \right\}.$$

Let

$$T_0(IA, IC) = \left\{ p \middle| p = \left(y, y^1, y^2, z\right)^T \text{ is an extreme point of } \overline{\overline{L}} \right\},$$

$$T_1(IA, IC) = \left\{ I_+(p) \middle| p \in T_0(IA, IC) \right\},$$

where $I_+(p) = \left\{ i \in \{1,...,m\} \middle| p_i > 0 \right\}$ and p_i is the *i*-th component of *p*;

$T_2(IA, IC)$ is a set consisting of all minimal elements of $T_1(IA, IC)$ by inclusion.

Let $T_2(A, C)$ be a set established based on $\tilde{L}(A, C)$ by a way similar to that used to establish $T_2(IA, IC)$ based on $\overline{\overline{L}}$.

It can be easily seen that the set $T_2(IA, IC)$ can be found by the method given in Tu [35] without determining all extreme points of $\overline{\overline{L}}$.

A relation between $T_2(IA, IC)$ and $T_2(A, C)$ is considered in the following property:

Property 2.5. For every $I \in T_2(A, C)$ there is $J \in T_2(IA, IC)$ such that $J \subseteq I$.

Proof. There is an extreme point of p^0 of $\tilde{L}(A, C)$ such that $I_+\left(p^0\right) = I$. Noting that $p^0 \in \tilde{L}(IA, IC)$, from (8) and Property 2.4 it follows that $p^0 \in \overline{L}$. Based on a proof similar to that of Property 2.4 in [34], it can be easily seen that there is an extreme point p^1 of $\overline{L}$ such that $I_+\left(p^1\right) \subseteq I_+\left(p^0\right)$. From the definition of $T_2(IA, IC)$ it follows that there is $J \in T_2(IA, IC)$ such that $J \subseteq I_+\left(p^1\right)$. Therefore, $J \subseteq I$. □

Let

$$S(I) = \left\{ x \in L(IA, ib) \middle| a_i x = b_i, i \in I \right\},$$

$$S(I, A, b) = \left\{ x \in L(A, b) \middle| a_i x = b_i, i \in I \right\},$$

where (a_i, b_i) is the *i*-th row of a matrix (A, b) defined in (4).

A formula to compute the efficient set of IMOLP problem (1)-(3) is shown in the following property:

Property 2.6. $E = \bigcup_{I \in T_2(IA, IC)} S(I)$.

Proof. For every element $x^0 \in E$ there are $A \in IA$, $C \in IC$ and $b \in ib$ such that $x^0 \in E(A, C, b)$. Based on Property 2.4 in [34], there is $J \in T_2(A, C)$ such that $J \subseteq ID\left(x^0, A, b\right)$, where

$$ID\left(x^0, A, b\right) = \left\{ i \in \{1,...,m\} \middle| a_i x^0 = b_i \right\}.$$

Therefore, $x^0 \in S(J, A, b)$. From Property 2.5 it follows that there is $I \in T_2(IA, IC)$ such that $I \subseteq J$. Thus, $x^0 \in S(J, A, b) \subseteq S(I, A, b) \subseteq S(I)$. Therefore, $E \subseteq \bigcup_{I \in T_2(IA, IC)} S(I)$.

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Conversely, for every element $x^0 \in \bigcup_{I \in T_2(IA,IC)} S(I)$ there is $I \in T_2(IA,IC)$ such that $x^0 \in S(I)$. Based on Property 2.4 and (8), from $I \in T_2(IA,IC)$ it follows that there are $A \in IA$, $C \in IC$ and an extreme point $p^0 = \left(y^0, z^0\right)^T \in \tilde{L}(A,C)$ such that $I_+\left(p^0\right) = I$. It is clear that $x^0 \in S\left(I, A, b^0\right)$, where $b^0 = Ax^0$. From $x^0 \in S\left(I, A, b^0\right)$ it follows that $I \subseteq ID\left(x^0, A, b^0\right)$. Noting that $I_+\left(p^0\right) = I$ and $p^0 = \left(y^0, z^0\right)^T$, it can be easily seen that $y^0 \in \tilde{F}\left(ID\left(x^0, A, b^0\right), x^0\right)$,

where

$$\tilde{F}\left(ID\left(x^0, A, b^0\right), x^0\right) = \left\{ y \in R^m \,\middle|\, \begin{array}{l} y^T A = \left(e + z^0\right)^T C, \\ y_i = 0 \text{ for all } i \in \{1, \ldots, m\} \setminus ID\left(x^0, A, b^0\right), \\ y_i \geq 0 \text{ for all } i \in ID\left(x^0, A, b^0\right) \end{array} \right\}.$$

Thus, based on the complementary theorem of linear programming, x^0 is an optimal solution of the linear programming problem $\max \left\{ \left(e + z^0\right)^T Cx \,\middle|\, x \in L\left(A, b^0\right) \right\}$. Therefore, $x^0 \in E\left(A, C, b^0\right)$. It is clear that $b^0 \in ib$. Therefore, $x^0 \in E$. The proof is complete. □

Since the solution set of an interval linear equation, in general, is not convex, see, for example, Hensen [15], Fiedler et al. [11], Rohn [28,30]. Therefore, the sets $S(I)$ defined in Property 2.6 can be not convex polyhedrons. This can cause difficulties in finding most preferred solutions from the efficient set of IMOLP problem (1)-(3).

Now we consider an IMOLP problem of which the efficient set can be computed by a union of a finite number of convex polyhedrons.

2.2. A Special Case

We consider the following IMOLP problem:

"maximize" Cx, (9)

$A^1 x \leq b^1$, $x \geq 0$, (10)

$C \in IC$, $A^1 \in IA^1$, $b^1 \in ib^1$, (11)

where $IA^1 = \left\{ A^1 \,\middle|\, \underline{A}^1 \leq A^1 \leq \overline{A}^1 \right\}$ is an $m_2 \times n$ interval matrix, $IC = \left\{ \underline{C} \leq C \leq \overline{C} \right\}$ is a $k \times n$ interval matrix, $ib^1 = \left\{ \underline{b}^1 \leq b^1 \leq \overline{b}^1 \right\}$ is an m_2 interval vector. Problem (9)-(11) is a special case of IMOLP problem (1)-(3) because its variables are restricted in sign. IMOLP problem (9)-(11) can be easily solved by the above presented method for solving problem (1)-(3). To do this, we restate problem (9)-(11) in the form of problem (1)-(3) by defining $A = \begin{pmatrix} A^1 \\ -I_n \end{pmatrix} \in R^{(m_2+n)\times n}$, $\underline{A} = \begin{pmatrix} \underline{A}^1 \\ -I_n \end{pmatrix} \in R^{(m_2+n)\times n}$, $\overline{A} = \begin{pmatrix} \overline{A}^1 \\ -I_n \end{pmatrix} \in R^{(m_2+n)\times n}$, $b = \begin{pmatrix} b^1 \\ O_{n\times 1} \end{pmatrix} \in R^{(m_2+n)\times 1}$, $\underline{b} = \begin{pmatrix} \underline{b}^1 \\ O_{n\times 1} \end{pmatrix} \in R^{(m_2+n)\times 1}$, $\overline{b} = \begin{pmatrix} \overline{b}^1 \\ O_{n\times 1} \end{pmatrix} \in R^{(m_2+n)\times 1}$ and $m = m_2 + n$, where I_n is the unit matrix in R^n and $O_{n\times 1}$ is the n column vector with components being 0. Thus, the efficient set of problem (9)-(11), denoted by E_+, can be computed by the formula given in Property 2.6. Now we represent this formula with using the data of problem (9)-(11).

Property 2.7. $E_+ = \bigcup_{I \in T_2(IA,IC)} S_+(I)$,

where

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$$S_+(I)=\left\{x\in R^n\left|\begin{array}{c}-\overline{a}_i^1x\le-\underline{b}_i^1, i\in I\cap\{1,\ldots,m_2\},\\ x_i=0,\ \text{if}\ (i+m_2)\in I\ \text{and}\ i\in\{1,\ldots,n\},\\ \underline{a}_i^1x+y_i\le\overline{b}_i^1, i\in\{1,\ldots,m_2\},\\ -x_{i-m_2}+y_i\le 0, i\in\{m_2+1,\ldots,m_2+n\}\cap I,\\ -\overline{a}_i^1x-y_i\le-\underline{b}_i^1, i\in\{1,\ldots,m_2\}\setminus I,\\ x_{i-m_2}-y_i=0, i\in\{m_2+1,\ldots,m_2+n\}\setminus I,\\ x\ge 0, y\ge 0, y\in R^{m_2+n}\end{array}\right.\right\},$$

$(\overline{a}_i^1,\underline{b}_i^1)$ and $(\underline{a}_i^1,\overline{b}_i^1)$ are the i-th row of the matrix $(\overline{A}^1,\underline{b}^1)$ and $(\underline{A}^1,\overline{b}^1)$, respectively.

Proof. Let $A(I), IA(I), b(I)$ and $ib(I)$ be the matrices obtained from the matrices A, IA, b and ib by dropping rows whose indices are not in I, respectively. Based on Properties 2.2 and 2.3, it can be easily seen that $S(I)=\{x\in L(IA,ib)\,|\,a_ix=b_i, i\in I\}$ = $\left\{x\in R^n\left|\begin{array}{l}a_ix=b_i, i\in I\\ Ax\le b, x\ge 0,\\ A\in IA, b\in ib\end{array}\right.\right\}$ =

$\left\{x\in R^n\left|\begin{array}{l}A(I)x=b(I), x\ge 0,\\ A(I)\in IA(I), b(I)\in ib(I),\\ Ax\le b,\\ A\in IA, b\in ib\end{array}\right.\right\}$ = $\left\{x\in R^n\left|\begin{array}{c}-\overline{a}_ix\le-\underline{b}_i, i\in I,\\ \underline{a}_ix\le\overline{b}_i, i\in I,\\ \underline{A}x+I_my\le\overline{b},\\ -\overline{A}x-I_my\le-\underline{b},\\ x\ge 0, y\ge 0\end{array}\right.\right\}$ ($(\underline{a}_i,\overline{b}_i)$ and $(\overline{a}_i,\underline{b}_i)$ are the i-th rows of the matrices $(\underline{A},\overline{b})$ and $(\overline{A},\underline{b})$, respectively) = $\left\{x\in R^n\left|\begin{array}{c}-\overline{a}_ix\le-\underline{b}_i, i\in I,\\ \underline{A}x+I_my\le\overline{b},\\ -\overline{a}_ix-y_i\le-\underline{b}_i, i\in\{1,\ldots,m\}\setminus I,\\ x\ge 0, y\ge 0\end{array}\right.\right\}$ =

$$\left\{x\in R^n\left|\begin{array}{c}-\overline{a}_i^1x\le-\underline{b}_i^1, i\in I\cap\{1,\ldots,m_2\},\\ x_i=0,\ \text{if}\ (i+m_2)\in I\ \text{and}\ i\in\{1,\ldots,n\},\\ \underline{a}_i^1x+y_i\le\overline{b}_i^1, i\in\{1,\ldots,m_2\},\\ -x_{i-m_2}+y_i\le 0, i\in\{m_2+1,\ldots,m_2+n\},\\ -\overline{a}_i^1x-y_i\le-\underline{b}_i^1, i\in\{1,\ldots,m_2\}\setminus I,\\ x_{i-m_2}-y_i\le 0, i\in\{m_2+1,\ldots,m_2+n\}\setminus I,\\ x\ge 0, y\ge 0, y\in R^{m_2+n}\end{array}\right.\right\}=\left\{x\in R^n\left|\begin{array}{c}-\overline{a}_i^1x\le-\underline{b}_i^1, i\in I\cap\{1,\ldots,m_2\},\\ x_i=0,\ \text{if}\ (i+m_2)\in I\ \text{and}\ i\in\{1,\ldots,n\},\\ \underline{a}_i^1x+y_i\le\overline{b}_i^1, i\in\{1,\ldots,m_2\},\\ -x_{i-m_2}+y_i\le 0, i\in\{m_2+1,\ldots,m_2+n\}\cap I,\\ -\overline{a}_i^1x-y_i\le-\underline{b}_i^1, i\in\{1,\ldots,m_2\}\setminus I,\\ x_{i-m_2}-y_i=0, i\in\{m_2+1,\ldots,m_2+n\}\setminus I,\\ x\ge 0, y\ge 0, y\in R^{m_2+n}\end{array}\right.\right\}.$$

Based on Property 2.6, the proof is complete. □

Remark 2.2. IMOLP problem (9)-(11) is a popular one used in investigating practical problems because the condition of the variables is natural. Since its efficient set can be computed by the union of convex polyhedrons, finding most preferred solutions based on IMOLP problem (9)-(11) has many advantages.

Remark 2.3. Interval linear programming (ILP) problems are extensively investigated by many researchers, for example, Garajova and Hladik [13], Hladik [16]. Since ILP problems are a special case of IMOLP problems, the above presented results also validate for ILP problems.

3. Examples

We consider problem (9)-(11) when

$IC=\begin{pmatrix}[0.5,1.8] & [-0.5,0.5]\\ [0.3,0.8] & [1,1.2]\end{pmatrix}$, $IA^1=\begin{pmatrix}[1.5,2.5] & [0.5,1]\\ [0.5,2] & [3,6]\end{pmatrix}$ and $ib^1=\begin{pmatrix}[6,10]\\ [14,16]\end{pmatrix}$.

This example is recalled from Oliveira and Antunes [24]. It is clear that

$$\bar{\bar{L}}=\left\{p=\left(y,y^1,y^2,z\right)^T\in R^{10}\;\middle|\;\begin{array}{l}1.5y_1+.5y_2-y_3+y_1^1-1.8z_1-.8z_2=2.6,\\ .5y_1+3y_2-y_4+y_2^1-.5z_1-1.2z_2=1.7,\\ -2.5y_1-2y_2+y_3+y_1^2+.5z_1+.3z_2=-.8,\\ -y_1-6y_2+y_4+y_2^2-.5z_1+z_2=-.5,\\ y=(y_1,y_2,y_3,y_4)^T\ge 0,y^1=\left(y_1^1,y_2^1\right)^T\ge 0,\\ y^2=\left(y_1^2,y_2^2\right)^T\ge 0,z=\left(z_1,z_2\right)^T\ge 0\end{array}\right\}.$$

Using Lips-1.9.4 software in MATLAB, based on the method given in Tu [35], we can easily obtain $T_2(IA,IC)=\{\{1\},\{2\}\}$ after solving 4 linear programming problems. Based on Property 2.7, we have $E_+=S_+(\{1\})\cup S_+(\{2\})$, where

$$S_+(\{1\})=\left\{x\in R^2\;\middle|\;\begin{array}{l}-2.5x_1-x_2\le -6,\\ 1.5x_1+.5x_2+y_1\le 10,\\ .5x_1+3x_2+y_2\le 16,\\ -2x_1-6x_2-y_2\le -14,\\ -x_1+y_3=0,\\ -x_2+y_4=0,\\ x_1\ge 0,x_2\ge 0,\left(y_1,y_2,y_3,y_4\right)\ge 0\end{array}\right\},$$

$$S_+(\{2\})=\left\{x\in R^2\;\middle|\;\begin{array}{l}-2x_1+6x_2\le -14,\\ 1.5x_1+.5x_2+y_1\le 10,\\ .5x_1+3x_2+y_2\le 16,\\ -2.5x_1-x_2-y_1\le -6,\\ -x_1+y_3=0,\\ -x_2+y_4=0,\\ x_1\ge 0,x_2\ge 0,\left(y_1,y_2,y_3,y_4\right)\ge 0\end{array}\right\}.$$

4. Conclusions

We propose a method to find all efficient solutions of an IMOLP problem with interval coefficients in objective functions, the constraint matrix and the right-hand side vector. The set of all efficient solutions of an IMOLP problem can completely supply information to the decision makers in finding their most preferred solutions but, up to now, only some solutions of an IMOLP problem can be found by known methods. The proposed methods are simple, is easy to implement and illustrated by a numerical example.

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