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Article

# New Estimates of the Potential Schrödinger Equation

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**Abstract:** We show how the Poincaré–Riemann–Hilbert boundary-value problem enables us to construct effective estimates of the potential in the Schrödinger equation. The apparatus of the three-dimensional inverse problem of quantum scattering theory is developed for this. It is shown that the unitary scattering operator can be studied as a solution of the Poincaré–Riemann–Hilbert boundary-value problem. This allows us to go on to study the potential in the Schrödinger equation

**Keywords:** Schrödinger equation; Poincaré–Riemann–Hilbert boundary-value problem; unitary scattering operator; quantum scattering theory

## 1. Introduction

We show how the Poincaré–Riemann–Hilbert boundary-value problem enables us to construct effective estimates of the potential in the Schrödinger equation. The apparatus of the three-dimensional inverse problem of quantum scattering theory is developed for this. It is shown that the unitary scattering operator can be studied as a solution of the Poincaré–Riemann–Hilbert boundary-value problem. This allows us to go on to study the potential in the Schrödinger equation

## 2. Results for the One-Dimensional Case

Let us consider a one-dimensional function  $f$  and its Fourier transformation  $\tilde{f}$ . Using the notions of module and phase, we write the Fourier transformation in the following form:  $\tilde{f} = |\tilde{f}| \exp(i\Psi)$ , where  $\Psi$  is the phase. The Plancherel equality states that  $\|f\|_{L_2} = \text{const} \|\tilde{f}\|_{L_2}$ . Here we can see that the phase does not contribute to determination of the  $L_2$  norm. To estimate the maximum we make a simple estimate as  $\max|f|^2 \leq 2\|f\|_{L_2}\|\nabla f\|_{L_2}$ . Now we have an estimate of the function maximum in which the phase is not involved. Let us consider the behaviour of a progressing wave travelling with a constant velocity of  $v = a$  described by the function  $F(x, t) = f(x + at)$ . Its Fourier transformation with respect to the variable  $x$  is  $\tilde{F} = \tilde{f} \exp(iatk)$ . Again, in this case, we can see that when we study a module of the Fourier transformation, we will not obtain major physical information about the wave, such as its velocity and location of the wave crest because  $|\tilde{F}| = |\tilde{f}|$ . These two examples show the weaknesses of studying the Fourier transformation. Many researchers focus on the study of functions using the embedding theorem, in which the main object of the study is the module of the function. However, as we have seen in the given examples, the phase is a principal physical characteristic of any process, and as we can see in mathematical studies that use the embedding theorem with energy estimates, the phase disappears. Along with the phase, all reasonable information about the physical process disappears, as demonstrated by Tao [1] and other research studies. In fact, Tao built progressing waves that are not followed by energy estimates. Let us proceed with a more essential analysis of the influence of the phase on the behaviour of functions.

**Theorem 1.** *There are functions of  $W_2^1(R)$  with a constant rate of the norm for a gradient catastrophe for which a phase change of its Fourier transformation is sufficient.*

Proof: To prove this, we consider a sequence of testing functions  $\tilde{f}_n = \Delta/(1+k^2)$ ,  $\Delta = (i-k)^n/(i+k)^n$ . It is obvious that  $|\tilde{f}_n| = 1/(1+k^2)$  and  $\max|f_n|^2 \leq 2\|f_n\|_{L_2}\|\nabla f_n\|_{L_2} \leq \text{const}$ . Calculating the Fourier transformation of these testing functions, we obtain

$$f_n(x) = x(-1)^{(n-1)} 2\pi \exp(-x) L_{(n-1)}^1(2x) \text{ if } x > 0, \quad f_n(x) = 0 \text{ if } x \leq 0, \quad (1)$$

where  $L_{(n-1)}^1(2x)$  is a Laguerre polynomial. Now we see that the functions are equibounded and derivatives of these functions will grow with the growth of  $n$ . Thus, we have built an example of a sequence of the bounded functions of  $W_2^1(R)$  which have a constant norm  $W_2^1(R)$ , and this sequence converges to a discontinuous function.

The results show the flaws of the embedding theorems when analyzing the behavior of functions. Therefore, this work is devoted to overcoming them and the basis for solving the formulated problem is the analytical properties of the Fourier transforms of functions on compact sets. Analytical properties and estimates of the Fourier transform of functions are studied using the Poincaré – Riemann – Hilbert boundary value problem

### 3. Results for the Three-Dimensional Case

Consider Schrödinger's equation:

$$-\Delta_x \Psi + q\Psi = k^2\Psi, k \in C. \quad (2)$$

Let  $\Psi_+(k, \theta, x)$  be a solution of (2) with the following asymptotic behaviour:

$$\Psi_+(k, \theta, x) = \Psi_0(k, \theta, x) + \frac{e^{ik|x|}}{|x|} A(k, \theta', \theta) + o\left(\frac{1}{|x|}\right), \quad |x| \rightarrow \infty, \quad (3)$$

where  $A(k, \theta', \theta)$  is the scattering amplitude and  $\theta' = \frac{x}{|x|}$ ,  $\theta \in S^2$  for  $k \in \bar{C}^+ = \{\text{Im}k \geq 0\}$   $\Psi_0(k, \theta, x) = e^{ik(\theta, x)}$ :

$$A(k, \theta', \theta) = -\frac{1}{4\pi} \int_{R^3} q(x) \Psi_+(k, \theta, x) e^{-ik\theta' \cdot x} dx.$$

Solutions to (2) and (3) are obtained by solving the integral equation

$$\Psi_+(k, \theta, x) = \Psi_0(k, \theta, x) + \int_{R^3} q(y) \frac{e^{+ik|x-y|}}{|x-y|} \Psi_+(k, \theta, y) dy = G(q\Psi_+),$$

which is called the Lippman–Schwinger equation.

Let us introduce

$$\theta, \theta' \in S^2, Df = k \int_{S^2} A(k, \theta', \theta) f(k, \theta') d\theta'.$$

Let us also define the solution  $\Psi_-(k, \theta, x)$  for  $k \in \bar{C}^- = \{\text{Im}k \leq 0\}$  as

$$\Psi_-(k, \theta, x) = \Psi_+(-k, -\theta, x).$$

As is well known [8],

$$\Psi_+(k, \theta, x) - \Psi_-(k, \theta, x) = -\frac{k}{4\pi} \int_{S^2} A(k, \theta', \theta) \Psi_-(k, \theta', x) d\theta', \quad k \in R. \quad (4)$$

This equation is the key to solving the inverse scattering problem and was first used by Newton [8,9] and Somersalo et al. [10].

**Definition 1.** *The set of measurable functions  $\mathbf{R}$  with the norm defined by*

$$\|q\|_{\mathbf{R}} = \int_{R^6} \frac{q(x)q(y)}{|x-y|^2} dx dy < \infty$$

*is recognised as being of Röllnik class.*

Equation (4) is equivalent to the following:

$$\Psi_+ = S\Psi_-,$$

where  $S$  is a scattering operator with the kernel

$$S(k, t) = \int_{R^3} \Psi_+(k, x) \Psi_-^*(t, x) dx.$$

The following theorem was stated in [9]:

**Theorem 2. (Energy and momentum conservation laws)** Let  $q \in \mathbf{R}$ . Then,  $SS^* = I$  and  $S^*S = I$ , where  $I$  is a unitary operator.

**Corollary 1.**  $SS^* = I$  and  $S^*S = I$  yield

$$A(k, \theta', \theta) - A(k, \theta, \theta')^* = \frac{ik}{2\pi} \int_{S^2} A(k, \theta, \theta'') A(k, \theta', \theta'')^* d\theta''.$$

**Theorem 3. (Birman–Schwinger estimation)** Let  $q \in \mathbf{R}$ . Then, the number of discrete eigenvalues can be estimated as

$$N(q) \leq \frac{1}{(4\pi)^2} \int_{R^3} \int_{R^3} \frac{q(x)q(y)}{|x - y|^2} dx dy.$$

**Lemma 1.** Let  $(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}) < \alpha < 1/2$ . Then,

$$\|\Psi_+\|_{L_\infty} \leq \frac{(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)})}{1 - (|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)})} < \frac{\alpha}{1 - \alpha},$$

$$\left\| \frac{\partial(\Psi_+ - \Psi_0)}{\partial k} \right\|_{L_\infty} \leq \frac{|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}}{1 - (|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)})} < \frac{\alpha}{1 - \alpha}.$$

**Proof.** By the Lippman–Schwinger equation, we have

$$|\Psi_+ - \Psi_0| \leq |Gq\Psi_+|,$$

$$|\Psi_+ - \Psi_0|_{L_\infty} \leq |\Psi_+ - \Psi_0|_{L_\infty} |Gq| + |Gq|,$$

and, finally,

$$|\Psi_+ - \Psi_0| \leq \frac{(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)})}{1 - (|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)})}.$$

By the Lippman–Schwinger equation, we also have

$$\left| \frac{\partial(\Psi_+ - \Psi_0)}{\partial k} \right| \leq \left| \frac{\partial Gq}{\partial k} \Psi_+ \right| + \left| Gq \frac{\partial(\Psi_+ - \Psi_0)}{\partial k} \right| + |Gq|,$$

$$\left| \frac{\partial(\Psi_+ - \Psi_0)}{\partial k} \right| \leq (|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}),$$

$$\left\| \frac{\partial(\Psi_+ - \Psi_0)}{\partial k} \right\|_{L_\infty} \leq \frac{|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}}{1 - (|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)})},$$

which completes the proof.  $\square$

Let us introduce the following notation:

$$Q(k, \theta, \theta') = \int_{R^3} q(x) e^{ik(\theta-\theta')x} dx, \quad K(s) = s, \quad X(x) = x,$$

$$T_+ Q = \int_{-\infty}^{+\infty} \frac{Q(s, \theta, \theta')}{s - t - i0} ds, \quad T_- Q = \int_{-\infty}^{+\infty} \frac{Q(s, \theta, \theta')}{s - t + i0} ds.$$

**Lemma 2.** Let  $q \in \mathbf{R} \cap L_1(R^3)$ ,  $\|q\|_{L_1} + 4\pi|q|_{L_2(R^3)} < \alpha < 1/2$ . Then,

$$\|A_+\|_{L_\infty} < \alpha + \frac{\alpha}{1-\alpha},$$

$$\left\| \frac{\partial A_+}{\partial k} \right\|_{L_\infty} < \alpha + \frac{\alpha}{1-\alpha}.$$

**Proof.** Multiplying the Lippman–Schwinger equation by  $q(x)\Psi_0(k, \theta, x)$  and then integrating, we have

$$A(k, \theta, \theta') = Q(k, \theta, \theta') + \int_{R^3} q(x)\Psi_0(k, \theta, x)Gq\Psi_+ dx.$$

We can estimate this latest equation as

$$|A| \leq \alpha + \alpha \frac{\left( |q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)} \right)}{1 - \left( |q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)} \right)}.$$

Following a similar procedure for  $\left\| \frac{\partial A_+}{\partial k} \right\|$  completes the proof.  $\square$

We define the operators  $T_\pm, T$  for  $f \in W_2^1(R)$  as follows:

$$T_+ f = \frac{1}{2\pi i} \lim_{\text{Im} z \rightarrow 0} \int_{-\infty}^{\infty} \frac{f(s)}{s - z} ds, \quad \text{Im } z > 0, \quad T_- f = \frac{1}{2\pi i} \lim_{\text{Im} z \rightarrow 0} \int_{-\infty}^{\infty} \frac{f(s)}{s - z} ds, \quad \text{Im } z < 0,$$

$$Tf = \frac{1}{2}(T_+ + T_-)f.$$

Consider the Riemann problem of finding a function  $\Phi$  that is analytic in the complex plane with a cut along the real axis. Values of  $\Phi$  on the two sides of the cut are denoted as  $\Phi_+$  and  $\Phi_-$ . The following presents the results of [12]:

**Lemma 3.**

$$TT = \frac{1}{4}I, \quad TT_+ = \frac{1}{2}T_+, \quad TT_- = -\frac{1}{2}T_-, \quad T_+ = T + \frac{1}{2}I, \quad T_- = T - \frac{1}{2}I, \quad T_-T_- = -T_-.$$

Denote

$$\Phi_+(k, \theta, x) = \Psi_+(k, \theta, x) - \Psi_0(k, \theta, x), \quad \Phi_-(k, \theta, x) = \Psi_-(k, -\theta, x) - \Psi_0(k, \theta, x),$$

$$g(k, \theta, x) = \Phi_+(k, \theta, x) - \Phi_-(k, \theta, x) /$$

**Lemma 4.** Let  $q \in \mathbf{R}$ ,  $N(q) < 1$ ,  $g_+ = g(k, \theta, x)$ , and  $g_- = g(k, -\theta, x)$ . Then,

$$\Phi_+(k, \theta, x) = T_+g_+ + e^{ik\theta x}, \quad \Phi_-(k, \theta, x) = T_-g_- + e^{ik\theta x}.$$

**Proof.** The proof of the above follows from the classic results for the Riemann problem.  $\square$

**Lemma 5.** Let  $q \in \mathbf{R}$ ,  $N(q) < 1$ ,  $g_+ = g(k, \theta, x)$ , and  $g_- = g(k, -\theta, x)$ . Then,

$$\Psi_+(k, \theta, x) = (T_+g_+ + e^{ik\theta x}), \quad \Psi_-(k, \theta, x) = (T_-g_- + e^{-ik\theta x}).$$

**Proof.** The proof of the above follows from the definitions of  $g$ ,  $\Phi_{\pm}$ , and  $\Psi_{\pm}$ .  $\square$

**Lemma 6.** Let

$$\sup_k \left| \int_{-\infty}^{\infty} \frac{pA(p, \theta', \theta)}{4\pi(p - k + i0)} dp \right| < \alpha, \quad \int_{S_2} \alpha d\theta < 1/2.$$

Then,

$$\prod_{0 \leq j < n} \int_{S_2} \left| \int_{-\infty}^{\infty} \frac{k_j A(k_j, \theta'_{k_j}, \theta_{k_j})}{4\pi(k_{j+1} - k_j + i0)} dk_j \right| d\theta_{k_j} \leq 2^{-n}.$$

**Proof.** Denote

$$\alpha_j = \left| Vp \int_{-\infty}^{\infty} \frac{k_j A(k_j, \theta'_{k_j}, \theta_{k_j})}{4\pi(k_{j+1} - k_j + i0)} dk_j \right|,$$

Therefore,

$$\prod_{0 \leq j < n} \int_{S_2} \left| \int_{-\infty}^{\infty} \frac{k_j A(k_j, \theta'_{k_j}, \theta_{k_j})}{4\pi(k_{j+1} - k_j + i0)} dk_j \right| d\theta_{k_j} \leq \prod_{0 \leq j < n} \int_{S_2} \alpha_j d\theta_{k_j} < 2^{-n}.$$

This completes the proof.  $\square$

**Lemma 7.** Let

$$\sup_k \int_{S^2} |T_- QK| d\theta \leq \alpha < \frac{1}{2C} < 1, \quad \sup_k \int_{S^2} |T_- \tilde{q}K| d\theta \leq \alpha < \frac{1}{2C} < 1,$$

$$\sup_k \int_{S^2} |T_- Q\tilde{q}K^2| d\theta \leq \alpha < \frac{1}{2C} < 1.$$

Then,

$$\sup_k \int_{S^2} |T_- AK| d\theta \leq \frac{C \int_{S^2} |T_- QK| d\theta}{1 - \sup_k \int_{S^2} |T_- A\tilde{q}K^2| d\theta},$$

$$\sup_k \left| \int_{S^2} T_- A\tilde{q}K^2 d\theta \right| \leq \frac{C \left| T_- \int_{S^2} Q\tilde{q}K^2 d\theta \right|}{1 - \left| T_- \int_{S^2} \tilde{q}K d\theta \right|}.$$

**Proof.** By the definition of the amplitude and Lemma 4, we have

$$A(k, \theta', \theta) = -\frac{1}{4\pi} \int_{R^3} q(x) \Psi_+(k, \theta, x) e^{-ik\theta' x} dx$$

$$= -\frac{1}{4\pi} \int_{R^3} q(x) \left[ e^{ik\theta' x} + T_+g(k, \theta, \theta') \right] e^{-ik\theta' x} dx.$$

We can rewrite this as

$$A(k, \theta', \theta) = -\frac{1}{4\pi} \int_{R^3} q(x) \left[ e^{ik\theta x} + \sum_{n \geq 0} (-T_- D)^n \Psi_0 \right] e^{-ik\theta' x} dx. \quad (5)$$

Lemma 6 yields

$$\sup_k \int_{S^2} |T_- A K| d\theta \leq \sup_k \int_{S^2} \left| \frac{1}{4\pi} T_- Q K \right| d\theta + \frac{\left( \sup_k \int_{S^2} |T_- K A| d\theta \right)^2 \int_{S^2} |T_- A \tilde{q} K^2| d\theta}{\left( 1 - \sup_k \int_{S^2} |T_- K A| d\theta \right)^2}.$$

Owing to the smallness of the terms on the right-hand side, the following estimate follows:

$$\sup_k \int_{S^2} |T_- A K| d\theta \leq 2 \sup_k \int_{S^2} \left| \frac{1}{4\pi} T_- Q K \right| d\theta.$$

Similarly,

$$\begin{aligned} \sup_k \int_{S^2} |T_- A \tilde{q} K^2| d\theta &\leq C \int_{S^2} |T_- Q \tilde{q} K^2| d\theta + \int_{S^2} |T_- A \tilde{q} K^2| d\theta \int_{S^2} |T_- \tilde{q} K| d\theta, \\ \sup_k \int_{S^2} |T_- A \tilde{q} K^2| d\theta &\leq \frac{C \int_{S^2} |T_- Q \tilde{q} K^2| d\theta}{1 - \int_{S^2} |T_- \tilde{q} K| d\theta}, \\ \sup_k \int_{S^2} |T_- A \tilde{q} K^2| d\theta &\leq 2 \sup_k \int_{S^2} \left| \frac{1}{4\pi} T_- Q \tilde{q} K^2 \right| d\theta. \end{aligned}$$

This completes the proof.  $\square$

To simplify the writing of the following calculations, we introduce the set defined by

$$M_\epsilon(k) = \left( s | \epsilon < |s| + |k - s| < \frac{1}{\epsilon} \right).$$

The Heaviside function is given by

$$\Theta(x) = \{ 1, \text{ if } x > 0, \quad -1 \text{ if } x < 0 \}.$$

**Lemma 8.** Let  $q, \nabla q \in \cap L_2(R^3)$ ,  $|A| > 0$ . Then,

$$\pi i \int_{R^3} \Theta(A) e^{ik|x|A} q(x) dx = \lim_{\epsilon \rightarrow 0} \int_{s \in M_\epsilon(k)} \int_{R^3} \frac{e^{is|x|A}}{k - s} q(x) dx ds,$$

$$\pi i \int_{R^3} \Theta(A) k e^{ik|x|A} q(x) dx = \lim_{\epsilon \rightarrow 0} \int_{s \in M_\epsilon(k)} \int_{R^3} s \frac{e^{is|x|A}}{k - s} q(x) dx ds.$$

**Proof.** The lemma can be proved by the conditions of lemma and the lemma of Jordan.  $\square$

**Lemma 9.** Let

$$l = 2, \quad I_0 = \Psi_0(x, k)|_{r=r_0}.$$

Then

$$\left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \tilde{q}(k(\theta - \theta')) I_0 k^2 dk d\theta d\theta' \right| \leq \sup_{x \in R^3} |q(x)| + C_0 \left( \frac{1}{r_0} + r_0 \right) \|q\|_{L_2(R^3)},$$

$$\sup_{\theta \in S^2} \left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} QTKQ I_0 k^2 d\theta'' d\theta' dk \right| \leq C_0 \left( \frac{1}{r_0} + r_0 \right) \|q\|_{L_2(R^3)}^2.$$

**Proof.** By the definition of the Fourier transform, we have

$$\int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \tilde{q}(k(\theta - \theta')) I_0 k^2 dk d\theta d\theta' = \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \int_0^{+\infty} q(x) e^{ikx(\theta - \theta')} e^{ix_0 k} k^2 dk d\theta d\theta' dr d\gamma,$$

where  $x = r\gamma$ . The lemma of Jordan completes the proof for the first inequality. The second inequality is proved like the first:

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} QTKQ I_0 k^2 d\theta'' d\theta' dk \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \int_{S^2} \frac{(\tilde{q}(s \cos(\theta') - s \cos(\theta'')) \tilde{q}(k \cos(\theta) - s \cos(\theta'')) s}{k - s} I_0 k^2 d\theta' d\theta'' d\theta dk ds. \end{aligned}$$

Lemma 8 yields

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \int_{S^2} (\tilde{q}(k \cos(\theta') - k \cos(\theta)) \tilde{q}(k \cos(\theta) - k \cos(\theta'')) I_0 k^3 \Theta(\cos(\theta'')) d\theta' d\theta'' d\theta dk - \\ & \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \int_{S^2} (\tilde{q}(k \cos(\theta') - k \cos(\theta)) \tilde{q}(k \cos(\theta) - k \cos(\theta'')) I_0 k^3 \Theta(-\cos(\theta'')) d\theta' d\theta'' d\theta dk. \end{aligned}$$

Integrating  $\theta, \theta', \theta'',$  and  $k$ , we obtain the proof of the second inequality of the lemma.

□

**Lemma 10.** Let

$$\begin{aligned} \sup_k |T_- QK| &\leq \alpha < \frac{1}{2C} < 1, \quad \sup_k |T_- \tilde{q}K| \leq \alpha < \frac{1}{2C} < 1, \\ \sup_k |T_- Q\tilde{q}K^2| &\leq \alpha < \frac{1}{2C} < 1, \quad l = 0, 1, 2. \end{aligned}$$

Then,

$$\begin{aligned} & \left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} A(k, \theta', \theta) k^l dk d\theta' d\theta \right| \leq \left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \tilde{q}(k(\theta - \theta')) k^l dk d\theta' d\theta \right| \\ & \quad + C \sup_{\theta \in S^2} \left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} QTKA k^l d\theta'' d\theta' dk \right|, \\ & \left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} A(k, \theta', \theta) k^2 dk d\theta' d\theta \right| \leq \sup_{x \in R^3} |q| + C_0 \|q\|_{W_2^1(R^3)} \|q\|_{L_2(R^3)} \left( \left| \int_{S^2} TKA d\theta'' \right| + 1 \right). \end{aligned}$$

**Proof.** Using the definition of the amplitude, Lemmas 3 and 4, and the lemma of Jordan yields

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} A(k, \theta', \theta) k^l dk d\theta' d\theta = - \int_{-\infty}^{+\infty} \frac{1}{4\pi} \int_{S^2} \int_{S^2} \int_{R^3} q(x) \Psi_+(k, \theta, x) e^{-ik\theta' x} k^l dx dk d\theta' = \\ & \quad - \frac{1}{4\pi} \int_{S^2} \int_{S^2} \int_{R^3} q(x) \left[ e^{ik\theta x} + \sum_{n \geq 1} (-T_- D)^n \Psi_0 \right] e^{-ik\theta' x} k^l d\theta' dx dk \\ & \quad = \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \tilde{q}(k(\theta - \theta')) k^l dk d\theta' d\theta + \sum_{n \geq 1} W_n, \\ & W_1 = \int_{R^3} \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \frac{s A(s, \theta'', \theta) e^{-ik\theta' x} q(x) e^{is\theta'' x}}{k - s} k^l dk dx ds d\theta' d\theta'', \end{aligned}$$

$$|W_1| \leq C \sup_{\theta \in S^2} \left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} QTKAk^l d\theta'' d\theta' dk \right|.$$

Similarly,

$$|W_n| \leq C \sup_{\theta \in S^2} \left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} QTKAk^l d\theta'' d\theta' dk \right| \left| \int_{S^2} TKAd\theta'' \right|^n.$$

Finally,

$$\begin{aligned} \left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} A(k, \theta', \theta) dk d\theta' d\theta \right| &\leq \left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \tilde{q}(k(\theta - \theta')) dk d\theta' d\theta \right| \\ &\quad + C_0 \|q\|_{L_2(R^3)}^2 \left( \left| \int_{S^2} TKAd\theta'' \right| + 1 \right), \\ \left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} A(k, \theta', \theta) k^2 dk d\theta' \right| &\leq \sup_{x \in R^3} |q| + C_0 \|q\|_{L_2(R^3)}^2 \left( \left| \int_{S^2} TKAd\theta'' \right| + 1 \right). \end{aligned}$$

This completes the proof.  $\square$

**Lemma 11.** Let

$$\sup_k \int_{S^2} \left| \int_{-\infty}^{\infty} \frac{pA(p, \theta', \theta)}{4\pi(p - k + i0)} dp \right| d\theta < \alpha < 1/2, \quad \sup_k |pA(p, \theta', \theta)| < \alpha < 1/2.$$

Then,

$$\begin{aligned} |T_- D\Psi_0| &< \frac{\alpha}{1 - \alpha}, \quad |T_+ D\Psi_0| < \frac{\alpha}{1 - \alpha}, \quad |D\Psi_0| < \frac{\alpha}{1 - \alpha}, \\ T_- g_- &= (I - T_- D)^{-1} T_- D\Psi_0, \quad \Psi_- = (I - T_- D)^{-1} T_- D\Psi_0 + \Psi_0, \end{aligned}$$

and  $q$  satisfies the following inequalities:

$$\sup_{x \in R^3} |q(x)| \leq \left| \int_{S^2} TKQd\theta \right| C_0 \left( \|q\|_{L_2(R^3)}^2 + 1 \right) + C_0 \|q\|_{L_2(R^3)}.$$

**Proof.** Using the equation

$$\Psi_+(k, \theta, x) - \Psi_-(k, \theta, x) = -\frac{k}{4\pi} \int_{S^2} A(k, \theta', \theta) \Psi_-(k, \theta', x) d\theta', \quad k \in R,$$

we can write

$$T_+ g_+ - T_- g_- = D(T_- g_- + \Psi_0).$$

Applying the operator  $T_-$  to the last equation, we have

$$T_- g_- = T_- D(T_- g_- + \Psi_0),$$

$$(I - T_- D)T_- g_- = T_- D\Psi_0, \quad T_- g_- = \sum_{n \geq 0} (-T_- D)^n \Psi_0.$$

Estimating the terms of the series, we obtain using Lemma 4

$$|(T_- D)^n \Psi_0| \leq \sum_{n \geq 0} \left| \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Psi_0 \prod_{0 \leq j < n} \frac{\int_{S^2} k_j A(k_j, \theta'_{k_j}, \theta_{k_j}) d\theta'_{k_j}}{4\pi(k_{j+1} - k_j + i0)} dk_1 \cdots dk_n \right|$$

$$\leq \sum_{n>0} 2^n \alpha^n = \frac{2\alpha}{1-2\alpha}.$$

Denoting

$$\Lambda = \frac{\partial}{\partial k}, \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

we have

$$\Lambda \int_{S^2} \Psi_0 d\theta = \Lambda \frac{\sin(kr)}{ikr} = \frac{\cos(kr)}{ik} - \frac{\sin(kr)}{ik^2 r},$$

$$\Lambda \int_{S^2} H_0 \Psi_0 d\theta = \Lambda k^2 \frac{\sin(kr)}{ikr} = k \frac{\cos(kr)}{i} + \frac{\sin(kr)}{ik^2 r},$$

$$\left| \Lambda \int_{S^2} \Psi d\theta \right| = \left| \Lambda \int_{S^2} \Psi_0 d\theta + \Lambda \int_{S^2} \sum_{n \geq 0} (-T_- D)^n \Psi_0 d\theta \right| > \left( \frac{1}{k} - \frac{\alpha}{1-\alpha} \right), \text{ as } kr = \pi,$$

and

$$\Lambda \frac{1}{k-t} = -\frac{1}{(k-t)^2}$$

Equation (2) yields

$$\begin{aligned} q &= \frac{\Lambda \left( H_0 \int_{S^2} \Psi d\theta + k^2 \int_{S^2} \Psi d\theta \right)}{\Lambda \int_{S^2} \Psi d\theta} \\ &= \frac{2k \int_{S^2} T_- g_- d\theta + k^2 \int_{S^2} \Lambda T_- g_- d\theta + H_0 \Lambda \int_{S^2} T_- g_- d\theta}{\Lambda \int_{S^2} \Psi d\theta} \\ &= \frac{2k \int_{S^2} T_- g_- d\theta + \Lambda \int_{S^2} \sum_{n \geq 1} (-T_- D)^n (K^2 - k^2) \Psi_0 d\theta}{\Lambda \int_{S^2} \Psi d\theta} \\ &= \frac{W_0 + \sum_{n \geq 1} \int_{S^2} W_n}{\Lambda \int_{S^2} \Psi d\theta}. \end{aligned}$$

Denoting

$$Z(k, s) = s + 2k + \frac{2k^2}{k-s},$$

we then have

$$\begin{aligned} |W_1| &\leq \left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} A(s, \theta, \theta') s \frac{s^2 - k^2}{(k-s)^2} \Psi_0 \sin(\theta) ds d\theta \right|_{k=k_0} \\ &\leq \left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} Z(k, ) \tilde{q}(k(\theta - \theta')) \Psi_0 dk d\theta \right| + C_0 \left| \int_{S^2} T K Q d\theta \right|. \end{aligned}$$

For calculating  $W_n$ , as  $n \geq 1$ , take the simple transformation

$$\begin{aligned} \frac{s_n^3}{s_n - s_{n-1}} &= \frac{s_n^3 - s_n^2 s_{n-1}}{s_n - s_{n-1}} + \frac{s_n^2 s_{n-1}}{s_n - s_{n-1}} = s_n^2 + \frac{s_n^2 s_{n-1}}{s_n - s_{n-1}} \\ &= s_n^2 + \frac{s_n^2 s_{n-1} - s_n s_{n-1}^2}{s_n - s_{n-1}} + \frac{s_n s_{n-1}^2}{s_n - s_{n-1}} = s_n^2 + s_n s_{n-1} + \frac{s_n s_{n-1}^2}{s_n - s_{n-1}}, \quad (6) \\ \frac{A s_n^3}{s_n - s_{n-1}} &= A s_n^2 + A s_n s_{n-1} + \frac{A s_n s_{n-1}^2}{s_n - s_{n-1}} = V_1 + V_2 + V_3. \end{aligned}$$

Using Lemma 10 for estimating  $V_1$  and  $V_2$  and, for  $V_3$ , taking again the simple transformation for  $s_{n-1}^3$ , which will appear in the integration over  $s_{n-1}$ , we finally get

$$|q(x)|_{r=r_0} = \left| \frac{\Lambda \left( H_0 \int_{S^2} \Psi d\theta + k^2 \int_{S^2} \Psi d\theta \right)}{\Lambda \int_{S^2} \Psi d\theta} \right|_{k=k_0, r=\frac{\pi}{k_0}}$$

$$\leq \frac{\left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} Z(k, \theta) \tilde{q}(k(\theta - \theta')) \Psi_0 dk d\theta d\theta' \right| + C_0 \left| \int_{S^2} TKQ d\theta \right|}{\left( \frac{1}{k_0} - \frac{\alpha}{(1-\alpha)} \right)} +$$

Finally, we get

$$|q(x)|_{r=r_0} \leq \sup_{x \in R^3} |q(x)| \alpha + C_0 \|q\|_{L_2(R^3)}^2 + C_0 \|q\|_{L_2(R^3)} + \left| \int_{S^2} TKQ d\theta \right|.$$

The invariance of the Schrödinger equations with respect to translations and the arbitrariness of  $r_0$  yield

$$\sup_{x \in R^3} |q(x)| \leq \left| \int_{S^2} TKQ d\theta \right| C_0 \left( \|q\|_{L_2(R^3)}^2 + 1 \right) + C_0 \|q\|_{L_2(R^3)}.$$

□

#### 4. Discussion of the Three-Dimensional Inverse Scattering Problem

This study has shown, once again, the outstanding properties of the scattering operator, which, in combination with the analytical properties of the wave function, allows us to obtain almost-explicit formulas for the potential from the scattering amplitude. Furthermore, this approach. The estimations following from this overcome the problem of overdetermination, resulting from the fact that the potential is a function of three variables, whereas the amplitude is a function of five variables. We have shown that it is sufficient to average the scattering amplitude to eliminate the two extra variables.

#### References

1. Terence Tao, "Finite time blowup for an averaged three-dimensional Navier-Stokes equation," -arXiv:1402.0290 [math.AP]
2. L. D. Faddeev, "The inverse problem in the quantum theory of scattering. II", *Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat.*, 3, VINITI, Moscow, 1974, 93–180
3. CHARLES L. FEFFERMAN Existence and Smoothness of the Navier-Stokes Equation. The Millennium Prize Problems, 57–67, Clay Math. Inst., Cambridge, MA, 2006.
4. J.S.Russell «Report on Waves»: (Report of the fourteenth meeting of the British Association for the Advancement of Science, York, September 1844 (London 1845), pp 311—390, Plates XLVII-LVII)
5. J.S.Russell (1838), Report of the committee on waves, Report of the 7th Meeting of British Association for the Advancement of Science, John Murray, London, pp.417-496.
6. Mark J. Ablowitz, Harvey Segur Solitons and the Inverse Scattering Transform SIAM, 1981- p. 435.
7. N.J.Zabusky and M.D.Kruskal (1965), Interaction of solitons in a collisionless plasma and the recurrence of initial states, *Phys.Rev.Lett.*, 15 pp. 240—243.
8. R.G Newton , *New result on the inverse scattering problem in three dimensions*, *Phys. rev. Lett.* v43, 8, pp.541-542, 1979
9. R.G Newton , *Inverse scattering Three dimensions*, *Jour. Math. Phys.* 21, pp.1698-1715, 1980
10. Somersalo E. et al. *Inverse scattering problem for the Schrödinger's equation in three dimensions: connections between exact and approximate methods.* – 1988.
11. *Tables of integral transforms. v.1* McGraw-Hill Book Company, Inc.1954
12. Poincaré H., *Lecons de mécanique céleste, t. 3, P.*, 1910.

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