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New Estimates of the Potential Schrödinger Equation

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Abstract: We demonstrate how the Poincaré-Riemann-Hilbert boundary-value problem enables us to construct effective estimates of the potential in the Schrödinger equation. For this purpose, the apparatus of the three-dimensional inverse problem of quantum scattering theory is developed. It is shown that the unitary scattering operator can be studied as a solution to the Poincaré-Riemann-Hilbert boundary-value problem. This allows us to proceed with the study of the potential in the Schrödinger equation. This research delves into the new invariants of the Schrödinger equation, illuminating their critically important consequences for theoretical physics and beyond. In particular, it investigates the behavior of scattering amplitudes and bound states depending on the choice of coordinate systems. The study reveals that while the energy of bound states remains invariant under coordinate transformations, the phase of scattering amplitudes undergoes changes, emphasizing its key role in theories dependent on phase normalization. However, these changes possess important properties that allow for effective potential estimates. Additionally, the invariance of the amplitude with respect to translations and rotations will enable other researchers to choose the appropriate coordinate system for their investigations.

Keywords: Poincaré–Riemann–Hilbert boundary-value problem; three-dimensional inverse problem; quantum scattering theory

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1. Introduction

The Schrödinger equation stands as a cornerstone in modeling various physical phenomena, transcending disciplines from quantum mechanics to applied sciences like economics and geophysics. Its solutions provide a profound insight into the behavior of quantum systems and serve as the basis for understanding fundamental principles governing matter and energy. In this pursuit, understanding the nuanced properties of its solutions becomes paramount, as they not only elucidate fundamental theoretical concepts but also have far-reaching implications across diverse scientific domains.

This study embarks on elucidating novel invariants of the Schrödinger equation, emphasizing their profound ramifications for theoretical physics and beyond. While the equation has been extensively studied since its inception, recent advancements have uncovered previously unnoticed symmetries and properties, offering new avenues for exploration and application. Of particular interest is the revelation that bound states' energy remains unaltered irrespective of the chosen coordinate system, underscoring a fundamental symmetry inherent in the equation.

Concurrently, the investigation unravels the intricate relationship between scattering amplitudes and coordinate system choice, highlighting the phase's sensitivity to such variations. Such insights not only deepen our theoretical understanding but also hold practical significance in various fields reliant on accurate phase normalization. These findings not only enrich our understanding of the Schrödinger equation but also open doors to novel applications in fields such as fluid dynamics, quantum mechanics, and medical imaging.

Moreover, beyond its theoretical implications, this research has practical applications that extend into applied sciences. By leveraging these newfound invariants, researchers can construct more robust models for complex physical systems, leading to advancements in areas such as fluid dynamics, material science, and medical imaging. For instance, in seismic exploration, tomography, and ultrasound imaging, these properties allow researchers to optimize phase selection, facilitating the most effective interpretation of measurement results. This strategic phase manipulation ensures clearer insights into subsurface structures, tissue composition, and fluid dynamics, with the flexibility to seamlessly transition back to the original coordinate system post-interpretation.

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Overall, this interdisciplinary synergy underscores the profound impact of fundamental theoretical research on advancing practical applications across diverse scientific domains. By uncovering new invariants and properties of the Schrödinger equation, this study not only contributes to our theoretical understanding of quantum mechanics but also paves the way for innovative solutions to real-world challenges, pushing the boundaries of human knowledge and technological capabilities. We show how the Poincaré–Riemann–Hilbert boundary-value problem enables us to construct effective estimates of the potential in the Schrödinger equation. The apparatus of the three-dimensional inverse problem of quantum scattering theory is developed for this. It is shown that the unitary scattering operator can be studied as a solution of the Poincaré–Riemann–Hilbert boundary-value problem. This allows us to go on to study the potential in the Schrödinger equation This study delves into the Schrödinger equation's new invariants, shedding light on their crucial implications for theoretical physics and beyond. Specifically, it explores the behavior of scattering amplitudes and bound states concerning the choice of coordinate systems. The research unveils that while the energy of bound states remains invariant under coordinate transformations, the phase of scattering amplitudes undergoes variations, underscoring its pivotal role in theories reliant on phase normalization.

Moreover, these findings have played a pivotal role in constructing estimates for three-dimensional Navier-Stokes equations, enhancing our ability to model complex fluid dynamics with greater precision and reliability. Additionally, in seismic exploration, tomography, and ultrasound imaging, these properties allow researchers to optimize phase selection, facilitating the most effective interpretation of measurement results. This strategic phase manipulation ensures clearer insights into subsurface structures, tissue composition, and fluid dynamics, with the flexibility to seamlessly transition back to the original coordinate system post-interpretation. This interdisciplinary synergy underscores the profound impact of fundamental theoretical research on advancing practical applications across diverse scientific domains

2. Problem Formulation

Let us consider a one-dimensional function f and its Fourier transformation $\tilde{f}$. Using the notions of module and phase, we write the Fourier transformation in the following form: $\tilde{f} = |\tilde{f}|\exp(i\Phi)$, where Φ is the phase. The Plancherel equality states that $||f||_{L_2} = \text{const}||\tilde{f}||_{L_2}$. Here we can see that the phase does not contribute to determination of the X norm. To estimate the maximum we make a simple estimate as $\max|f|^2 \leq 2||f||_{L_2}||\nabla f||_{L_2}$. Now we have an estimate of the function maximum in which the phase is not involved. Let us consider the behaviour of a progressing wave travelling with a constant velocity of $v = a$ described by the function $F(x,t) = f(x + at)$. Its Fourier transformation with respect to the variable x is $\tilde{F} = \tilde{f}\exp(iatk)$. Again, in this case, we can see that when we study a module of the Fourier transformation, we will not obtain major physical information about the wave, such as its velocity and location of the wave crest because $|\tilde{F}| = |\tilde{f}|$. These two examples show the weaknesses of studying the Fourier transformation. Many researchers focus on the study of functions using the embedding theorem, in which the main object of the study is the module of the function. However, as we have seen in the given examples, the phase is a principal physical characteristic of any process, and as we can see in mathematical studies that use the embedding theorem with energy estimates, the phase disappears. Along with the phase, all reasonable information about the physical process disappears, as demonstrated by Tao [1] and other research studies. In fact, Tao built progressing waves that are not followed by energy estimates . Let us proceed with a more essential analysis of the influence of the phase on the behaviour of functions.

Theorem 1. *There are functions of $W_2^1(R)$ with a constant rate of the norm for a gradient catastrophe for which a phase change of its Fourier transformation is sufficient.*

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Proof: To prove this, we consider a sequence of testing functions $\tilde{f_n} = \Delta/(1+k^2)$, $\Delta = (i-k)^n/(i+k)^n$. It is obvious that $|\tilde{f_n}| = 1/(1+k^2)$ and $\max|f_n|^2 \leq 2||f_n||_{L_2}||\nabla f_n||_{L_2} \leq$ const. Calculating the Fourier transformation of these testing functions, we obtain

$$f_n(x) = x(-1)^{(n-1)} 2\pi \exp(-x) L^1_{(n-1)}(2x) \text{as } x > 0, \ f_n(x) = 0 \ as \ x \leq 0, \tag{1}$$

where $L^1_{(n-1)}(2x)$ is a Laguerre polynomial. Now we see that the functions are equibounded and derivatives of these functions will grow with the growth of n. Thus, we have built an example of a sequence of the bounded functions of $W_2^1(R)$ which have a constant norm $W_2^1(R)$, and this sequence converges to a discontinuous function.

The results show the flaws of the embedding theorems when analyzing the behavior of functions. Therefore, this work is devoted to overcoming them and the basis for solving the formulated problem is the analytical properties of the Fourier transforms of functions on compact sets. Analytical properties and estimates of the Fourier transform of functions are studied using the Poincaré – Riemann – Hilbert boundary value problem

3. Results

Consider Schrödinger's equation:

$$-H_0\Phi + q\Phi = k^2\Phi, H_0 = \Delta_x, \ k \in C. \tag{2}$$

Let $\Phi_+(k,\theta,x)$ be a solution of (2) with the following asymptotic behaviour:

$$\Phi_+(k,\theta,x) = \Phi_0(k,\theta,x) + \frac{e^{ik|x|}}{|x|} A(k,\theta',\theta) + 0\left(\frac{1}{|x|}\right), \ |x| \to \infty, \tag{3}$$

where $A(k,\theta',\theta)$ is the scattering amplitude and $\theta' = \frac{x}{|x|}$, $\theta \in S^2$ for $k \in \bar{C}^+ = \{\mathrm{Im}k \geq 0\}$ $\Phi_0(k,\theta,x) = e^{ik(\theta,x)}$:

$$A(k,\theta',\theta) = -\frac{1}{4\pi}\int_{R^3} q(x)\Phi_+(k,\theta,x)e^{-ik\theta' x}dx.$$

Solutions to (2) and (3) are obtained by solving the integral equation

$$\Phi_+(k,\theta,x) = \Phi_0(k,\theta,x) + \int_{R^3} q(y)\frac{e^{+ik|x-y|}}{|x-y|}\Phi+(k,\theta,y)dy = G(q\Phi_+),$$

which is called the Lippman–Schwinger equation.

Let us introduce

$$\theta,\theta' \in S^2, Df = k\int_{S^2} A(k,\theta',\theta)f(k,\theta')d\theta'.$$

Let us also define the solution $\Phi_-(k,\theta,x)$ for $k \in \bar{C}^- = \{\mathrm{Im}k \leq 0\}$ as

$$\Phi_-(k,\theta,x) = \Phi_+(-k,-\theta,x).$$

As is well known [2],

$$\Phi_+(k,\theta,x) - \Phi_-(k,\theta,x) = -\frac{k}{4\pi}\int_{S^2} A(k,\theta',\theta)\Phi_-(k,\theta',x)d\theta', \ k \in R. \tag{4}$$

This equation is the key to solving the inverse scattering problem and was first used by Newton [2,3] and Somersalo et al. [4].

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Definition 1. *The set of measurable functions* **R** *with the norm defined by*

$$||q||_{\mathbf{R}} = \int_{R^6} \frac{q(x)q(y)}{|x-y|^2} dxdy < \infty$$

is recognised as being of Rollnik class.

Equation (4) is equivalent to the following:

$$\Phi_+ = S\Phi_-,$$

where S is a scattering operator with the kernel

$$S(k, ł) = \int_{R^3} \Phi_+(k,x)\Phi_-^*(ł,x)dx.$$

The following theorem was stated in [3]:

Theorem 2. ***(Energy and momentum conservation laws)*** *Let* $q \in \mathbf{R}$*. Then,* $SS^* = I$ *and* $S^*S = I$*, where* I *is a unitary operator.*

Corollary 1. $SS^* = I$ *and* $S^*S = I$ *yield*

$$A(k,\theta',\theta) - A(k,\theta,\theta')^* = \frac{ik}{2\pi} \int_{S^2} A(k,\theta,\theta'')A(k,\theta',\theta'')^* d\theta''.$$

Theorem 3. ***(Birmann–Schwinger estimation)*** *Let* $q \in \mathbf{R}$*. Then, the number of discrete eigenvalues can be estimated as*

$$N(q) \leq \frac{1}{(4\pi)^2} \int_{R^3} \int_{R^3} \frac{q(x)q(y)}{|x-y|^2} dxdy.$$

Lemma 1. *Let* $\left(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}\right) < \alpha < 1/2$*. Then,*

$$\|\Phi_+\|_{L_\infty} \leq \frac{\left(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}\right)}{1 - \left(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}\right)} < \frac{\alpha}{1-\alpha},$$

$$\left\|\frac{\partial(\Phi_+ - \Phi_0)}{\partial k}\right\|_{L_\infty} \leq \frac{|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}}{1 - \left(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}\right)} < \frac{\alpha}{1-\alpha}.$$

Proof. By the Lippman–Schwinger equation, we have

$$|\Phi_+ - \Phi_0| \leq |Gq\Phi_+|,$$

$$|\Phi_+ - \Phi_0|_{L_\infty} \leq |\Phi_+ - \Phi_0|_{L_\infty}|Gq| + |Gq|,$$

and, finally,

$$|\Phi_+ - \Phi_0| \leq \frac{\left(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}\right)}{1 - \left(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}\right)}.$$

By the Lippman–Schwinger equation, we also have

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$$\left|\frac{\partial(\Phi_+ - \Phi_0)}{\partial k}\right| \leq \left|\frac{\partial Gq}{\partial k}\Phi_+\right| + \left|Gq\frac{\partial(\Phi_+ - \Phi_0)}{\partial k}\right| + |Gq|,$$

$$\left|\frac{\partial(\Phi_+ - \Phi_0)}{\partial k}\right| \leq \left(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}\right),$$

$$\left\|\frac{\partial(\Phi_+ - \Phi_0)}{\partial k}\right\|_{L_\infty} \leq \frac{|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}}{1 - \left(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}\right)},$$

which completes the proof. □

Let us introduce the following notation:

$$A_0(k,\theta,\theta') = \int_{R^3} q(x)e^{ik(\theta-\theta')x}dx, \ K(s) = s, \ X(x) = x,$$

$$\mathcal{H}_+A_0 = \int_{-\infty}^{+\infty} \frac{A_0(s,\theta,\theta')}{s-t-i0}ds, \ \mathcal{H}_-A_0 = \int_{-\infty}^{+\infty} \frac{A_0(s,\theta,\theta')}{s-t+i0}ds.$$

Lemma 2. *Let* $q \in \mathbf{R} \cap L_1(R^3)$, $\|q\|_{L_1} + 4\pi|q|_{L_2(R^3)} < \alpha < 1/2$. *Then,*

$$\|A_+\|_{L_\infty} < \alpha + \frac{\alpha}{1-\alpha},$$

$$\left\|\frac{\partial A_+}{\partial k}\right\|_{L_\infty} < \alpha + \frac{\alpha}{1-\alpha}.$$

Proof. Multiplying the Lippman–Schwinger equation by $q(x)\Phi_0(k,\theta,x)$ and then integrating, we have

$$A(k,\theta,\theta') = A_0(k,\theta,\theta') + \int_{R^3} q(x)\Phi_0(k,\theta,x)Gq\Phi_+dx.$$

We can estimate this latest equation as

$$|A| \leq \alpha + \alpha\frac{\left(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}\right)}{1 - \left(|q|_{L_1(R^3)} + 4\pi|q|_{L_2(R^3)}\right)}.$$

Following a similar procedure for $\left\|\frac{\partial A_+}{\partial k}\right\|$ completes the proof. □

We define the operators $\mathcal{H}_\pm, \mathcal{H}$ for $f \in W_2^1(R)$ as follows:

$$\mathcal{H}_+f = \frac{1}{2\pi i}\lim_{\text{Im}z\to 0}\int_{-\infty}^{\infty}\frac{f(s)}{s-z}ds, \ \text{Im}\, z > 0, \ \mathcal{H}_-f = \frac{1}{2\pi i}\lim_{\text{Im}z\to 0}\int_{-\infty}^{\infty}\frac{f(s)}{s-z}ds, \ \text{Im}\, z < 0,$$

$$\mathcal{H}f = \frac{1}{2}(\mathcal{H}_+ + \mathcal{H}_-)f.$$

Consider the Riemann problem of finding a function Φ that is analytic in the complex plane with a cut along the real axis. Values of Φ on the two sides of the cut are denoted as Φ_+ and Φ_-. The following presents the results of [5]:

Lemma 3.

$$\mathcal{H}\mathcal{H} = \frac{1}{4}I, \ \mathcal{H}\mathcal{H}_+ = \frac{1}{2}\mathcal{H}_+, \ \mathcal{H}\mathcal{H}_- = -\frac{1}{2}\mathcal{H}_-, \ \mathcal{H}_+ = \mathcal{H} + \frac{1}{2}I, \ \mathcal{H}_- = \mathcal{H} - \frac{1}{2}I, \ T_-T_- = -T_-.$$

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Denote

$$\Phi_+(k,\theta,x) = \Phi_+(k,\theta,x) - \Phi_0(k,\theta,x),\ \ \Phi_-(k,\theta,x) = \Phi_-(k,-\theta,x) - \Phi_0(k,\theta,x),$$

$$g(k,\theta,x) = \Phi_+(k,\theta,x) - \Phi_-(k,\theta,x)/$$

Lemma 4. *Let $q \in \mathbf{R}$, $N(q) < 1$, $g_+ = g(k,\theta,x)$, and $g_- = g(k,-\theta,x)$. Then,*

$$\Phi_+(k,\theta,x) = \mathcal{H}_+ g_+ + e^{ik\theta x},\ \Phi_-(k,\theta,x) = \mathcal{H}_- g_+ + e^{ik\theta x}.$$

Proof. The proof of the above follows from the classic results for the Riemann problem. □

Lemma 5. *Let $q \in \mathbf{R}$, $N(q) < 1$, $g_+ = g(k,\theta,x)$, and $g_- = g(k,-\theta,x),)$. Then,*

$$\Phi_+(k,\theta,x) = (\mathcal{H}_+ g_+ + e^{ik\theta x}),\ \Phi_-(k,\theta,x) = (\mathcal{H}_- g_- + e^{-ik\theta x}).$$

Proof. The proof of the above follows from the definitions of g, $\Phi_\pm$, and $\Phi_\pm$. □

Lemma 6. *Let*

$$\sup_k \left| \int\limits_{-\infty}^{\infty} \frac{pA(p,\theta',\theta)}{4\pi(p-k+i0)} dp \right| < \alpha,\ \int_{S_2} \alpha d\theta < 1/2.$$

Then,

$$\prod_{0\le j<n} \int_{S_2} \left| \int_{-\infty}^{\infty} \frac{k_j A(k_j,\theta'_{k_j},\theta_{k_j})}{4\pi(k_{j+1}-k_j+i0)} dk_j \right| d\theta_{k_j} \le 2^{-n}.$$

Proof. Denote

$$\alpha_j = \left| Vp \int_{-\infty}^{\infty} \frac{k_j A(k_j,\theta'_{k_j},\theta_{k_j})}{4\pi(k_{j+1}-k_j+i0)} dk_j \right|,$$

Therefore,

$$\prod_{0\le j<n} \int_{S_2} \left| \int_{-\infty}^{\infty} \frac{k_j A(k_j,\theta'_{k_j},\theta_{k_j})}{4\pi(k_{j+1}-k_j+i0)} dk_j \right| d\theta_{k_j} \le \prod_{0\le j<n} \int_{S_2} \alpha_j d\theta_{k_j} < 2^{-n}.$$

This completes the proof. □

Lemma 7. *Let*

$$\sup_k \int_{S^2} |\mathcal{H}_- A_0 K| d\theta \le \alpha < \frac{1}{2C} < 1,\ \ \sup_k \int_{S^2} |\mathcal{H}_- \tilde{q} K| d\theta \le \alpha < \frac{1}{2C} < 1,$$

$$\sup_k \int_{S^2} \left| \mathcal{H}_- A_0 \tilde{q} K^2 \right| d\theta \le \alpha < \frac{1}{2C} < 1.$$

Then,

$$\sup_k \int_{S^2} |\mathcal{H}_- A K| d\theta \le \frac{C \int_{S^2} |\mathcal{H}_- A_0 K| d\theta}{1 - \sup_k \int_{S^2} |\mathcal{H}_- A \tilde{q} K^2| d\theta},$$

$$\sup_k \left| \int_{S^2} \mathcal{H}_- A \tilde{q} K^2 d\theta \right| \le \frac{C \left| \mathcal{H}_- \int_{S^2} A_0 \tilde{q} K^2 d\theta \right|}{1 - \left| \mathcal{H}_- \int_{S^2} \tilde{q} K d\theta \right|}.$$

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Proof. By the definition of the amplitude and Lemma 4, we have

$$A(k,\theta',\theta) = -\frac{1}{4\pi}\int_{R^3} q(x)\Phi_+(k,\theta,x)e^{-ik\theta' x}dx$$

$$= -\frac{1}{4\pi}\int_{R^3} q(x)\left[e^{ik\theta' x} + \mathcal{H}_+ g(k,\theta,\theta')\right]e^{-ik\theta' x}dx.$$

We can rewrite this as

$$A(k,\theta',\theta) = -\frac{1}{4\pi}\int_{R^3} q(x)\left[e^{ik\theta x} + \sum_{n\geq 0}(-\mathcal{H}_- D)^n\Phi_0\right]e^{-ik\theta' x}dx. \quad (5)$$

Lemma 6 yields

$$\sup_k \int_{S^2}|\mathcal{H}_- AK|d\theta \leq \sup_k \int_{S^2}\left|\frac{1}{4\pi}\mathcal{H}_- A_0 K\right|d\theta + \frac{\left(\sup_k \int_{S^2}|\mathcal{H}_- KA|d\theta\right)^2 \int_{S^2}|\mathcal{H}_- A\tilde{q}K^2|d\theta}{\left(1-\sup_k \int_{S^2}|\mathcal{H}_- KA|d\theta\right)^2}.$$

Owing to the smallness of the terms on the right-hand side, the following estimate follows:

$$\sup_k \int_{S^2}|\mathcal{H}_- AK|d\theta \leq 2\sup_k \int_{S^2}\left|\frac{1}{4\pi}\mathcal{H}_- A_0 K\right|d\theta.$$

Similarly,

$$\sup_k \int_{S^2}\left|\mathcal{H}_- A\tilde{q}K^2\right|d\theta \leq C\int_{S^2}\left|\mathcal{H}_- A_0\tilde{q}K^2\right|d\theta + \int_{S^2}\left|\mathcal{H}_- A\tilde{q}K^2\right|d\theta \int_{S^2}|\mathcal{H}_-\tilde{q}K|d\theta,$$

$$\sup_k \int_{S^2}\left|\mathcal{H}_- A\tilde{q}K^2\right|d\theta \leq \frac{C\int_{S^2}|\mathcal{H}_- A_0\tilde{q}K^2|d\theta}{1-\int_{S^2}|\mathcal{H}_-\tilde{q}K|d\theta},$$

$$\sup_k \int_{S^2}\left|\mathcal{H}_- A\tilde{q}K^2\right|d\theta \leq 2\sup_k \int_{S^2}\left|\frac{1}{4\pi}\mathcal{H}_- A_0\tilde{q}K^2\right|d\theta.$$

This completes the proof. □

To simplify the writing of the following calculations, we introduce the set defined by

$$M_\epsilon(k) = \left(s|\epsilon < |s| + |k-s| < \frac{1}{\epsilon}\right).$$

The Heaviside function is defined as:

$$\theta(x) = \begin{cases} 1 & \text{if } x>0, \\ 0.5 & \text{if } x=0, \\ 0 & \text{if } x<0. \end{cases}$$

Lemma 8. *Let* $q, \nabla q \in \cap L_2(R^3)$, $|A|>0$. *Then,*

$$\pi i\int_{R^3}\theta(A)e^{ik|x|A}q(x)dx = \lim_{\epsilon\to 0}\int_{s\in M_\epsilon(k)}\int_{R^3}\frac{e^{is|x|A}}{k-s}q(x)dxds,$$

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$$\pi i \int_{R^3} \theta(A) k e^{ik|x|A} q(x) dx = \lim_{\epsilon \to 0} \int_{s \in M_\epsilon(k)} \int_{R^3} s \frac{e^{is|x|A}}{k-s} q(x) dx ds.$$

Proof. The lemma can be proved by the conditions of lemma and the lemma of Jordan. □

Lemma 9. *Let*

$$I_0 = \Phi_0(x,k)|_{r=r_0}.$$

Then

$$\left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \tilde{q}(k(\theta-\theta')) I_0 k^2 dk d\theta d\theta' \right| \le \sup_{x \in R^3} |q(x)| + C_0(\frac{1}{r_0} + r_0) \|q\|_{L_2(R^3)},$$

$$\sup_{\theta \in S^2} \left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} A_0 \mathcal{H} K A_0 I_0 k^2 d\theta'' d\theta' dk \right| \le C_0(\frac{1}{r_0} + r_0) \|q\|^2_{L_2(R^3)}.$$

Proof. By the definition of the Fourier transform, we have

$$\int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \tilde{q}(k(\theta-\theta')) I_0 k^2 dk d\theta d\theta' = \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \int_0^{+\infty} q(x) e^{ikx(\theta-\theta')} e^{ix_0 k} k^2 dk d\theta d\theta' dr d\gamma,$$

where $x = r\gamma$ The lemma of Jordan completes the proof for the first inequality. The second inequality is proved like the first:

$$\int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} A_0 \mathcal{H} K A_0 I_0 k^2 d\theta'' d\theta' dk$$

$$= V.P \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \int_{S^2} \frac{(\tilde{q}(s\cos(\theta') - s\cos(\theta''))\tilde{q}(k\cos(\theta) - s\cos(\theta''))s}{k-s} I_0 k^2 d\theta' d\theta'' d\theta dk ds.$$

Lemma 8 yields

$$\int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \int_{S^2} (\tilde{q}(k\cos(\theta') - k\cos(\theta))\tilde{q}(k\cos(\theta) - k\cos(\theta'')) I_0 k^3 \theta(\cos(\theta'')) d\theta' d\theta'' d\theta dk -$$

$$\int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \int_{S^2} (\tilde{q}(k\cos(\theta') - k\cos(\theta))\tilde{q}(k\cos(\theta) - k\cos(\theta'')) I_0 k^3 \theta(-\cos(\theta'')) d\theta' d\theta'' d\theta dk.$$

Integrating θ, θ', θ'', and k, we obtain the proof of the second inequality of the lemma. □

Lemma 10. *Let*

$$\sup_k |\mathcal{H}_- A_0 K| \le \alpha < \frac{1}{2C} < 1, \quad \sup_k |\mathcal{H}_- \tilde{q} K| \le \alpha < \frac{1}{2C} < 1,$$

$$\sup_k \left| \mathcal{H}_- A_0 \tilde{q} K^2 \right| \le \alpha < \frac{1}{2C} < 1, \quad l = 0,1,2.$$

Then,

$$\left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} A(k,\theta',\theta) k^l dk d\theta' d\theta \right| \le \left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} \tilde{q}(k(\theta-\theta')) k^l dk d\theta' d\theta \right|$$

$$+ C \sup_{\theta \in S^2} \left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} A_0 \mathcal{H} K A k^l d\theta'' d\theta' dk \right|,$$

$$\left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} A(k,\theta',\theta) k^2 dk d\theta' d\theta \right| \le \sup_{x \in R^3} |q| + C_0 \|q\|_{W_2^1(R^3)} \|q\|_{L_2(R^3)} \left(\left| \int_{S^2} \mathcal{H} K A d\theta'' \right| + 1 \right).$$

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Proof. Using the definition of the amplitude, Lemmas 3 and 4, and the lemma of Jordan yields

$$\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2} A(k,\theta',\theta)k^l dkd\theta' d\theta = -\int_{-\infty}^{+\infty}\frac{1}{4\pi}\int_{S^2}\int_{S^2}\int_{R^3} q(x)\Phi_+(k,\theta,x)e^{-ik\theta' x}k^l dxdkd\theta' =$$

$$-\frac{1}{4\pi}\int_{S^2}\int_{S^2}\int_{R^3} q(x)\left[e^{ik\theta x}+\sum_{n\geq 1}(-\mathcal{H}_- D)^n\Phi_0\right]e^{-ik\theta' x}k^l d\theta' dxdk$$

$$= \int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2}\tilde{q}(k(\theta-\theta'))k^l dkd\theta' d\theta + \sum_{n\geq 1} W_n,$$

$$W_1 = V.P\int_{R^3}\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2}\frac{sA(s,\theta'',\theta)e^{-ik\theta' x}q(x)e^{is\theta'' x}}{k-s}k^l dkdxdsd\theta' d\theta'',$$

$$|W_1| \leq C\sup_{\theta\in S^2}\left|\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2} A_0\mathcal{H}KAk^l d\theta'' d\theta' dk\right|.$$

Similarly,

$$|W_n| \leq C\sup_{\theta\in S^2}\left|\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2} A_0\mathcal{H}KAk^l d\theta'' d\theta' dk\right|\left|\int_{S^2}\mathcal{H}KAd\theta''\right|^n.$$

Finally,

$$\left|\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2} A(k,\theta',\theta)dkd\theta' d\theta\right| \leq \left|\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2}\tilde{q}(k(\theta-\theta'))dkd\theta d\theta'\right|$$

$$+C_0\|q\|^2_{L_2(R^3)}\left(\left|\int_{S^2}\mathcal{H}KAd\theta''\right|+1\right),$$

$$\left|\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2} A(k,\theta',\theta)k^2 dkd\theta'\right| \leq \sup_{x\in R^3}|q| + C_0\|q\|^2_{L_2(R^3)}\left(\left|\int_{S^2}\mathcal{H}KAd\theta''\right|+1\right).$$

This completes the proof. □

Lemma 11. *Let*

$$\sup_k\int_{S^2}\left|\int_{-\infty}^{\infty}\frac{pA(p,\theta',\theta)}{4\pi(p-k+i0)}dp\right|d\theta < \alpha < 1/2,\quad \sup_k\left|pA(p,\theta',\theta)\right| < \alpha < 1/2.$$

Then,

$$|\mathcal{H}_- D\Phi_0| < \frac{\alpha}{1-\alpha},\quad |\mathcal{H}_+ D\Phi_0| < \frac{\alpha}{1-\alpha},\quad |D\Phi_0| < \frac{\alpha}{1-\alpha},$$

$$\mathcal{H}_- g_- = (I-\mathcal{H}_- D)^{-1}\mathcal{H}_- D\Phi_0,\qquad \Phi_- = (I-\mathcal{H}_- D)^{-1}\mathcal{H}_- D\Phi_0 + \Phi_0,$$

and q satisfies the following inequalities:

$$\sup_{x\in R^3}|q(x)| \leq \left|\int_{S^2}\mathcal{H}KA_0 d\theta\right| C_0\left(\|q\|^2_{L_2(R^3)}+1\right) + C_0\|q\|_{L_2(R^3)}.$$

Proof. Using the equation

$$\Phi_+(k,\theta,x) - \Phi_-(k,\theta,x) = -\frac{k}{4\pi}\int_{S^2} A(k,\theta',\theta)\Phi_-(k,\theta',x)d\theta',\ k\in R,$$

we can write

$$\mathcal{H}_+ g_+ - \mathcal{H}_- g_- = D(\mathcal{H}_- g_- + \Phi_0).$$

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Applying the operator $\mathcal{H}_-$ to the last equation, we have

$$\mathcal{H}_-g_- = \mathcal{H}_-D(\mathcal{H}_-g_- + \Phi_0),$$

$$(I - \mathcal{H}_-D)\mathcal{H}_-g_- = \mathcal{H}_-D\Phi_0,\ \ \mathcal{H}_-g_- = \sum_{n\geq 0}(-\mathcal{H}_-D)^n\Phi_0.$$

Estimating the terms of the series, we obtain using Lemma 4

$$|(\mathcal{H}_-D)^n\Phi_0| \leq \sum_{n\geq 0}\left|\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Phi_0\prod_{0\leq j<n}\frac{\int_{S^2}k_jA(k_j,\theta'_{k_j},\theta_{k_j})d\theta'_{k_j}}{4\pi(k_{j+1})-k_j+i0)}dk_1\ldots d_{k_n}\right|$$

$$\leq \sum_{n>0}2^n\alpha^n = \frac{2\alpha}{1-2\alpha}.$$

Denoting

$$\Lambda = \frac{\partial}{\partial k},\ r = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

we have

$$\Lambda\int_{S^2}\Phi_0 d\theta = \Lambda\frac{\sin(kr)}{ikr} = \frac{\cos(kr)}{ik} - \frac{\sin(kr)}{ik^2r},$$

$$\Lambda\int_{S^2}H_0\Phi_0 d\theta = \Lambda k^2\frac{\sin(kr)}{ikr} = k\frac{\cos(kr)}{i} + \frac{\sin(kr)}{ik^2r},$$

$$\left|\Lambda\int_{S^2}\Phi d\theta\right| = \left|\Lambda\int_{S^2}\Phi_0 d\theta + \Lambda\int_{S^2}\sum_{n\geq 0}(-\mathcal{H}_-D)^n\Phi_0 d\theta\right| > \left(\frac{1}{k} - \frac{\alpha}{1-\alpha}\right),\text{ as } kr = \pi,$$

and

$$\Lambda\frac{1}{k-t} = -\frac{1}{(k-t)^2}$$

Equation (2) yields

$$q = \frac{\Lambda\left(H_0\int_{S^2}\Phi d\theta + k^2\int_{S^2}\Phi d\theta\right)}{\Lambda\int_{S^2}\Phi d\theta}$$

$$= \frac{2k\int_{S^2}\mathcal{H}_-g_-d\theta + k^2\int_{S^2}\Lambda\mathcal{H}_-g_-d\theta + H_0\Lambda\int_{S^2}\mathcal{H}_-g_-d\theta}{\Lambda\int_{S^2}\Phi d\theta}$$

$$= \frac{2k\int_{S^2}\mathcal{H}_-g_-d\theta + \Lambda\int_{S^2}\sum_{n\geq 1}(-\mathcal{H}_-D)^n(K^2-k^2)\Phi_0 d\theta}{\Lambda\int_{S^2}\Phi d\theta}$$

$$= \frac{W_0 + \sum_{n\geq 1}\int_{S^2}W_n}{\Lambda\int_{S^2}\Phi d\theta}.$$

Denoting

$$Z(k,s) = s + 2k + \frac{2k^2}{k-s},$$

we then have

$$|W_1| \leq \left|\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2}A(s,\theta,\theta')s\frac{s^2-k^2}{(k-s)^2}\Phi_0\sin(\theta)dsd\theta\right|_{k=k_0}$$

$$\leq \left|\int_{-\infty}^{+\infty}\int_{S^2}\int_{S^2}Z(k,)\tilde{q}(k(\theta-\theta'))\Phi_0 dkd\theta\right| + C_0\left|\int_{S^2}\mathcal{H}KA_0 d\theta\right|.$$

For calculating W_n, as $n \geq 1$, take the simple transformation

$$\frac{s_n^3}{s_n - s_{n-1}} = \frac{s_n^3 - s_n^2 s_{n-1}}{s_n - s_{n-1}} + \frac{s_n^2 s_{n-1}}{s_n - s_{n-1}} = s_n^2 + \frac{s_n^2 s_{n-1}}{s_n - s_{n-1}}$$

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$$= s_n^2 + \frac{s_n^2 s_{n-1} - s_n s_{n-1}^2}{s_n - s_{n-1}} + \frac{s_n s_{n-1}^2}{s_n - s_{n-1}} = s_n^2 + s_n s_{n-1} + \frac{s_n s_{n-1}^2}{s_n - s_{n-1}}, \tag{6}$$

$$\frac{A s_n^3}{s_n - s_{n-1}} = A s_n^2 + A s_n s_{n-1} + \frac{A s_n s_{n-1}^2}{s_n - s_{n-1}} = V_1 + V_2 + V_3.$$

Using Lemma 10 for estimating V_1 and V_2 and, for V_3, taking again the simple transformation for s_{n-1}^3, which will appear in the integration over s_{n-1}, we finally get

$$|q(x)|_{r=r_0} = \left| \frac{\Lambda\left(H_0 \int_{S^2} \Phi d\theta + k^2 \int_{S^2} \Phi d\theta\right)}{\Lambda \int_{S^2} \Phi d\theta} \right|_{k=k_0, r=\frac{\pi}{k_0}}$$

$$\leq \frac{\left| \int_{-\infty}^{+\infty} \int_{S^2} \int_{S^2} Z(k,) \tilde{q}(k(\theta - \theta'))\Phi_0 dk d\theta d\theta' \right| + C_0 \left| \int_{S^2} \mathcal{H} K A_0 d\theta \right|}{\left(\frac{1}{k_0} - \frac{\alpha}{(1-\alpha)}\right)} +$$

Finally, we get

$$|q(x)|_{r=r_0} \leq \sup_{x \in R^3} |q(x)|\alpha + C_0 \|q\|^2_{L_2(R^3)} + C_0 \|q\|_{L_2(R^3)} + \left| \int_{S^2} \mathcal{H} K A_0 d\theta \right|.$$

The invariance of the Schrödinger equations with respect to translations which will below be and the arbitrariness of r_0 yield

$$\sup_{x \in R^3} |q(x)| \leq \left| \int_{S^2} \mathcal{H} K A_0 d\theta \right| C_0 \left(\|q\|^2_{L_2(R^3)} + 1 \right) + C_0 \|q\|_{L_2(R^3)}.$$

□

To complete the construction of estimates, we need to prove the invariance of the Schrödinger equation with respect to shifts and coordinate transformations. To do this, we introduce the following notations and definitions:

$$q_{Ua}(x) = q(Ux + a)$$

where

$$UU' = U'U = I$$

$$a \in R^3$$

The corresponding amplitude and wave functions, denoted as

$$A_{Ua}, \Phi_{Ua}, E_{Ua}$$

are associated with these potentials.

Theorem 4. *The wave function Φ_{Ua_+} can be expressed as:*

$$\Phi_{Ua+}(k, \theta, x) = \Phi_0(k, \theta, x) + \sum (G q_{Ua})^n \Phi_0 \tag{7}$$

$$A_{Ua+}(k, \theta', \theta) = -(1/(4\pi)) \int q_{Ua}(x) \Phi_0(k, \theta, x) \Phi_{Ua}(k, -\theta', x) dx \tag{8}$$

Proof. The theorem follows directly from the representations (7) and (8).

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Theorem 5. *: The poles of the functions Φ_{Ua_+} and Φ_+ coincide, i.e.*

$$E_{Ua} = E.$$

Proof: From the representations (7) and (8).

$$\Phi_{Ua+}(k,\theta,x_1) = \Phi_0(k,\theta,x_1) + \sum_{n=1}^{\infty}\prod_{k=1}^{n}\int \frac{q_{Ua}(x_{k+1})e^{ik|x_k-x_{k+1}|}}{|x_k-x_{k+1}|}\Phi_0(k,\theta,x_{n+1})dx_2...dx_{n+1} \tag{9}$$

$$\Phi_{Ua+}(k,\theta,x_1) = \Phi_0(k,\theta,x_1)+$$
$$e^{ik(\theta,a)}\sum_{n=1}^{\infty}\prod_{k=1}^{n}\int \frac{q(x_k)e^{ik|x_k-x_{k+1}|}}{|x_k-x_{k+1}|}\Phi_0(k,\theta,Ux_{n+1})dx_2...dx_{n+1}$$
$$\Phi_{Ua+}(k,\theta,x_1) = \Phi_0(k,\theta,x_1)+$$
$$e^{ik(\theta,a)}\sum_{n=1}^{\infty}\prod_{k=1}^{n}\int \frac{q(x_k)e^{ik|x_k-x_{k+1}|}}{|x_k-x_{k+1}|}\Phi_0(k,\theta,Ux_{n+1})dx_2...dx_{n+1}$$
$$\Phi_{Ua+}(k,\theta,x_1) = \Phi_0(k,\theta,x_1)+$$
$$e^{ik(\theta,a)}[\Phi_0(k,U'\theta,x_1) + \sum_{n=1}^{\infty}\prod_{k=1}^{n}\int \frac{q(x_k)e^{ik|x_k-x_{k+1}|}}{|x_k-x_{k+1}|}\Phi_0(k,U'\theta,x_{n+1})dx_2...dx_{n+1}]$$
$$-e^{ik(\theta,a)}\Phi_0(k,U'\theta,x_1)$$
$$\Phi_{Ua+}(k,\theta,x_1) = \Phi_0(k,\theta,x_1) + e^{ik(\theta,a)}\Phi_+(k,U'\theta,x_1) - e^{ik(\theta,a)}\Phi_0(k,U'\theta,x_1)$$

From the last equation, it follows that the poles of the function on the right and left coincide. □

:

Theorem 6. *: Amplitudes of the functions Φ_{Ua_+} and Φ_+ can calculates as .*

$$A_{Ua+}(k,\theta',\theta) = e^{ik(\theta'-\theta)}A(k,U'\theta',U'\theta))$$

Proof. : From the Theorem (4)

$$A_{Ua+}(k,\theta',\theta) = -(1/(4\pi))\int q_{Ua}(x_1)\Phi_0(k,\theta,x_1)\Phi_{Ua}(k,-\theta',x_1)dx_1 \tag{10}$$

from Theorem (5)

$$A_{Ua+}(k,\theta',\theta) = -(1/(4\pi))\int q_{Ua}(x_1)\Phi_0(k,\theta,x)[\Phi_0(k,-\theta',x_1)+$$
$$\sum_{n=1}^{\infty}\prod_{k=1}^{n}\int \frac{q_{Ua}(x_{k+1})e^{ik|x_k-x_{k+1}|}}{|x_k-x_{k+1}|}\Phi_0(k,-\theta',x_{n+1})]dx_2...dx_{n+1}dx_1$$
$$A_{Ua+}(k,\theta',\theta) = e^{ik(\theta'-\theta)}A(k,U'\theta',U'\theta))$$

□

which simplifies to our theorem's statement through a series of mathematical manipulations, revealing the direct relationship between the transformed and original amplitudes via a phase shift.

This theorem and its proof not only highlight the mathematical elegance underlying the scattering process but also serve as a crucial tool for analyzing the physical implications of spatial transformations

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on wave functions. It underscores the intrinsic link between geometry, through the transformation parameters U and a, and the observable characteristics of scattering, embodied in the amplitude A.

Our calculation offers a fresh perspective on the dynamics of wave scattering, providing a robust framework for predicting and understanding the outcomes of various scattering scenarios. It stands as a testament to the power of mathematical physics in unraveling the complexities of quantum phenomena.

4. Discussion of the Three-Dimensional Inverse Scattering Problem

This study has shown, once again, the outstanding properties of the scattering operator, which, in combination with the analytical properties of the wave function, allows us to obtain almost-explicit formulas for the potential from the scattering amplitude. Furthermore, this appro. The estimations following from this overcome the problem of overdetermination, resulting from the fact that the potential is a function of three variables, whereas the amplitude is a function of five variables. We have shown that it is sufficient to average the scattering amplitude to eliminate the two extra variables.

5. Applications

New Invariants of the Schrödinger Equation and Problems of Measurement Interpretation

- This study unveils a significant connection regarding the invariance of eigenvalue discreteness for a family of potentials obtained through linear transformations of variables. The simplicity of these transformations reveals that many achievements obtained using Lax pairs are inherent properties resulting from linear transformations of variables in Schrödinger equations.
- For the analysis and optimal utilization of seismic data.
- For the analysis and optimal utilization of ultrasound scan data.
- For the analysis and optimal utilization of electromagnetic scan data.
- For the analysis and optimal utilization of nonlinear scan fluctuation data.

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