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Posted Date: 3 April 2024

doi: 10.20944/preprints202404.0268.v1

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Covers of Finitely Generated Acts over Monoids

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Abstract: In (Semigroup Forum 77: 325-338, 2008) Mahmoudi M. and Renshaw J. solved a study that covers of cyclic S-acts over monoids. This article is an attempt to initiate the covers of finitely generated S-acts. We give a necessary and sufficient condition for a monoid to have the properties that n-generated S-acts have strongly flat covers, Condition (P) covers and projective covers. The main conclusions extend some known results. We show also that Condition (P) covers of finitely generated S-acts are not unique, unlike the situation for strongly flat covers. Additionally, we demonstrate that the property of Enochs' $\mathcal{X}$-precover of S-act A, where $\mathcal{X}$ denotes a class of S-acts that are closed under isomorphisms.

Keywords: cover; coproduct; finitely generated; $\mathcal{X}$-precover

MSC: 20M30, 20M50

1. Introduction and Preliminaries

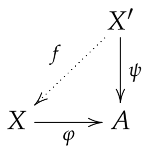
Let S be a monoid. By a right S-act we mean a non-empty set A together with an action $A \times S \to A$ given by $(a, s) \mapsto as$ such that for all $a \in A, s, t \in S, a1 = a$ and $a(st) = (as)t$. We refer the reader to [1,7,9] for all undefined terms concerning acts over monoids.

Mahmoudi and Renshaw introduced coessential epimorphisms and covers of cyclic S-acts in [9], gave a description of coessential epimorphisms in terms of congruence classes and a method of constructing covers from left unitary submonoids, and defined a strongly flat cover, Condition (P) cover, projective cover and provided a necessary and sufficient condition for a cyclic S-act to have a strongly flat cover, Condition (P) cover and projective cover. Now we restrict our attention to these notions.

Let S be a monoid. A right S-act B is called a cover of a right S-act A if there exists an epimorphism $f : B \to A$ such that for any proper subact C of B the restriction $f|_C$ is not an epimorphism. Moreover, we call an S-epimorphism $f : B \to A$ coessential in [9] if for each S-act C and each S-map $g : C \to B$, if fg is an epimorphism, then g is an epimorphism. And we shall say that an act B together with an S-epimorphism $f : B \to A$ is a X-cover of A if B satisfies property X and f is coessential. It is easy to see that $f : B \to A$ is a cover of A if and only if it is a coessential epimorphism. Let $\mathcal{X}$ be a class of right S-acts. We assume that $\mathcal{X}$ is closed under isomorphisms, i.e., if $A \in \mathcal{X}$ and $B \cong A$, then $B \in \mathcal{X}$.

The concept of cover used in [5] is slightly different from that given above. (Enochs' notion) For a right S-act A, an S-act $X \in \mathcal{X}$ is called an $\mathcal{X}$-cover of A if there is a homomorphism $\varphi : X \to A$ such that

(1) for any homomorphism ψ: $X' \to A$ with $X' \in \mathcal{X}$, there is a homomorphism f: $X' \to X$ with $\psi = \varphi f$. In other words, the following diagram



commutes;

 doi:10.20944/preprints202404.0268.v1

(2) If an endomorphism $f : X \to X$ is such that $\varphi = \varphi f$, then f must be an automorphism.

If (1) holds, we call φ: $X \to A$ an $\mathcal{X}$-precover.

Recall that an S-act A_S is called strongly flat if the tensor functor $A \otimes -$ preserves pullbacks and equalizers. An S-act A_S is said to satisfy Condition (P) if for all $a, a' \in A, s, s' \in S$ such that $as = a's'$, then there exist $a'' \in A, u, v \in S$ with $a = a''u$, $a' = a''v$ and $us = vs'$. And recall that an S-act A_S satisfies Condition (E) if $as = as'$ for $a \in A, s, s' \in S$, implies that there exist $a' \in A, u \in S$ such that $a = a'u$ and $us = us'$. It is well known that an act A is strongly flat if and only if it satisfies Condition (P) and Condition (E). We are principally concerned with finitely generated S-acts here and with the concepts of strongly flatness and Condition (P) and in these cases there are useful characterizations.

A monoid S is called right reversible if for all $s, t \in S$, there exist $p, q \in S$ such that $ps = qt$. A monoid S is called left collapsible if for all $s, t \in S$, there exists $u \in S$ such that $us = ut$. A submonoid T of S is said to be left unitary if and only if whenever $t, ts \in T$ then $s \in T$.

Throughout this paper, S denotes a monoid, $A = \langle a_1, a_2, \cdots, a_n \rangle = \bigcup\limits_{i=1}^{n} a_i S$ is said that A is a n-generated right S-act and $I = \{1, 2, \cdots, n\}$. And $\coprod\limits_{i=1}^{n} a_i S$ will denote a coproduct of cyclic subacts $a_i S$ for all $i \in I$. We shall deal exclusively with right S-acts and simply refer to them as S-acts. We shall also consistently write our maps on the left so that gf means f followed by g.

The main results here are descriptions of finitely generated acts and coproducts in Section 2, and we provide a necessary and sufficient condition for a finitely generated S-act to have a cover in Section 3. Next in Section 4 we concentrate on strongly flat covers, Condition (P) covers and projective covers, and the main conclusions extend some known results in [9]. In Section 5 we study Enochs' notion of cover in the category of acts over monoids and focus on $\mathcal{X}$-precovers, where $\mathcal{X}$ is a class of S-acts closed under isomorphisms. Basic results on covers of acts over monoids can be found in [3,4,6,11].

2. Finite Generation and Coproducts

This section aims to describe the finite generation and coproduct of S-acts. A subset U of an S-act A is a generating set for A if for any $a \in A$, there exist $u \in U, s \in S$ such that $a = us$. In other words, U is a set of generating elements for A_S if $\langle U \rangle := \bigcup\limits_{u \in U} uS = A_S$, where $uS = \{us \mid s \in S\}$. An S-act A is said to be finitely generated (resp. cyclic) if it has a finite (resp. one-element) generating set. Further details may be found, for example, in [7].

Proposition 2.1. *Let S be a monoid and $A = \bigcup\limits_{i=1}^{n} A_i$ be S-act, A_i is a subact of A. For every $i \in I$, A_i is finitely generated, then A is finitely generated.*

Proof. Let $A_i = \langle X_i \rangle$. Then $A = \langle \bigcup\limits_{i=1}^{n} X_i \rangle$. $\square$

Proposition 2.2. *Let S be a monoid and $A = \bigcup\limits_{i=1}^{n} A_i$ an S-act, A_i are subacts of A for $i \in I$. And for $i \neq j \in I$, suppose that $A_i \bigcap A_j$ is either finitely generated or empty. If A is finitely generated, then A_i for all $i \in I$, is finitely generated.*

Proof. Let $I = \{1, 2\}$, it is clear by Lemma 5.3 of [10]. If $I_1 = \{1, 2, \cdots, n-1\}$ the assertion is hold, that is, for $i \neq j \in I_1$, $A_i \bigcap A_j$ is either finitely generated or empty, if A is finitely generated, then $A_i, i \in I$, is finitely generated. Assume that $I = \{1, 2, \cdots, n\} = I_1 \bigcup \{n\}$. If $A_i \bigcap A_j = \emptyset, i \neq j \in I$, obviously. Otherwise, take $j = n$, for $i \in I_1$, $A_i \bigcap A_n$ is finitely generated, by Proposition 2.1, $(\bigcup\limits_{i \in I_1} A_i) \bigcap A_n$ is finitely generated. Since $A = (\bigcup\limits_{i \in I_1} A_i) \bigcup A_n$ is finitely generated, $\bigcup\limits_{i \in I_1} A_i$ and A_n are finitely generated. By assumption, $A_i, i \in I_1$, is finitely generated. Therefore, for all $i \in I$, A_i is finitely generated. $\square$

Corollary 2.1. *Let S be a monoid. S-act $A = \coprod_{i=1}^{n} A_i$ is finitely generated if and only if S-act A_i is finitely generated.*

It is important for monoids to consider the covers of finitely generated acts. Now we first aim to investigate the cover of coproducts of acts.

Proposition 2.3. *Suppose that $f_i : B_i \to A_i, i \in I$ are a family of S-morphisms, where $I = \{1, 2, \cdots, n\}$. Let $B = \coprod_{i=1}^{n} B_i$, $A = \coprod_{i=1}^{n} A_i$, and $f : B \to A$ satisfies $f|_{B_i} = f_i$. Then $f : B \to A$ is a cover of A if and only if $f_i : B_i \to A_i$ are covers of A_i for all $i \in I$.*

Proof. Sufficiency. Since $f_i : B_i \to A_i, i \in I$ are a family of epimorphisms, it is clear that $f : B \to A$ is epimorphic. Assume that f is not coessential, that is, there exists a proper subact $\bar{B}$ of B such that $f|_{\bar{B}}$ is epimorphic. Therefore, there is $j \in I$ such that $B'_j = B_j \bigcap \bar{B}$ is a proper subact of B_j. Thus, $f_j|_{B'_j}$ is not epimorphism since $f_j : B_j \to A_j$ is a coessential epimorphism, so there exists $a_j \in A_j$ such that $f_j(b'_j) \neq a_j$ for all $b'_j \in B'_j$. However, by the surjectivity of $f|_{\bar{B}}$, we obtain that $f(\bar{b}) = a_j$ for some $\bar{b} \in \bar{B}$. Thus there exists $k \in I$ such that $\bar{b} \in B_k, k \neq j$. Then $a_j = f(\bar{b}) \in A_k$ which contradicts $a_j \in A_j$.

Necessity. Let $f : \coprod_{i=1}^{n} B_i \to \coprod_{i=1}^{n} A_i$ be a coessential epimorphism. Then we obtain that $f_i : B_i \to A_i$ are a family of epimorphisms for all $i \in I$. To show that f_i are coessential for all $i \in I$, assume that there exists a map $f_j : B_j \to A_j$ such that f_j is not coessential, where $j \in I$. Then $f_j|_{C_j} : C_j \to A_j$ is an epimorphism for some proper subact C_j of B_j. Thus $f|_{(\coprod_{i \neq j} B_i) \coprod C_j} : (\coprod_{i \neq j} B_i) \coprod C_j \to \coprod_{i=1}^{n} A_i$ is an epimorphism. But $(\coprod_{i \neq j} B_i) \coprod C_j$ is a proper subact of $\coprod_{i=1}^{n} B_i$ and by assumption $f|_{(\coprod_{i \neq j} B_i) \coprod C_j}$ is not epimorphic, a contradiction. So $f_i : B_i \to A_i$ are a family of coessential epimorphisms for all $i \in I$. □

Now we concentrate on covers of finitely generated of S-acts and we begin with some notations. For an element a of a right S-act A we denote by $R(a)$ the set $\{s \in S : as = a\}$, the submonoid of right identities of a, and it is clear that $R(a)$ is a left unitary submonoid of S. We denote by $r(a, b)$ the set $\{(s, t) \in S \times S : as = bt\}$. In particular, $r(a) = r(a, a) = \{(s, t) \in S \times S : as = at\}$, the right annihilator congruence ρ_a of a. It is clear that aS is isomorphic to S/ρ_a under the S-isomorphism $as \mapsto [s]$, where $[s]$ denote the class of $s \in S$ with respect to an equivalence relation ρ.

Lemma 2.1 ([7], Proporsition III.13.14). *Let A_S be a finitely generated S-act. If A_S satisfies Condition (P), then A_S is a coproduct of cyclic subacts.*

Let S be a monoid and $A = \coprod_{i=1}^{n} A_i$, where A_i, $i \in I$ are right S-acts. Then A is strongly flat (resp. satisfies Condition (P)) if and only if A_i is strongly flat (resp. satisfies Condition (P)) for every $i \in I$. The following results turn out very helpful in the following section.

Proposition 2.4. *If the finitely generated S-act $A = \langle a_1, \cdots, a_n \rangle$ is strongly flat(resp. satisfis Conditon (P)), then $R(a_i)$ is a left collapsible(resp. right reversible) submonoid of S.*

Proof. Let A satisfy Conditon (P). By Lemma 2.1, a_iS satisfies Conditon (P) for $i \in I$. Thus $R(a_i)$ is right reversible. And in a similar way we obtain that $R(a_i)$ is a left collapsible submonoid of S if A is strongly flat. □

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Proposition 2.5. *Let T be a submonoid of a monoid S and A a right S-act, $a \in A$. If $r(a) = T \times T$, then $T = R(a)$.*

Proof. Since $r(a) = \{(s,t) \in S \times S | as = at\} = T \times T$, for any $t \in T$ and $1 \in T$, then $(1,t) \in T \times T = r(a)$. Thus $a = at$, that is, $t \in R(a)$. Therefore $T \subseteq R(a)$. It is clear that $R(a) \subseteq T$. □

Lemma 2.2. *Let S be a monoid and P a left collapsible(resp. right reversible) submonoid of S, $A = \langle a_1, \cdots, a_n\rangle$ is n-generated. If $r(a_i) = P \times P$, then $R(a_i)$ is left collapsible(resp. right reversible) and $\coprod_{i=1}^{n} a_iS$ is strongly flat(resp. satisfies Condition (P)).*

Proof. Since $r(a_i) = P \times P$, $P = R(a_i)$ by Proposition 2.5 and so $R(a_i)$ is left collapsible(resp. right reversible). Notice that in either case, P is right reversible and so defined by $bs = bt$ if and only if there exists $p, q \in P$ with $ps = qt$. Notice that $b = a_i$. In fact, if $a_is = a_it$ then there exist $p, q \in P$ with $ps = qt$ and so $bs = bt$. Conversely, if $bs = bt$, then there are $p, q \in P$ with $ps = qt$ and $a_i = a_ip = a_iq$ from $P \subseteq R(a_i)$. So $a_is = a_ips = a_iqt = a_it$ and hence $b = a_i$. It is clear that a_iS satisfies Condition (P). Then $\coprod_{i=1}^{n} a_iS$ satisfies Condition (P). If in addition P is left collapsible, then $\coprod_{i=1}^{n} a_iS$ is strongly flat. □

It is clear that any cover of a cyclic right S-act is cyclic by Lemma 2.3 of [9]. The following theorem concerns the covers of n-generated right S-acts.

Theorem 2.1. *Let $f : B_S \to A_S$ be a cover of A_S. Then B_S is n-generated if and only if A_S is n-generated.*

Proof. Necessity. Assume that B is n-generated, $\bar{B} = \{b_1, b_2, \cdots, b_n\}$ is a generating set of B. Since $f : B \to A$ is an epimorphism, A is generated by n elements. Now we are ready to prove n is the smallest. Suppose that $A = \langle y_1, \cdots, y_k\rangle, k < n$. Then $f(z_i) = y_i$, $i = 1, \cdots, k$. Therefore $B = \bigcup_{i=1}^{k} z_iS$, which contradicts that B is n-generated. Thus, A is n-generated.

Sufficiency. Suppose $A = \langle a_1, \cdots, a_n\rangle$. Since f is an epimorphism, there exists $b_i \in B$ such that $f(b_i) = a_i$, $i = 1, 2, \cdots, n$. Take $C = \bigcup_{i=1}^{n} b_iS$, clearly $C \subseteq B$ and $f(C) = A$. By the definition of cover, we have $C = B$. Therefore B is generated by n elements. Now we are ready to prove a result that n is the smallest. If $B = \langle x_1, \cdots, x_k\rangle, k < n$, and $f(B) = A$, we have A is generated by $f(x_i)$, $i = 1, 2, \cdots, k$, a contradiction. □

Let $A = \bigcup_{i=1}^{n} A_i$ be n-generated and $B = \coprod_{i=1}^{n} B_i$. The following Proposition 2.6 shows that $f : B \to A$ is a cover of A implies that $f_i : B_i \to A_i$, $i \in I$, are a family of covers of A_i.

Proposition 2.6. *Let $A = \bigcup_{i=1}^{n} A_i, B = \coprod_{i=1}^{n} B_i$. Let $f_i : B_i \to A_i$, $i \in I$, be a family of S-morphisms, and $f : B \to A$ satisfies $f|_{B_i} = f_i$. Then $f : B \to A$ is a cover of A implies $f_i : B_i \to A_i$, $i \in I$, are a family of covers of A_i.*

Proof. If $A_i \bigcap A_j = \emptyset$ for any $i, j \in I$, it is clear by Proposition 2.3. Otherwise, suppose $a \in A_i \bigcap A_j$, $i \neq j$, $i, j \in I$. Since f is an epimorphism, for $a \in A_i$, there exists $b \in B_i$ such that $f(b) = f_i(b) = a$. If $a \in A_j$, we have $f(b) = f_j(b) = a$ for some $b \in B_j$. But $B_i \bigcap B_j = \emptyset$, a contradiction. Therefore $A_i \bigcap A_j = \emptyset$, we obtain that $A = \coprod_{i=1}^{n} A_i$. By Proposition 2.3, $f : B \to A$ is a cover of A implies that $f_i : B_i \to A_i$ are covers of A_i. □

Coversely, if B is n-generated and A is a disjoint union of n subacts, the argument always holds in Proposition 2.7.

Proposition 2.7. *Suppose that $B = \bigcup_{i=1}^{n} B_i$, $A = \coprod_{i=1}^{n} A_i$, $f_i : B_i \to A_i$, $i \in I$ are a family of S-morphisms, and $f : B \to A$ satisfies $f|_{B_i} = f_i$. Then $f : B \to A$ is a cover of A if and only if $f_i : B_i \to A_i$ are a family of covers of A_i for every $i \in I$.*

Proof. Let $f : B \to A$ be a cover of A. If there exist $i, j \in I$ such that $B_i \bigcap B_j \neq \emptyset$, that is, $f(b_i) = f(b_j) = a$, where $b_i \in B_i, b_j \in B_j$, $a \in A$, then $f|_{(B_i \backslash \{b_i\}) \bigcup (\bigcup_{k \neq i} B_k)} : (B_i \backslash \{b_i\}) \bigcup (\bigcup_{k \neq i} B_k) \to \coprod_{i=1}^{n} A_i$ is an epimorphism, a contradiction. So f is one-to-one. Assume $b \in B_i \bigcap B_j$. Then $f(b) = f_i(b) \in A_i$, and $f(b) = f_j(b) \in A_j$ which contradicts $A_i \bigcap A_j = \emptyset$. Hence $B_i \bigcap B_j = \emptyset$ or the element in $B_i \bigcap B_j$ has no image under the action of f. If the latter, then $f|_{B \backslash (B_i \bigcap B_j)} : B \backslash (B_i \bigcap B_j) \to A$ is an epimorphism, a contradiction. Therefore, $B = \coprod_{i=1}^{n} B_i$. Conversely, let $f_i : B_i \to A_i$ be a family of covers of A_i, we obtain the same result $B = \coprod_{i=1}^{n} B_i$. By Proposition 2.3, $f : B \to A$ is a cover of A if and only if $f_i : B_i \to A_i$ are a family of covers of A_i. □

Remark 2.1. *From the above several propositions it is shown that if A and B are the union(coproduct) of n S-acts A_i and B_i, respectively, then $f : B \to A$ is a cover of A if and only if $f_i : B_i \to A_i$ are a family of covers of A_i for $i \in \{1, 2, \cdots, n\}$.*

Let S be a monoid and X an S-act. We say that X is Noetherian if every congruence on X is finitely generated, and we say that a monoid S is Noetherian if it is Noetherian as an S-act over itself.

Proposition 2.8. *Let S be a Noetherian monoid and $f : B \to A$ be a cover. Then A is Noetherian if and only if B is Noetherian.*

Proof. Let $f : B \to A$ be a cover of A. If B is Noetherian. By Lemma 7.6 of [2], then B is finitely generated, and so is A by Theorem 2.1. All finitely generated S-acts over a Noetherian monoid are Noetherian, thus A is Noetherian. Coversely, it is obvious. □

3. Covers of Finitely Generated Acts

Let S be a monoid and ρ, σ right congruences on S. In Mahmoudi and Renshaw [9], it is proved that a cyclic act S/ρ has a cover S/σ if and only if $\sigma \subseteq \rho$ and for all $u \in [1]_\rho$, $uS \cap [1]_\sigma \neq \emptyset$. The next there propositions are needed to get the result.

Proposition 3.1. *Let S be a monoid. Then $A' = \langle a'_1, a'_2, \cdots, a'_n \rangle$ is isomorphic to a finitely generated subact of $A = \langle a_1, a_2, \cdots, a_n \rangle$ if and only if there exist $u_i, u_p \in S$, such that $r(a'_i, a'_p) = r(a_i u_i, a_p u_p)$ for any $i, p \in \{1, 2, \cdots, n\}$.*

Proof. Let $f : A' \to A$ be an S-monomorphism and $I = \{1, 2, \cdots, n\}$. For $a'_i \in A'$, there is $u_j \in S$ such that $f(a'_i) = a_j u_j \in a_j S \subseteq A$, $i, j \in I$. Since f is a monomorphism, then $j = i$. Thus, we have $f(a'_i) = a_i u_i$ for some $a_i \in \{a_1, \cdots, a_n\}$ and $u_i \in S$. And so $f(a'_i s) = a_i u_i s$ for any $s \in S$. Let $a'_i s = a'_p t$, $i, p \in I, s, t \in S$. Then $a_i u_i s = a_p u_p t$. In addition, suppose that $a_i u_i s = a_p u_p t$, and we have $a'_i s = a'_p t$ since f is monomorphic. Therefore $r(a'_i, a'_p) = r(a_i u_i, a_p u_p)$.

Conversely, the mapping $f : A' \to A$ defined by $a'_i s \mapsto a_i u_i s$ is well-defined and f is an S-monomorphism. □

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Lemma 3.1. *Let S be a monoid and $u_i \in S$. Consider the S-monomorphism $f : \langle a'_1, a'_2, \cdots, a'_n \rangle \to \langle a_1, a_2, \cdots, a_n \rangle$ given by $f(a'_i s) = a_i u_i s$, $s \in S$. Then f is onto if and only if $u_i S \bigcap R(a_i) \neq \emptyset$ for every $i \in I$.*

Proof. Since f is an S-epimorphism. For any $a_i \in \{a_1, \cdots, a_n\}$, there exist $a'_p \in \{a'_1, \cdots, a'_n\}$ and $m \in S$ such that $f(a'_p m) = a_i$. And since $f(a'_p m) = a_p u_p m$, it follows that $a_i = a_p u_p m \in a_p S$, namely $a_i S \subseteq a_p S$, a contradiction. Thus $p = i$ and so $a_i = a_i u_i m$, hence $u_i S \bigcap R(a_i) \neq \emptyset$.

Conversely, since $u_i S \bigcap R(a_i) \neq \emptyset$, there exists $s \in S$ such that $a_i = a_i u_i s$ for every $i \in I$. So $f(a'_i s) = a_i u_i s = a_i$. It is easy to see that f is an epimorphism. □

Lemma 3.2. *Let S be a monoid, $B = \langle b_1, b_2, \cdots, b_n \rangle$ and $A = \langle a_1, a_2, \cdots, a_n \rangle$. If $f : B \to A$ is a coessential epimorphism then there exists $u_i \in S$ such that $g : B' \to B$, $b'_i s \mapsto b_i u_i s$, is isomorphic, where $B' = \langle b'_1, b'_2, \cdots, b'_n \rangle$. And $f' : B' \to A$ given by $f(b'_i s) = a_i s$ is a coessential S-epimorphism. In particular, $r(b'_i, b'_p) \subseteq r(a_i, a_p)$.*

Proof. Since $f : B \to A$ is a coessential epimorphism, it follows that A and B are n-generated. For any $a_i \in \{a_1, \cdots, a_n\}$, there exist $b_i \in \{b_1, \cdots, b_n\}$ and $u_i \in S$ such that $f(b_i u_i) = a_i$. Suppose that $g : B' \to B$ given by $f(b'_i s) = b_i u_i s$ is a monomorphism whose composite with f is clearly onto. Since B is a cover of A, $B' \cong B$ and so $f' : B' \to A$, $b'_i s \mapsto a_i s$, is a coessential S-epimorphism. It then follows that $r(b'_i, b'_p) \subseteq r(a_i, a_p)$. □

We now present a fundamental theorem that yields the main result of this section.

Theorem 3.1. *Let S be a monoid and $A = \langle a_1, a_2, \cdots, a_n \rangle$ a n-generated S-act. The map $f : \langle b_1, b_2, \cdots, b_n \rangle \to \langle a_1, a_2, \cdots, a_n \rangle$ given by $b_i s \mapsto a_i s$ is a coessential epimorphism if and only if $r(b_i, b_j) \subseteq r(a_i, a_j)$, $i, j \in I$, and for all $u_i \in R(a_i)$, $u_i S \bigcap R(b_i) \neq \emptyset$.*

Proof. Let $b_i s = b_j t$ for $b_i, b_j \in \{b_1, b_2, \cdots, b_n\}$, $s, t \in S$ and $i, j \in I$. Then $a_i s = a_j t$ because f is well-defined. Hence $r(b_i, b_j) \subseteq r(a_i, a_j)$. For $a_i \in \{a_1, a_2, \cdots, a_n\}$, there exist $b_j \in \{b_1, \cdots, b_n\}$ and $u_j \in S$ such that $f(b_j u_j) = a_i \in a_i S$, but $f(b_j u_j) = a_j u_j \in a_j S$, we obtain $a_i S \subseteq a_j S$, a contradiction. Therefore $j = i$ and so $f(b_i u_i) = a_i$. Since $f : \langle b_1, b_2, \cdots, b_n \rangle \to \langle a_1, a_2, \cdots, a_n \rangle$ is a coessential epimorphism, by Lemma 3.2, there is a subact $\langle b'_1, b'_2, \cdots, b'_n \rangle$ such that $\langle b'_1, b'_2, \cdots, b'_n \rangle \cong \langle b_1, b_2, \cdots, b_n \rangle$ and $f|_{\langle b'_1, b'_2, \cdots, b'_n \rangle} : \langle b'_1, b'_2, \cdots, b'_n \rangle \to \langle a_1, a_2, \cdots, a_n \rangle$ is a coessential epimorphism. So we have $b_i = b'_i m$ for some $m \in S$. And since $g : \langle b'_1, b'_2, \cdots, b'_n \rangle \to \langle b_1, b_2, \cdots, b_n \rangle$ is isomorphic by Lemma 3.2, it follows that $b_i u_i m = b_i$, thus $u_i S \bigcap R(b_i) \neq \emptyset$ as required.

Conversely, if the given conditions hold, then clearly f is well-defined. Let B be an S-subact of $\langle b_1, b_2, \cdots, b_n \rangle$ and suppose that $f|_B$ is onto. Then for $a_i \in \{a_1, a_2, \cdots, a_n\}$, there exists $b_p u_p \in B$ such that $f(b_p u_p) = a_i$ and we obtain $p = i$, so $f(b_i u_i) = a_i$. However $f(b_i u_i) = a_i u_i$, and hence $u_i \in R(a_i)$. By assumption, there exists $m \in S$ such that $b_i = b_i u_i m$, in the sense that $b_i s = (b_i u_i m)s = (b_i u_i)(ms) \in BS \subseteq B$ for an arbitrary $s \in S$. So $\langle b_1, b_2, \cdots, b_n \rangle \subseteq B$, that is, $\langle b_1, b_2, \cdots, b_n \rangle = B$, and f is a cover. □

If $n = 1$, we can easily obtain the Theorem 2.7 in [9].

Theorem 3.2. *Let S be a monoid and $A = \langle a_1, a_2, \cdots, a_n \rangle$ a n-generated S-act. If the natural map $f : \bigcup_{i=1}^{n} S_i \to A$ is a coessential epimorphism, where $S_i \cong S$. Then $R(a_i), i \in I$, is a subgroup of S.*

Proof. Since $f : \bigcup_{i=1}^{n} S_i \to \bigcup_{i=1}^{n} a_i S$ is a coessential epimorphism. Then for an arbitrary element $u_i \in R(a_i)$, $u_i S \bigcap R(1) \neq \emptyset$. And $R(1) = \{t \in S | 1t = 1\} = \{1\}$. We can verify that $u_i s_i = 1$ for $s_i \in S$. Moreover, $a_i = a_i u_i$, it is easy to see that $a_i s_i = a_i u_i s_i = a_i$ and so $s_i \in R(a_i)$. Consequently, $R(a_i)$ is a subgroup of S. □

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Proposition 3.2. *Let S be a monoid. Then the map $f : \bigcup_{i=1}^{n} S_i \to \bigcup_{i=1}^{n} a_iS$ is a coessential epimorphism, where $S_i \cong S$, if and only if $R(a_i), i \in I$, is a subgroup of S and $r(1_i, 1_p) \subseteq r(a_i, a_p)$ for $i, p \in I$.*

Proof. Let $f : \bigcup_{i=1}^{n} S_i \to \bigcup_{i=1}^{n} a_iS$, $s \mapsto a_is$, be a coessential epimorphism. Let $1_is = 1_pt$ for $s, t \in S$ and $i, p \in I$. We have $a_is = a_is$ since f is well-defined. By Theorem 3.2, $R(a_i)$ is a subgroup of S.

Conversely, suppose that $R(a_i)$ is a subgroup of S. For any $u_i \in R(a_i)$, there exists $u_i^{-1} \in R(a_i)$ such that $u_iu_i^{-1} = \{1\}$. Thus $u_iS \bigcap R(1) \neq \emptyset$. In addition, since $r(1_i, 1_p) \subseteq r(a_i, a_p)$ for $i, p \in I$, we can verify that f is a coessential epimorphism by Theorem 3.1. □

Proposition 3.3. *Let S be a right simple semigroup with a 1 adjoined and $B = \langle b_1, b_2, \cdots, b_n \rangle$, $A = \langle a_1, a_2, \cdots, a_n \rangle$. If $r(b_i, b_p) \subseteq r(a_i, a_p)$ for $i, p \in I$ and $R(b_i) \neq \{1\}$. Then $f : B \to A$ given by $b_is \mapsto a_is$ is a coessential epimorphism.*

Proof. Since $R(b_i) \neq \{1\}$, we can verify that $R(b_i) \subseteq S \backslash \{1\} = T$, where T is a right simple semigroup. So $tT = T$ for every $t \in T$. Then for $u_i \in R(a_i)$, $u_iT \bigcap R(b_i) = T \bigcap R(b_i) = R(b_i) \neq \emptyset$. It is easy to see that f is a coessential epimorphism since $r(b_i, b_p) \subseteq r(a_i, a_p)$. □

Remark 3.1. *It follows from the above that covers of finitely generated S-acts need not be unique. If S is a group then every n-generated S-act has $\bigcup_{i=1}^{n} S_i$ as a cover, where $S_i \cong S$. And so proper n-generated S-acts do not have unique covers. In fact, if the map $\langle b_1, b_2, \cdots, b_n \rangle \to \langle a_1, a_2, \cdots, a_n \rangle$ is onto, then $\langle b_1, b_2, \cdots, b_n \rangle$ is trivially a cover of $\langle a_1, a_2, \cdots, a_n \rangle$.*

4. *N*-Generated Flat Covers

In Mahmoudi and Renshaw [9] it is proved that an equivalent characterization of a cyclic act having flat properties covers, such as strongly flat, Condition (P), projective. The next theorems are to investigate flatness covers of n-generated S-acts.

Theorem 4.1. *Let S be a monoid. Then the n-generated S-act $A = \langle a_1, a_2, \cdots, a_n \rangle$ has a strongly flat cover B if and only if $B = \coprod_{i=1}^{n} b_iS$ and $R(a_i)$ contains a left collapsible submonoid R such that for all $u_i \in R(a_i)$, $u_iS \bigcap R \neq \emptyset$.*

Proof. Suppose that A has a strongly flat cover B. By Theorem 2.1, $B = \langle b_1, b_2, \cdots, b_n \rangle$. Then by Theorem 3.1 we can assume that $R = R(b_i) \subseteq R(a_i)$ and that for all $u_i \in R(a_i)$, $u_iS \bigcap R \neq \emptyset$. Moreover, since B is strongly flat, $B = \coprod_{i=1}^{n} b_iS$ by Lemma 2.1 and R is left collapsible by Proposition 2.4.

Conversely, suppose that $B = \coprod_{i=1}^{n} b_iS$ and R is a left collapsible submonoid of $R(a_i)$ such that for all $u_i \in R(a_i)$, $u_iS \bigcap R \neq \emptyset$. Define $r(b_i) = R \times R$, then $R \subseteq R(b_i)$ and further $B = \coprod_{i=1}^{n} b_iS$ is strongly flat by Lemma 2.2. Define a map $f : \langle b_1, b_2, \cdots, b_n \rangle \to \langle a_1, a_2, \cdots, a_n \rangle$ given by $b_is \mapsto a_is$ and note that f is a well-defined S-epimorphism. To see this, first notice that if f is well-defined, then it is clearly an S-map which is onto. Since $R \subseteq R(a_i)$, it is easy to see that $R \times R \subseteq r(a_i)$ but $r(b_i) = R \times R$ and hence $r(b_i) \subseteq r(a_i)$. And $r(b_i, b_j) \subseteq r(a_i, a_j)$ for any $i \neq j, i, j \in I$, is not necessary to consider because $B = \coprod_{i=1}^{n} b_iS$. Further, for all $u_i \in R(a_i)$, $u_iS \bigcap R \neq \emptyset$ so f is coessential by Theorem 3.1. □

We can easily obtain the Theorem 3.2 of [9] if $n = 1$ in Theorem 4.1.

 doi:10.20944/preprints202404.0268.v1

Example 4.1. *Let S be a left cancellative monoid. Then n-generated S-act $A = \langle a_1, a_2, \cdots, a_n \rangle$ has a strongly flat cover if and only if $R(a_i)$, $i \in I$, is a subgroup of S and in this case $\bigcup_{i=1}^{n} S_i$ is a strongly flat cover of A, where $S_i \cong S$. First notice that the only strongly flat cover of a n-generated S-act(assuming that it has one) is then $\bigcup_{i=1}^{n} S$. Notice also that not all finitely generated S-acts need have a strongly flat cover(see, for example, Remark 3.6 in [9]).*

Theorem 4.2. *Let S be a monoid. Then the n-generated S-act $A = \langle a_1, a_2, \cdots, a_n \rangle$ has a (P)-cover B if and only if $B = \coprod_{i=1}^{n} b_iS$ and $R(a_i)$ contains a right reversible submonoid R such that for all $u_i \in R(a_i)$, $u_iS \bigcap R \neq \emptyset$.*

Proof. Suppose that A has a (P)-cover, $B = \langle b_1, b_2, \cdots, b_n \rangle$. Then by Theorem 3.1 we can assume that $R = R(b_i) \subseteq R(a_i)$ and that for all $u_i \in R(a_i)$, $u_iS \bigcap R \neq \emptyset$. Moreover, since B satisfies Condition (P), $B = \coprod_{i=1}^{n} b_iS$ by Lemma 2.1 and R is right reversible by Proposition 2.4.

Conversely, suppose that $B = \coprod_{i=1}^{n} b_iS$ and R is a right reversible submonoid of $R(a_i)$ such that for all $u_i \in R(a_i)$, $u_iS \bigcap R \neq \emptyset$. Define $r(b_i) = R \times R$, then $R \subseteq R(b_i)$ and further $B = \coprod_{i=1}^{n} b_iS$ satisfies Condition (P) by Lemma 2.2. Define a map $f : \langle b_1, b_2, \cdots, b_n \rangle \to \langle a_1, a_2, \cdots, a_n \rangle$ given by $b_is \mapsto a_is$ and note that f is a well-defined S-epimorphism. Since $R \subseteq R(a_i)$, it follows that $R \times R \subseteq r(a_i)$ but $r(b_i) = R \times R$ and hence $r(b_i) \subseteq r(a_i)$. And $r(b_i, b_j) \subseteq r(a_i, a_j)$ for any $i \neq j, i, j \in I$, is not necessary to consider because $B = \coprod_{i=1}^{n} b_iS$. Further, since for all $u_i \in R(a_i)$, $u_iS \bigcap R \neq \emptyset$, then f is coessential by Theorem 3.1. □

Similarly, if $n = 1$, we can easily obtain the Theorem 4.2 of [9].

Proposition 4.1. *Let S be a group. Then every n-generated S-act $A(A \neq S)$ has at least two (P)-covers but only has a unique strongly flat cover.*

Proof. Let $A = \langle a_1, a_2, \cdots, a_n \rangle$. Since S is a group, then all S-acts satisfy condition (P) and so in particular A is a (P)-cover of itself. But $R(a_i), i \in I$, is a subgroup of S, thus $\bigcup_{i=1}^{n} S_i$ is a cover of A by Proposition 3.2, where $S_i \cong S$. While the latter part follows immediately from the Example 4.1. □

Lemma 4.1 ([8], Corollary 3.8). *A right S-act P_S is projective if and only if $P = \coprod_{i \in I} P_i$ where $P_i \cong e_iS$ for idempotents $e_i \in S$, $i \in I$.*

Theorem 4.3. *Let S be a monoid and $A = \langle a_1, a_2, \cdots, a_n \rangle$ a n-generated S-act. Then the following are equivalent:*
(1) *A has a projective cover;*
(2) *There exits a submonoid R of $R(a_i)$ which has a left zero element and for all $u_i \in R(a_i)$, $u_iS \bigcap R \neq \emptyset$.*
(3) *The submonoid $R(a_i)$ of S has a minimal right ideal generated by an idempotent.*

Proof. (1)⇒(2) Let $B = \langle b_1, b_2, \cdots, b_n \rangle$ be a projective cover of A. Then by Theorem 3.1, take $R = R(b_i)$, for all $u_i \in R(a_i)$, $u_iS \bigcap R \neq \emptyset$. Moreover by Lemma 4.1 there exists $e_i \in E(S)$ with $B = \coprod_{i=1}^{n} b_iS \cong \coprod_{i=1}^{n} e_iS$ and thus obviously $e_i \in R(e_i)$ and $R(e_i) = R(b_i)$. If $u_i \in R(e_i)$, then $e_iu_i = e_i$ and so e_i is a left zero element of $R(b_i)$.

(2)⇒(3) Assume that $z \in R$ is a left zero element of R and consider the right ideal $zR(a_i)$ of $R(a_i)$. If this is not minimal, then there exists $u_i \in R(a_i)$ with $zuR(a_i) \neq zR(a_i)$. By assumption there exists

 doi:10.20944/preprints202404.0268.v1

$s \in S$ with $u_i s \in R \subseteq R(a_i)$, and since $R(a_i)$ is a left unitary submonoid, then $s \in R(a_i)$. Since z is a left zero element of R, $z = z(u_i s) \in zu_i R(a_i)$ and so $zR(a_i) = zu_i R(a_i)$, which is a contradiction. Therefore the submonoid $zR(a_i)$ of $zR(a_i)$ is a minimal right ideal generated by an idempotent.

(3)⇒(1) Suppose that $e_i \in E(S)$ with $e_i R(a_i)$ is a minimal right ideal of $R(a_i)$. Clearly $\coprod_{i=1}^{n} e_i S$ is projective. Define the map $f : \coprod_{i=1}^{n} e_i S \to \bigcup_{i=1}^{n} a_i S$ by $f(e_i s) = a_i s$. Then f is well defined since if $e_i s = e_j t$, $i, j \in I$ and $s, t \in S$, then $i = j$ and $a_i s = a_i e_i s = a_i e_i t = a_i t$. In addition, it is also coessential since if $u_i \in R(a_i)$, then $e_i u_i R(a_i) = e_i R(a_i)$ by the minimality of $e_i R(a_i)$, and so $e_i = e_i u_i t$ for some $t \in R(a_i)$. But then $u_i t \in R(e_i)$ and so $u_i S \bigcap R(e_i) \neq \emptyset$. Hence the result follows by Theorem 3.1. □

If $n = 1$ in Theorem 4.3, we can get the Theorem 5.2 in [9].

5. $\mathcal{X}$-Precovers

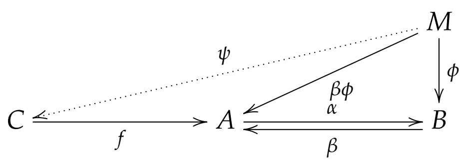
In this section we study Enochs' notion of cover [1] in the category of acts over monoids and focus on $\mathcal{X}$-precovers and $\mathcal{X}$-covers, where $\mathcal{X}$ will be a class of S-acts closed under isomorphisms. For example, we denote the class of all projective S-acts by $\mathcal{P}$. Clearly the concept of cover by Enochs' is slightly different from that the coessential cover.

Proposition 5.1. *Let S be a monoid and let $\mathcal{X}$ satisfy the property that $X \in \mathcal{X} \Leftrightarrow X \otimes M \in \mathcal{X}$ for ${}_S M_S \in S$-act-S. If A has an $\mathcal{X}$-precover, then $A \otimes M$ has an $\mathcal{X}$-precover.*

Proof. Let $g : C \to A$ be $\mathcal{X}$-precover of A. Define $g \otimes 1 : C \otimes M \to A \otimes M$ to be the obvious induced map. Let $X \otimes M \in \mathcal{X}$ and let $h \otimes 1 : X \otimes M \to A \otimes M$. Now, by the hypothesis, $X \in \mathcal{X}$, so, since C is an $\mathcal{X}$-precover of A, there exists $f \in \mathrm{Hom}_S(X, C)$ such that $gf = h$. So define $f \otimes 1 : X \otimes M \to C \otimes M$ to be the induced map, and clearly $(g \otimes 1)(f \otimes 1) = gf \otimes 1 = h \otimes 1$. Thus, $A \otimes M$ has an $\mathcal{X}$-precover. □

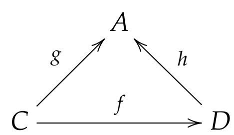
Proposition 5.2. *Let A be an S-act and $\mathcal{X}$ a class of S-acts. Any retract of any act satisfying $\mathcal{X}$ satisfies $\mathcal{X}$. Then the $\mathcal{X}$-precover of A, if it exists, is the $\mathcal{X}$-precover of the retract of A.*

Proof. Please see the following diagram



Let $f : C \to A$ be the $\mathcal{X}$-precover and B the retract of A. Then $\alpha : A \to B$ and $\beta : B \to A$ satisfy $\alpha\beta = 1_B$. For S-map $\phi : M \to B$, $M \in \mathcal{X}$, by the $\mathcal{X}$-precover property, there exists an S-map $\psi : M \to C$ such that $f\psi = \beta\phi$. We claim that $\alpha f \psi = \alpha\beta\phi = 1_B\phi = \phi$. Hence, $\alpha f : C \to B$ is the $\mathcal{X}$-precover of B. □

Proposition 5.3. *Let $\mathcal{X}$ be a class of acts and $D \in \mathcal{X}$. Consider the following commutative diagram of S-acts:*



If $g : C \to A$ is an $\mathcal{X}$-precover of S-act A, then $h : D \to A$ is also a $\mathcal{X}$-precover of A.

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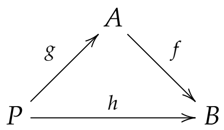
Proof. Let $\psi : M \to A$ be an S-homomorphism for $M \in \mathcal{X}$. By the $\mathcal{X}$-precover property, there exists an S-map $\phi : M \to C$ such that $g\phi = \psi$. Hence, $hf\phi = \psi$ from $hf = g$. So $h : D \to A$ is a $\mathcal{X}$-precover of A. $\square$

Theorem 5.1. *Let $\mathcal{X}$ be a class of S-acts and $f : A \to B$ be a monomorphism. Consider the following commutative diagram of S-acts:*

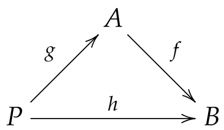


If $h : P \to B$ is an $\mathcal{X}$-precover of B, then $g : P \to A$ is an $\mathcal{X}$-precover of A.

Proof. For every S-map $h' : P' \to A$, where $P' \in \mathcal{X}$. Since $h : P \to B$ is an $\mathcal{X}$-precover of B, there exists an S-map $\varphi : P' \to P$ with $fg\varphi = h\varphi = fh'$. Then $g\varphi = h$ since f is a monomorphism, and we are done. $\square$

Dually, if $f : A \to B$ is an S-epimorphism, we obtain the following Theorem 5.2.

Theorem 5.2. *Let S be a monoid and $\mathcal{P}$ the class of all projective S-acts. Let $f : A \to B$ be an S-epimorphism. Consider the following commutative diagram of S-acts:*



If $g : P \to A$ is a $\mathcal{P}$-precover of A, then $h : P \to B$ is a $\mathcal{P}$-precover of B.

Proof. Let $g : P \to A$ be $\mathcal{P}$-precover of A and $f : A \to B$ an S-epimorphism. To show that $h : P \to B$ is a $\mathcal{P}$-precover of B, we assume that any S-map $\varphi : P' \to B$, where $P' \in \mathcal{P}$. By assumption, there exists $\psi : P' \to A$ such that $f\psi = \varphi$. Since $g : P \to A$ is a $\mathcal{P}$-precover of A, we obtain that $g\phi = \psi$ for some $\phi : P' \to P$. So $\varphi = f\psi = fg\phi = h\phi$. Therefore, $h : P \to B$ is a $\mathcal{P}$-precover of B. $\square$

Remark 5.1. *It is clear that if $g : P \to A$ is an $\mathcal{X}$-cover of A, then $h : P \to B$ is an $\mathcal{X}$-cover of B in Theorem 5.2. A question that could be brought up is whether Theorem 5.2 is valid for free S-acts. It is easy to answer this question positively.*

Theorem 5.3. *Let S be a monoid and $\mathcal{X}$ a class of S-acts. If $A \in \mathcal{X}$ is an $\mathcal{X}$-precover of B and $C \in \mathcal{X}$ is an $\mathcal{X}$-precover of A. Then C is an $\mathcal{X}$-precover of B.*

Proof. Suppose that $f : A \to B$ is an $\mathcal{X}$-precover of B and $g : C \to A$ is an $\mathcal{X}$-precover of A. For any S-map $h : P \to B$, for $P \in \mathcal{X}$, there is an S-map $\varphi : P \to A$ with $f\varphi = h$ since $f : A \to B$ is an $\mathcal{X}$-precover of B. Similarly, because $g : C \to A$ is an $\mathcal{X}$-precover of A, we have $g\psi = \varphi$ for some S-map $\psi : P \to C$. Thus, we can verify that $h = f\varphi = (fg)\psi$. So $fg : C \to B$ is an $\mathcal{X}$-precover of B, that is, C is an $\mathcal{X}$-precover of B. $\square$

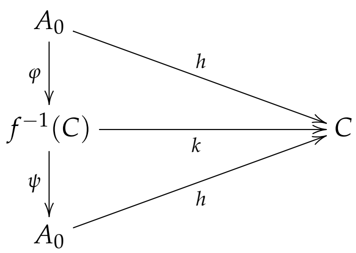
Proposition 5.4. *Let S be a monoid and B a right S-act. Let $\mathcal{X}$ be a class of S-acts closed under factor acts. If $f : A \to B$ is a $\mathcal{X}$-precover for B, then $f' : A/\rho \to B$ is a $\mathcal{X}$-precover for B, where $\rho \subseteq ker(f)$ is a congruence on A.*

Proof. Let ρ be a congruence on A contained in $ker(f)$ and $A \in \mathcal{X}$. First note that $A/\rho \in \mathcal{X}$. By Homomorphism Theorem for acts, there exists a homomorphism $f' : A/\rho \to B$, defined by

$f'(a\rho) = f(a)$, with $f'\pi = f$, where $\pi : A \to A/\rho$. For each morphism $h : C \to B$ with $C \in \mathcal{X}$, there exists a morphism $g : C \to A$ such that $fg = h$. Thus $f'(\pi g) = h$, that is, $f' : A/\rho \to B$ is a $\mathcal{X}$-precover for B. □

Proposition 5.5. *Let $\mathcal{X}$ be a class of S-acts which is closed under subacts. And suppose that all subacts of B have $\mathcal{X}$-covers. If $A \in \mathcal{X}$ and $f : A \to B$ is an $\mathcal{X}$-precover and C is a proper subact of B. Then there exists an S-subact A^* of A and a homomorphism $g : A^* \to C$ such that it is an $\mathcal{X}$-cover of C.*

Proof. By assumption, we note that $f^{-1}(C) \in \mathcal{X}$. Thus $k : f^{-1}(C) \to C$ is an $\mathcal{X}$-precover of C. Let $h : A_0 \to C$ be an $\mathcal{X}$-cover of C. Then we have the commutative diagram



such that $k\varphi = h$ and $h\psi = k$, and so $h\psi\varphi = h$. Since h is an $\mathcal{X}$-cover of C, it follows that $\psi\varphi$ is an automorphism of A_0. Thus φ is monomorphism. Take $A^* = \varphi(A_0) \cong A_0$ is an S-subact of A, then $g = h\psi : A^* \to C$ is an $\mathcal{X}$-cover of C. □

6. Conclusions

Based on the covers of cyclic S-acts over monoids(see [9]), we have introduced the projective covers (resp. strongly flat covers, (P)-covers) of the finitely generated S-acts over monoids in this work. Then we pose the following questions for consideration:

1.What is the equivalent conditions of the Condition (P')-cover and (PF'')-cover of the finitely generated S-acts?

2.What is the flat covers of finitely presented S-acts?

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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