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Innovative Potential Estimates in Schrödinger Equations: Bridging Quantum Mechanics and Fluid Dynamics

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Abstract: This manuscript presents a novel approach to estimating the potential within Schrödinger equations, with a particular focus on applications to the Navier-Stokes problem in fluid dynamics. By establishing new theoretical estimates, we delve into the intricate dynamics of fluid flow, aiming to unveil previously obscured aspects of the Navier-Stokes equations. Through a rigorous mathematical framework, we explore how these new potential estimates can provide a fresh perspective on fluid mechanics, contributing to the ongoing quest to solve some of its most persistent challenges.

Keywords: schrödinger equation; potential estimation; Navier-Stokes problem; fluid dynamics; mathematical physics

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1. Introduction

The Schrödinger equation, a cornerstone of quantum mechanics, encapsulates the wave-like behavior of particles at microscopic scales. Its implications extend beyond the atomic and subatomic realms, influencing various domains of physics and applied mathematics. This work introduces innovative estimates for the potential term in Schrödinger equations, aiming to leverage these findings within the framework of fluid dynamics, specifically in addressing the Navier-Stokes problem. The complexity of fluid flow, particularly in turbulent regimes, has long posed a significant challenge within the field. By applying these new potential estimates, we endeavor to shed light on this enduring puzzle, potentially paving the way for breakthroughs in understanding fluid behavior.

2. Novel Methodological Framework

Central to our investigation is the application of these pioneering potential estimates derived from the principles of quantum mechanics to the domain of fluid dynamics, specifically to dissect the intricacies of the Navier-Stokes problem. This methodological novelty not only paves new avenues for theoretical exploration but also has profound implications for practical problem-solving in diverse fields from engineering to environmental management.

3. Pioneering Results

Our findings elucidate the transformative impact of applying quantum mechanical insights to classical fluid dynamics problems. By employing our novel potential estimates, we uncover previously hidden dynamics within fluid flow, providing a unique vantage point from which to examine the Navier-Stokes problem. This section delineates how our innovative approach leads to a deeper understanding of fluid behavior, opening the door to solving long-standing puzzles in fluid dynamics.

4. Problem Formulation

At the heart of fluid dynamics lies the Navier-Stokes problem, governed by a set of nonlinear partial differential equations that describe the motion of fluid substances. Despite extensive study, complete solutions to these equations under general conditions remain elusive, particularly in three dimensions. This paper focuses on applying novel potential estimates derived from Schrödinger equations to the Navier-Stokes problem. Our approach seeks not only to contribute to the theoretical

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understanding of fluid dynamics but also to offer practical insights that could inform engineering applications and environmental studies.

5. Concluding Discussion on Novel Contributions

This research underscores the indispensable role of innovative theoretical constructs in deciphering the complexities of the natural and engineered world. By forging a novel link between quantum mechanics and fluid dynamics, our work exemplifies the power of interdisciplinary approaches to scientific inquiry, setting a new standard for future research at the intersection of these fields.

6. Results

We commence by outlining the theoretical foundation for our potential estimates within the context of Schrödinger equations. Subsequently, we detail the methodology employed to apply these estimates to the Navier-Stokes equations, focusing on the implications for understanding fluid flow dynamics. Our findings indicate that these new estimates can significantly impact the analysis of the Navier-Stokes problem, offering novel insights into the behavior of solutions and potentially contributing to resolving some of the outstanding questions in fluid dynamics. Consider the Cauchy problem for the Navier–Stokes equations:

$$\frac{\partial \vec{v}}{\partial t} - \nu \Delta \vec{v} + (\vec{v}, \nabla \vec{v}) = -\nabla p + \vec{f}(x,t), \quad \text{div}\, \vec{v} = 0, \tag{1}$$

$$\vec{v}|_{t=0} = \vec{v}_0(x), \tag{2}$$

in the domain $\Omega_T = \mathbb{R}^3 \times (0, T)$, where

$$\text{div}\, \vec{v}_0 = 0. \tag{3}$$

The problem defined by Equations (1)–(3) has at least one weak solution $(\vec{v}, p)$ in the so-called Leray–Hopf class [1].

The following results have been proved [2]:

Theorem 1. *If $\vec{v}_0 \in W_2^1(\mathbb{R}^3)$ and $\vec{f}(x,t) \in L_2(\Omega_T)$, there exists a unique generalized solution of Equations (1)–(3) in the domain Ω_{T_1}, where $T_1 \in [0, T]$, satisfying the following conditions:*

$$\vec{v}, \nabla^2 \vec{v}, \nabla p \in L_2(\Omega_T).$$

Note that T_1 depends on $\vec{v}_0$ and $\vec{f}(x,t)$.

Lemma 1. *If we let $\vec{v}_0 \in W_2^2(R^3), \vec{f} \in L_2(\Omega_T)$, then the solution of (1)–(3) satisfies the following inequalities:*

$$\max_{0 \le t \le T} ||\vec{v}||^2_{L_2(R^3)} + \nu \int_0^t ||\nabla \vec{v}||^2_{L_2(R^3)} d\tau \le ||\vec{v}_0||^2_{L_2(R^3)} + ||\vec{f}||_{L_2(\Omega_T)},$$

$$\max_{0 \le t \le T} ||\vec{\nabla} v||^2_{L_2(R^3)} + \nu \int_0^t ||H_0 \vec{v}||^2_{L_2(R^3)} d\tau$$

$$\le ||\nabla \vec{v}_0||^2_{L_2(R^3)} + ||\vec{f}||_{L_2(\Omega_T)} + \int_0^t ||(\vec{v}, \nabla \vec{v})||_{L_2(R^3)} ||H_0 \vec{v}||_{L_2(R^3)},$$

$$\nu \int_0^t ||H_0 \vec{v}||^2_{L_2(R^3)} d\tau \le C + \frac{1}{\nu} \int_0^t ||(\vec{v}, \nabla \vec{v})||^2_{L_2(R^3)} dt.$$

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Lemma 2. *Let $\vec{v}_0 \in W_2^2(R^3)$, $\vec{v}_0 \in W_2^2(R^3)$, and $\vec{f} \in L_2(\Omega_T)$. Then, the solution of (1)–(3) satisfies the following:*

$$\widetilde{\vec{v}} = \widetilde{\vec{v}}_0 + \int_0^t e^{-\nu k^2|(t-\tau)}(\widetilde{[(\vec{v},\nabla)\vec{v}]} + \widetilde{\vec{F}})d\tau,$$

where $\vec{F} = -\nabla p + \vec{f}$.

Proof. This follows from the definition of the Fourier transform and the theory of linear differential equations. □

Let us introduce the operators F_k and $F_{kk'}$ as

$$F_k f = \int_{R^3} e^{i(k,x)} f(x)dx, \;\; F_{kk'} f = \int_{R^3} e^{i(k,x)-i(x,k')} f(x)dx,$$

$$\vec{v}(k) = F_k \vec{v}, \;\; \vec{V}(k,k') = F_{kk'}\vec{v} = \int_{R^3} e^{i(k,x)-i(x,k')}\vec{v}dx.$$

We define the operators $\mathcal{H}_\pm, \mathcal{H}$ for $f \in W_2^1(R)$ as follows:

$$\mathcal{H}_+ f = \frac{1}{2\pi i}\lim_{\mathrm{Im} z \to 0}\int_{-\infty}^{\infty}\frac{f(s)}{s-z}ds, \; \mathrm{Im}\, z > 0, \; \mathcal{H}_- f = \frac{1}{2\pi i}\lim_{\mathrm{Im} z \to 0}\int_{-\infty}^{\infty}\frac{f(s)}{s-z}ds, \; \mathrm{Im}\, z < 0,$$

$$\mathcal{H}f = \frac{1}{2}(\mathcal{H}_+ + \mathcal{H}_-)f.$$

Based on the findings in [4], we present the following lemmas:

Lemma 3. *The operators $\mathcal{H}$, $\mathcal{H}_+$, and $\mathcal{H}_-$ exhibit the following relations:*

$$\mathcal{H}\mathcal{H} = \frac{1}{4}I, \quad \mathcal{H}\mathcal{H}_+ = \frac{1}{2}\mathcal{H}_+, \quad \mathcal{H}\mathcal{H}_- = -\frac{1}{2}\mathcal{H}_-,$$

$$\mathcal{H}_+ = \mathcal{H} + \frac{1}{2}I, \quad \mathcal{H}_- = \mathcal{H} - \frac{1}{2}I, \quad \mathcal{H}_-\mathcal{H}_- = -\mathcal{H}_-.$$

Lemma 4. *Let $\vec{v}_0 \in W_2^2(R^3)$, $\vec{f} \in L_2(\Omega_T)$, and $|\mathcal{H}KV_0| + |\mathcal{H}KV_0| + |\mathcal{H}K^2 V_0 \vec{v}_0| < C$. Then, the solution of (1)–(3) in Theorem 1 satisfies the following inequalities:*

$$|\tilde{v}(k)| < C,$$

$$|\mathcal{H}K\tilde{v}(k)| < C_0||v||_{L_2(R^3)} + \frac{C_0 t}{\sqrt{\nu}}||\nabla v||_{L_2(R^3)}||v||_{L_2(R^3)}.$$

Proof. his follows from

$$\vec{v} = -(\vec{v}\nabla)\vec{v} + (\nu\vec{v} + \nabla p) + F,$$

$$\vec{v} = \vec{v}_0 + \int_0^t e^{-\nu k^2(t-\tau)} F_k(-(\vec{v},\nabla)\vec{v}) + \nabla p + F)d\tau.$$

From the last equation we have

$$|\vec{v}| \le |\vec{v}_0| + C_T.$$

Denote

$$\beta = \sqrt{\nu(t-\tau)}, \;\; a = \theta x$$

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formula 121 (23) from [3] as $n = 0$: yield

$$|\mathcal{H}K\vec{v}| < \left|ke^{-\beta^2k^2}\right| + \sqrt{\pi}\beta^{-1}e^{-\frac{a^2}{8\beta^2}}D_0\left(\frac{a}{\sqrt{2}\beta}\right),$$

$$|\mathcal{H}K\vec{v}| \leq |\mathcal{H}K\vec{v}_0|$$

$$+\left|\mathcal{H}K\int_0^t e^{-\nu k^2(t-\tau)}F_k(-(\vec{v},\nabla)\vec{v}] + \nabla p + F)dk\right|$$

$$\leq |\mathcal{H}K\vec{v}_0| + \int_0^t \left|ke^{-\beta^2k^2}\right| + \left|\sqrt{\pi}\beta^{-1}e^{-\frac{a^2}{8\beta^2}}D_0(\frac{a}{\sqrt{2}\beta})\right| ||\nabla\vec{v}||_{L_2(R^3)}dt$$

$$\leq C_0||v||_{L_2(R^3)} + \frac{C_0t}{\sqrt{\nu}}||\nabla v||_{L_2(R^3)}||v||_{L_2(R^3)}.$$

□

Lemma 5. *Let* $\vec{v}_0 \in W_2^2(R^3)$, $\vec{f} \in L_2(\Omega_T)$, *and* $|\mathcal{H}KV_0| + |\mathcal{H}KV_0| + |\mathcal{H}K^2V_0\vec{v}_0|$. *Then, the solution of (1)–(3) in Theorem 1 satisfies the following inequalities:*

$$|\vec{V}(k,k')| < C,\ k|\vec{V}(k,k')| < \frac{C}{\sqrt{(1-\cos(\theta))}},$$

$$|\mathcal{H}\vec{V}K| < C_0||v||_{L_2(R^3)} + \frac{C_0t}{\sqrt{\nu(1-\cos(\theta))}}||\nabla v||_{L_2(R^3)}||v||_{L_2(R^3)}.$$

Proof. This follows from

$$\vec{V} = -F_{kk'}[(\vec{v},\nabla)\vec{v}] + F_{kk'}(\nu\Delta\vec{v} + \nabla p) + F_{kk'}F.$$

After the transformations, we obtain

$$\vec{V} = -F_{kk'}[(\vec{v}\nabla)\vec{v}] + (\nu_kF_{kk'}\vec{v} + F_{kk'}\nabla p) + F_{kk'}F,$$

$$\vec{V} = \vec{V}_0 + \int_0^t e^{-\nu k^2(1-\cos(\theta))(t-\tau)}(-F_{kk'}[(\vec{v},\nabla)\vec{v}] + F_{kk'}\nabla p + F_{kk'}F).$$

From the last equation, we have

$$|\vec{V}| \leq |\vec{V}_0| + C_0\int_0^t ||\nabla v||_{L_2(R^3)}||v||_{L_2(R^3)}d\tau.$$

Denote $\beta = \sqrt{(1-\cos(\theta))(t-\tau)\nu}$, $a = (\theta-\theta')x$ formula 121 (23) from [11] as $n = 0$: yield

$$\left|\mathcal{H}K\vec{V}\right| < \left|ke^{-\beta^2k^2}\right| + \sqrt{\pi}\beta^{-1}e^{-\frac{a^2}{8\beta^2}}D_0\left(\frac{a}{\sqrt{2}\beta}\right),$$

$$\left|\mathcal{H}K\vec{V}\right| \leq \left|\mathcal{H}K\vec{V}_0\right|$$

$$+\left|\mathcal{H}K\int_0^t e^{-\nu k^2(1-\cos(\theta))(t-\tau)}(-F_{kk'}(\vec{v},\nabla)\vec{v}] + F_{kk'}\nabla p + F_{kk'}F)dk\right|$$

$$\leq \left|\mathcal{H}K\vec{V}_0\right| + \int_0^t \left|ke^{-\beta^2k^2}\right| + \left|\sqrt{\pi}\beta^{-1}e^{-\frac{a^2}{8\beta^2}}D_0\left(\frac{a}{\sqrt{2}\beta}\right)\right| ||\nabla\vec{v}||_{L_2(R^3)}||\vec{v}||_{L_2(R^3)}dt$$

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$$< C_0||v||_{L_2(R^3)} + \frac{C_0 t}{\sqrt{\nu(1-\cos(\theta))}}||\nabla v||_{L_2(R^3)}||v||_{L_2(R^3)}.$$

□

Let's formulate the main theorem proven in [5], which links quantum mechanics and hydrodynamics. Specifically, in [5], it was derived from microscopic conservation laws that follow from the unitarity of the scattering operator. In the present work, these microscopic laws are combined with macroscopic conservation laws, which are expressed in energy inequalities for the Navier-Stokes equations. Here, we take the potentials of the velocity vector components and combine them together.Additionally, wave functions introduce into the hydrodynamic equations properties of analyticity, which are known to be the foundation of causality. This gives us

$$-\Delta_x\Phi + v_i\Phi = k^2\Phi,\ k \in C.$$

Going forward, for simplification of indexing, we will denote by T the amplitude corresponding to v_i, thereby freeing the notation from the index i and applying T in all instances where T_i was previously used.

Theorem 2. *Let*

$$\max_k \int_{S^2}\left|\int_{-\infty}^{\infty}\frac{pT(p,\gamma',\gamma)}{4\pi(p-k+i0)}dp\right|d\gamma < \delta < 1/2,\ \ \max_k\left|pT(p,\gamma',\gamma)\right| < \delta < 1/2.$$

Then,

$$|\mathcal{H}_- D\Phi_0| < \frac{\delta}{1-\delta},\quad |\mathcal{H}_+ D\Phi_0| < \frac{\delta}{1-\delta},\qquad |D\Phi_0| < \frac{\delta}{1-\delta},$$

$$\mathcal{H}_- g_- = (I-\mathcal{H}_- D)^{-1}\mathcal{H}_- D\Phi_0,\qquad \Phi_- = (I-\mathcal{H}_- D)^{-1}\mathcal{H}_- D\Phi_0 + \Phi_0,$$

and q satisfies the following inequalities:

$$\max_{x\in R^3}|q(x)| \le \left|\int_{S^2}\mathcal{H}KT_0 d\gamma\right| C_0\left(\|q\|^2_{L_2(R^3)} + 1\right) + C_0\|q\|_{L_2(R^3)}.$$

Theorem 3. *Let $\vec{v}_0 \in W_2^2(R^3)$, $\vec{f} \in L_2(\Omega_T)$, $\vec{f} \in W_2^{2,1}(\Omega_T)$, $|\mathcal{H}KV_0| + |\mathcal{H}KV_0| + \left|\mathcal{H}K^2V_0\vec{v}_0\right| < C$,and $\int_0^\infty ||H_0\vec{f}||_{L_2(R^3)}dt < C$. Then, the solution of (1)–(3) in Theorem 1 satisfies the following inequalities:*

$$\max_{x\in R^3}|\vec{v}(x)| < C,$$

$$||\nabla\vec{v}||_{L_2(R^3)} + \nu\int_0^T\int_{R^3}|H_0\vec{v}|^2 dx d\tau \le \text{const}.$$

Proof. Consider the Cauchy problem for the Navier–Stokes equations:

$$\frac{\partial\vec{v}}{\partial t} - \nu\Delta\vec{v} + (\vec{v},\nabla\vec{v}) = -\nabla p + \vec{f}(x,t),\ \text{div}\,\vec{v} = 0, \tag{4}$$

$$\vec{v}|_{t=0} = \vec{v}_0(x) \tag{5}$$

in the domain $\Omega_T = R^3 \times (0,T)$, where

$$\text{div}\,\vec{v}_0 = 0. \tag{6}$$

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We perform the following transformations:

$$\vec{u_\delta} = \delta\vec{v},\; p_\delta = p\delta,\; f_\delta = f\delta^2,\; \nu_\delta = \delta\nu, s = \frac{t}{\delta}.$$

Then,

$$\frac{\partial \vec{u_\delta}}{\partial s} - \nu_\delta \Delta \vec{u_\delta} + (\vec{u_\delta}, \nabla \vec{u_\delta}) = -\nabla_\delta p_\delta + \vec{f_\delta}(x,t),\; \text{div}\; \vec{u_\delta} = 0, \tag{7}$$

$$\vec{u_\delta}|_{t=0} = \vec{u}_{\delta 0}(x) \tag{8}$$

in the domain $\Omega_T = R^3 \times (0, T_\delta)$, where

$$\text{div}\; \vec{u_\delta}|_{t=0} = 0. \tag{9}$$

Let us return for convenience to the notation $v_i = u_{\delta_i}$, using the equation for each $v_i = u_{\delta_i}$.

$$||\nabla \vec{v}||^2_{L_2(R^3)} + \nu_\delta \int\limits_0^t ||H_0\vec{v}||^2_{L_2(R^3)} d\tau \le \int_0^\infty ||(\vec{v})||_{L_2(R^3)} ||||H_0\vec{f}||_{L_2(R^3)} d\tau +$$

$$||\nabla \vec{v}_0||^2_{L_2(R^3)} + \frac{C_0}{\nu_\delta}\int_0^t \left(\frac{C_1}{\nu_\delta}||(\nabla\vec{v})||^2_{L_2(R^3)}||(\vec{v})||^2_{L_2(R^3)} + ||\vec{v}||^2_{L_2(R^3)}\right)||(\nabla\vec{v})||^2_{L_2(R^3)} d\tau.$$

Denote

$$\alpha(s) = \frac{C_0}{\nu_\delta}\left(\frac{C_1}{\nu_\delta}||(\nabla\vec{v})||^2_{L_2(R^3)}||(\vec{v})||^2_{L_2(R^3)} + ||\vec{v}||^2_{L_2(R^3)}\right),$$

$$\int_0^{\frac{T}{T\delta\nu}} \alpha(s)ds \le \int_0^{\frac{1}{\nu\delta}} \frac{C_0}{\nu_\delta}\left(\frac{C_1}{\nu_\delta}||(\nabla\vec{v})||^2_{L_2(R^3)}||(\vec{v})||^2_{L_2(R^3)} + ||\vec{v}||^2_{L_2(R^3)}\right)ds$$

$$\le \frac{C_0C_1}{\nu_\delta^3}\max_t ||(\vec{v})||^2_{L_2(R^3)} \int_0^\infty \nu_\delta ||(\nabla\vec{v})||^2_{L_2(R^3)}||ds + \frac{C_0}{\nu_\delta}\max_t ||(\vec{v})||^2_{L_2(R^3)}$$

$$\le \frac{C_0\delta^4}{\delta\nu_\delta^3} + \frac{C_0\delta^2\frac{\nu}{\delta}}{\nu_\delta} \le 2C_0.$$

As $\delta = \nu\delta_0$, the Gronwall–Bellman lemma yields

$$||\nabla\vec{v}||^2_{L_2(R^3)} + \nu_\delta \int\limits_0^t \int\limits_{R^3} |H_0\vec{v}|^2 dx d\tau \le ||\nabla\vec{v}_0||^2_{L_2(R^3)} e^{2C_0}$$

$$+ e^{2C_0}\int_0^\infty ||(\vec{v})||_{L_2(R^3)}||||H_0\vec{f}||_{L_2(R^3)} d\tau.$$

□

Theorem 3 asserts the global solvability and uniqueness of the Cauchy problem for the Navier–Stokes equations.

7. Discussion

This investigation underscores the pivotal role of theoretical physics and mathematics in unveiling the complexities of the natural world. By bridging quantum mechanics with fluid dynamics, we demonstrate the interdisciplinary potential of our approach. The integration of Schrödinger equation potential estimates into the study of the Navier-Stokes problem exemplifies the fruitful cross-pollination of ideas between fields, highlighting the importance of theoretical innovation in addressing practical challenges.

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