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Valuation of Currency Option based on Uncertain Fractional Differential Equation

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Abstract Uncertain fractional differential equation (UFDE) is an excellent tool for describing complicated dynamic systems. This paper analyzes the valuation problems of currency option based on UFDE under optimistic value criterion. Firstly, an uncertain fractional currency model is formulated to describe the dynamics of foreign exchange rate. Then, the pricing formulae of European, American and Asian currency option are obtained under optimistic value criterion, respectively. Finally, numerical simulations are provided to illustrate the validity of our theoretical results.

Keywords Uncertainty; Fractional-order differential equation; Currency option; Optimistic value

1. Introduction

A currency option is a contract that offers its holder the entitlement to purchase a certain amount of foreign currency with a predetermined exchange rate at maturity or within the validity period. This kind of option not only has the characteristic of nonlinear profits, it also has the functions of investment, hedging and risk avoidance. Compared to traditional forward foreign exchange contract, currency option is a more flexible derivative. The first development of pricing currency option may be traced back to the study of Garman and Kohlhagen (1983). They assumed that the exchange rate followed a geometric Brownian motion and derived the pricing formula of European currency option. Despite the popularity of Garman and Kohlhagen (G-K) model, there were still some inconsistencies in the model. From then on, many methodologies of pricing currency option were introduced by modifying G-K model, such as Baldeauxa et al. (2015), Lian and Chen (2022). Considering the fluctuation of interest rate over time, Grabbe (1983) assumed that both interest rates and exchange rate to be stochastic and constructed a new pricing model for currency option. Motivated by this idea, many scholars gave numerous extensions of Grabbe model, such as Carr and Wu (2007), Van and Pelsser (2011).

As mentioned above, the previous studies on currency option pricing are based on probability theory. A precondition of using probabilistic theory is that the distribution function is close enough to the frequency in the future, but numerous empirical studies show that the real world is far from frequency stability. This fact makes the distribution function obtained in practice usually deviate from the frequency. Based on this situation, we cannot but rely on the belief degrees of domain experts for each indeterminate event. To deal with belief degree reasonably, Liu (2007) established uncertainty theory based on normality, duality, subadditivity and product axioms. Nowadays it has become a branch of axiomatic mathematics to simulate human uncertainty and received wide applications.

Owing to some antinomies in random theory (Liu 2013), uncertain financial theory on the basis of uncertain differential equations (UDEs, Liu 2008) emerges as the times require. If the underlying stock price followed a geometric Liu process, Liu (2009) presented an uncertain stock model and derived its European option pricing formulas. Afterwards, Liu et al. (2015) applied UDE to model

exchange rate and first put forward an uncertain currency model. Shen and Yao (2016) extended this model to a mean-reverting currency model and derived the corresponding currency option prices. Considering the fierce drifts of exchange rate, Ji and Wu (2017) provided a currency exchange rate model with jumps. Li and Sheng (2022) gave an uncertain exponential currency model and calculated the currency option prices. For the recent research on uncertain finance, readers can refer to references (Deng and Qin 2021; Yan et al. 2022; Jin et al. 2023).

These models are built on the differential equations of first order. In realistic financial markets, the future asset price not only relies on the current price, but it also correlates with the past condition. Different from integer-order differential equations, fractional differential equations (FDEs) have the characteristics of long memory and non-locality, which can better model the fluctuations of asset prices. Zhu (2015) first introduced uncertain factors into FDEs and presented the definition of UFDE. Then, Lu and Zhu (2019) put forward the concept of α -path and gave an efficient numerical method. Based on UFDE, Lu et al. (2019a) presented a new type of uncertain stock model and calculated its European option prices. Lu et al. (2019b) discussed the prices of Asian option under uncertain fractional stock model. Following that, Liu et al. (2022) used UFDE to describe exchange rate and proposed an uncertain fractional currency model. In addition, other financial problems were also discussed based on UFDE, such as barrier option (Jin et al. 2021), barrier swaption (Lu et al. 2022), and vulnerable option (Lei et al. 2023).

Because of the possibility of drastic fluctuation in asset price, investors would not always take expected value criteria as their sole investment guideline. Instead, they usually consider the best case at a certain confidence level. Based on the analysis above, this paper will study the valuation issues of currency option under optimistic criterion based on UFDE. To the best of our knowledge, the research on this topic has not yet been done. This paper is structured as follows. Section 2 reviews some useful results on UFDE and an uncertain fractional currency model. Section 3 presents a new currency model and computes the prices of European, American and Asian currency option based on optimistic value criterion, respectively. Section 4 performs numerical examples to illustrate our theoretical results. Section 5 provides a concise conclusion.

2. Preliminaries

In this part, we recall some important results on UFDE and an uncertain fractional currency model. For more information about uncertainty theory, readers may refer to Liu (2015) for a better understanding. Unless stated otherwise, the fractional order p is always assumed to satisfy $n-1 < p \leq n, \ \ n \in N_{+}$.

2.1. UFDE with Caputo Type

Zhu (2015) proposed two kinds of UFDEs: Riemann-Liouville type and Caputo type. As previously discussed (Ford and Simpson 2001; Diethelm et al. 2002), Caputo fractional derivative is more appropriate to model uncertain dynamic systems than Riemann-Liouville derivative. Thus, this paper only focuses on the Caputo type of UFDE.

Definition 2.1 (Zhu 2015) Assume g_1, g_2 are two functions defined on $[0,+\infty) \times R$ and C_t is a Liu process. Then

$$^{c}D^{p}Y_t = g_1(t,Y_t) + g_2(t,Y_t)\frac{dC_t}{dt} \tag{1}$$

is called the UFDE of Caputo type.

Definition 2.2 (Lu et al. 2019a) Assume g_1, g_2 are two functions defined on $[0,+\infty) \times R$. Then the solution of the related equation with initial conditions

$$\begin{cases} ^{c}D^{p}Y_t = g_1(t,Y_t) + g_2(t,Y_t)\dfrac{dC_t}{dt} \\ Y^{(l)}(0) = y_l, l = 0,1,\ldots,n-1 \end{cases} \tag{2}$$

is an uncertain process Y_t that satisfies

$$Y_t = \sum_{l=0}^{n-1} \frac{y_l t^l}{\Gamma(l+1)} + \frac{1}{\Gamma(p)} \int_0^t (t-v)^{p-1} g_1(v, Y_v) dv + \frac{1}{\Gamma(p)} \int_0^t (t-v)^{p-1} g_2(v, Y_v) dC_v. \quad (3)$$

Definition 2.3 (Lu and Zhu 2019) An UFDE (2) is considered to have an α-path Y_t^α $(0 < \alpha < 1)$ if it is the solution of the associated FDE

$$\begin{cases} {}^c D^p Y_t^\alpha = g_1(t, Y_t^\alpha) + \left| g_2(t, Y_t^\alpha) \right| \gamma^{-1}(\alpha), \\ Y^{(l)}(0) = y_l, l = 0,1,\ldots,n-1 \end{cases} \quad (4)$$

where $\gamma^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}$.

Theorem 2.1 *(Lu and Zhu 2019) Assume UFDE (2) has a solution Y_t and α-path Y_t^α. Then*

$$M\left\{Y_t \leq Y_t^\alpha, \forall t\right\} = \alpha, \quad (5)$$

$$M\left\{Y_t > Y_t^\alpha, \forall t\right\} = 1 - \alpha. \quad (6)$$

Moreover, Y_t has an inverse uncertainty distribution (IUD)

$$I_t^{-1}(\alpha) = Y_t^\alpha. \quad (7)$$

Theorem 2.2 *(Jin et al. 2019) Assume UFDE (2) has a solution Y_t and α-path Y_t^α. Then the supremum $\sup_{0 \leq t \leq T} H(Y_t)$ has an IUD*

$$\Psi_T^{-1}(\alpha) = \sup_{0 \leq t \leq T} H(Y_t^\alpha) \quad \left(\zeta_T^{-1}(\alpha) = \sup_{0 \leq t \leq T} H(Y_t^{1-\alpha}) \right), \quad (8)$$

where $H(y)$ is a strictly increasing (decreasing) function.

Theorem 2.3 *(Jin et al. 2020) Assume UFDE (2) has a solution Y_t and α-path Y_t^α. Then the integral $\int_0^T H(Y_t) dt$ has an IUD*

$$\Psi_T^{-1}(\alpha) = \int_0^T H(Y_t^\alpha) dt \quad \left(\zeta_T^{-1}(\alpha) = \int_0^T H(Y_t^{1-\alpha}) dt \right), \quad (9)$$

where $H(y)$ is a strictly increasing (decreasing) function.

2.2. Uncertain Fractional Currency Model

In realistic financial markets, the future exchange rate not only relies on the current value, it also correlates with the past state. Unlike integer order differential equation, FDE provides an excellent tool to reflect the memory and hereditary properties of exchange rate. Assume X_t and Y_t are the domestic riskless currency and foreign riskless currency, respectively, and that foreign exchange rate Z_t obeys an UFDE of Caputo type, Liu et al. (2022) introduced an uncertain fractional currency model

$$\begin{cases} dX_t = r_d X_t dt \\ dY_t = r_f Y_t dt \\ {}^c D^p Z_t = a Z_t + b Z_t \dfrac{dC_t}{dt} \\ Z_t^{(l)} \big|_{t=0} = z_l, l = 0,1,\ldots,n-1 \end{cases} \quad (10)$$

where r_d and r_f denote the domestic interest rate and foreign interest rate, respectively, a and b are the drift and diffusion of foreign exchange rate, respectively, and C_t is a canonical Liu process.

3. Currency Option Pricing

In real financial markets, investors would not always take expected value criteria as their sole guideline due to the possibility of drastic fluctuation in asset price. Based on the confidence in the change trend of future asset price, investors usually consider the maximum benefit at a certain confidence level. Lu et al. (2019b) first introduced the idea of VaR into the research of option pricing and discussed the valuation of Asian option based on optimistic criterion. Motivated by these ideas, this section studies the prices of currency option under optimistic criterion.

3.1. Mean-Reverting Uncertain Fractional Currency Model

In model (10), Liu et al. (2022) analyzed the short-run fluctuations in exchange rate. However, the actual exchange rate oscillates periodically around a certain level in a long run. Considering this situation, we present a mean-reverting currency model for the long term

$$\begin{cases} dX_t = r_d X_t dt \\ dY_t = r_f Y_t dt \\ {}^c D^p Z_t = \left(a - b Z_t\right) + c \dfrac{dC_t}{dt} \\ Z_t^{(l)}\big|_{t=0} = z_l, l = 0,1,\ldots,n-1 \end{cases} \quad (11)$$

where $a,b,c > 0$, b is the reversion rate, a/b is the average exchange rate, c is the exchange rate's volatility.

3.2. European Currency Option

Suppose a European currency option's striking price is K and that maturity date is T. Set f_1 in domestic currency as the contract price. At initial time 0, the investor pays f_1 for purchasing this contract and receives the maximum benefit at time T based on optimistic criterion $[(Z_T - K)^+]_{\sup}(\delta)$, where δ is the confidence level. Hence, the investor obtains the net return at time 0

$$- f_1 + \exp\left(- r_d T\right)[(Z_T - K)^+]_{\sup}(\delta).$$

On the other hand, the bank obtains profit f_1 for selling this contract at time 0, and pays $\left[\left(1 - \dfrac{K}{Z_T}\right)^+\right]_{\sup}(\delta)$ in foreign currency at time T. Hence, the bank can receive the net earning at time 0

$$f_1 - Z_0 \exp\left(- r_f T\right)\left[\left(1 - \frac{K}{Z_T}\right)^+\right]_{\sup}(\delta).$$

It follows from the fair price principle that the definition of a European currency option price is given as follows.

Definition 3.1 Assuming that a European currency option's striking price is K and maturity date is T. Then the option price under confidence level δ based on optimistic criterion is

$$f_1 = \frac{1}{2}\exp\left(- r_d T\right)[(Z_T - K)^+]_{\sup}(\delta) + \frac{Z_0}{2}\exp\left(- r_f T\right)\left[\left(1 - \frac{K}{Z_T}\right)^+\right]_{\sup}(\delta). \quad (12)$$

Theorem 3.1 *Suppose a European currency option for model (11) has a striking price K and that its expiration time is T. Then the option price under confidence level δ based on optimistic value criteria is*

$$f_1 = \frac{1}{2}\exp(-r_d T)\left[\Psi_T^{-1}(1-\delta) - K\right]^+ + \frac{Z_0}{2}\exp(-r_f T)\left[1 - \frac{K}{\Psi_T^{-1}(1-\delta)}\right]^+, \qquad (13)$$

$$\Psi_T^{-1}(1-\delta) = \sum_{l=0}^{n-1} z_l T^l E_{p,(l+1)}\left(-bT^p\right) + \left(a + \frac{c\sqrt{3}}{\pi}\ln\frac{1-\delta}{\delta}\right)T^p E_{p,(p+1)}\left(-bT^p\right),$$

where

$$E_{x,y}(q) = \sum_{u=0}^{\infty} \frac{q^u}{\Gamma(ux+y)}.$$

Proof. Based on Lu and Zhu (2019), we get the IUD of foreign exchange rate Z_t

$$\Psi_t^{-1}(\alpha) = Z_t^\alpha$$

$$= \sum_{l=0}^{n-1} z_l t^l E_{p,(l+1)}\left(-bt^p\right) + \left(a + \frac{c\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)t^p E_{p,(p+1)}\left(-bt^p\right).$$

According to the definition of optimistic value for uncertain variable, we get

$$\left[(Z_T - K)^+\right]_{\sup}(\delta) = \sup_{\eta \geq 0}\left\{\eta \middle| M\left\{(Z_T - K)^+ \geq \eta\right\} \geq \delta\right\}$$

$$= \sup_{\eta > 0}\left\{\eta \middle| M\left\{Z_T - K \geq \eta\right\} \geq \delta\right\} \vee 0$$

$$= \sup_{\eta > 0}\left\{\eta \middle| 1 - M\left\{Z_T < K + \eta\right\} \geq \delta\right\} \vee 0$$

$$= \sup_{\eta > 0}\left\{\eta \middle| 1 - \Psi_T\left(K + \eta\right) \geq \delta\right\} \vee 0$$

$$= \sup_{\eta > 0}\left\{\eta \middle| \Psi_T\left(K + \eta\right) \leq 1 - \delta\right\} \vee 0$$

$$= \sup_{\eta > 0}\left\{\eta \middle| \Psi_T^{-1}\left(\Psi_T\left(K + \eta\right)\right) \leq \Psi_T^{-1}\left(1 - \delta\right)\right\} \vee 0$$

$$= \sup_{\eta > 0}\left\{\eta \middle| K + \eta \leq \Psi_T^{-1}\left(1 - \delta\right)\right\} \vee 0$$

$$= \sup_{\eta > 0}\left\{\eta \middle| \eta \leq \Psi_T^{-1}\left(1 - \delta\right) - K\right\} \vee 0$$

$$= \left[\Psi_T^{-1}\left(1 - \delta\right) - K\right]^+.$$

By use of the similar method, we can get

$$\left[\left(1 - \frac{K}{Z_T}\right)^+\right]_{\sup}(\delta) = \sup_{\xi \geq 0}\left\{\xi \middle| M\left\{\left(1 - \frac{K}{Z_T}\right)^+ \geq \xi\right\} \geq \delta\right\}$$

$$= \sup_{\xi > 0}\left\{\xi \middle| M\left\{1 - \frac{K}{Z_T} \geq \xi\right\} \geq \delta\right\} \vee 0$$

$$= \sup_{\xi > 0}\left\{\xi \middle| M\left\{(1 - \xi)Z_T \geq K\right\} \geq \delta\right\} \vee 0$$

$$= \sup_{0 < \xi < 1}\left\{\xi \middle| M\left\{Z_T \geq \frac{K}{1 - \xi}\right\} \geq \delta\right\} \vee 0$$

$$= \sup_{0 < \xi < 1}\left\{\xi \middle| 1 - M\left\{Z_T < \frac{K}{1 - \xi}\right\} \geq \delta\right\} \vee 0$$

$$= \sup_{0<\xi<1}\left\{\xi \left| 1 - \Psi_T\left(\frac{K}{1-\xi}\right) \geq \delta \right.\right\} \vee 0$$

$$= \sup_{0<\xi<1}\left\{\xi \left| \Psi_T\left(\frac{K}{1-\xi}\right) \leq 1-\delta \right.\right\} \vee 0$$

$$= \sup_{0<\xi<1}\left\{\xi \left| \Psi_T^{-1}\left(\Psi_T\left(\frac{K}{1-\xi}\right)\right) \leq \Psi_T^{-1}(1-\delta) \right.\right\} \vee 0$$

$$= \sup_{0<\xi<1}\left\{\xi \left| \frac{K}{1-\xi} \leq \Psi_T^{-1}(1-\delta) \right.\right\} \vee 0$$

$$= \sup_{0<\xi<1}\left\{\xi \left| \xi \leq 1 - \frac{K}{\Psi_T^{-1}(1-\delta)} \right.\right\} \vee 0$$

$$= \left[1 - \frac{K}{\Psi_T^{-1}(1-\delta)} \right]^+.$$

It follows from Definition 3.1 that we finally have

$$f_1 = \frac{1}{2}\exp(-r_d T)\big[(Z_T - K)^+\big]_{\sup}(\delta) + \frac{Z_0}{2}\exp(-r_f T)\left[\left(1 - \frac{K}{Z_T}\right)^+\right]_{\sup}(\delta)$$

$$= \frac{1}{2}\exp(-r_d T)\big[\Psi_T^{-1}(1-\delta) - K\big]^+ + \frac{Z_0}{2}\exp(-r_f T)\left[1 - \frac{K}{\Psi_T^{-1}(1-\delta)}\right]^+,$$

$$\Psi_T^{-1}(1-\delta) = \sum_{l=0}^{n-1} z_l T^l E_{p,(l+1)}\left(-bT^p\right) + \left(a + \frac{c\sqrt{3}}{\pi}\ln\frac{1-\delta}{\delta}\right)T^p E_{p,(p+1)}\left(-bT^p\right).$$

where

Hence, the result (13) is proved. □

3.3. American Currency Option

Suppose an American currency option's striking price is K and that maturity date is T. Set f_2 in domestic currency as the contract price. At time 0, the investor pays f_2 for purchasing this contract and receives the maximum benefit based on optimistic value criterion $\left[\sup_{0\leq t\leq T}\exp(-r_d t)(Z_t - K)^+\right]_{\sup}(\delta),$ where δ is the confidence level. Hence, the investor obtains the net return at time 0

$$-f_2 + \left[\sup_{0\leq t\leq T}\exp(-r_d t)(Z_t - K)^+\right]_{\sup}(\delta).$$

Meanwhile, the bank obtains profit f_2 for selling this contract at time 0, and pays $\left[\sup_{0\leq t\leq T}\exp(-r_f t)Z_0\left(1 - \frac{K}{Z_t}\right)^+\right]_{\sup}(\delta).$ Hence, the bank can receive the net earning at time 0

$$f_2 - \left[\sup_{0\leq t\leq T}\exp(-r_f t)Z_0\left(1 - \frac{K}{Z_t}\right)^+\right]_{\sup}(\delta).$$

It follows from the fair price principle that the definition of an American currency option price is given as follows.

Definition 3.2 Assuming that an American currency option's striking price is K and maturity date is T. Then the option price under confidence level δ based on optimistic criterion is

$$f_2 = \frac{1}{2}\left[\sup_{0\leq t\leq T}\exp(-r_d t)(Z_t - K)^+\right]_{\sup}(\delta) + \frac{Z_0}{2}\left[\sup_{0\leq t\leq T}\exp(-r_f t)\left(1 - \frac{K}{Z_t}\right)^+\right]_{\sup}(\delta). \quad (14)$$

Theorem 3.2 *Suppose that an American currency option of model (11) has a striking price K and its maturity date is T. Then the option price under confidence level δ based on optimistic value criteria is*

$$f_2 = \frac{1}{2}\left[\sup_{0\leq t\leq T}\exp(-r_d t)\left(\Psi_t^{-1}(1-\delta) - K\right)^+\right] + \frac{Z_0}{2}\left[\sup_{0\leq t\leq T}\exp(-r_f t)\left(1 - \frac{K}{\Psi_t^{-1}(1-\delta)}\right)^+\right], \quad (15)$$

$$\Psi_t^{-1}(1-\delta) = \sum_{l=0}^{n-1} z_l t^l E_{p,(l+1)}\left(-bt^p\right) + \left(a + \frac{c\sqrt{3}}{\pi}\ln\frac{1-\delta}{\delta}\right)t^p E_{p,(p+1)}\left(-bt^p\right).$$

where

Proof. Based on Lu and Zhu (2019), we can easily obtain that Z_t has an IUD

$$\Psi_t^{-1}(\alpha) = Z_t^{\alpha}$$

$$= \sum_{l=0}^{n-1} z_l t^l E_{p,(l+1)}\left(-bt^p\right) + \left(a + \frac{c\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)t^p E_{p,(p+1)}\left(-bt^p\right).$$

It follows from Theorem 2.2 that the IUD of $\sup\limits_{0\leq t\leq T}\exp(-r_d t)(Z_t - K)^+$ is

$$\phi_T^{-1}(\alpha) = \sup_{0\leq t\leq T}\exp(-r_d t)\left(\Psi_t^{-1}(\alpha) - K\right)^+,$$

and the IUD of $\sup\limits_{0\leq t\leq T}\exp(-r_f t)\left(1 - \frac{K}{Z_t}\right)^+$ is

$$\Omega_T^{-1}(\alpha) = \sup_{0\leq t\leq T}\exp(-r_f t)\left(1 - \frac{K}{\Psi_t^{-1}(\alpha)}\right)^+.$$

According to the definition of optimistic value for uncertain variable, we have

$$\left[\sup_{0\leq t\leq T}\exp(-r_d t)(Z_t - K)^+\right]_{\sup}(\delta) = \sup_{\varpi\geq 0}\left\{\varpi\left|\mathrm{M}\left\{\sup_{0\leq t\leq T}\exp(-r_d t)(Z_t - K)^+ \geq \varpi\right\} \geq \delta\right.\right\}$$

$$= \sup_{\varpi\geq 0}\left\{\varpi\left|1 - \mathrm{M}\left\{\sup_{0\leq t\leq T}\exp(-r_d t)(Z_t - K)^+ < \varpi\right\} \geq \delta\right.\right\}$$

$$= \sup_{\varpi\geq 0}\left\{\varpi\left|1 - \phi_T(\varpi) \geq \delta\right.\right\}$$

$$= \sup_{\varpi\geq 0}\left\{\varpi\left|\phi_T(\varpi) \leq 1 - \delta\right.\right\}$$

$$= \sup_{\varpi\geq 0}\left\{\varpi\left|\phi_T^{-1}(\phi_T(\varpi)) \leq \phi_T^{-1}(1-\delta)\right.\right\}$$

$$= \sup_{\varpi\geq 0}\left\{\varpi\left|\varpi \leq \phi_T^{-1}(1-\delta)\right.\right\}$$

$$= \phi_T^{-1}(1-\delta).$$

By use of the similar method, we can get

$$\left[\sup_{0\leq t\leq T}\exp\left(-r_f t\right)\left(1 - \frac{K}{Z_t}\right)^+\right]_{\sup}(\delta) = \sup_{\beta\geq 0}\left\{\beta\left|\mathrm{M}\left\{\sup_{0\leq t\leq T}\exp\left(-r_f t\right)\left(1 - \frac{K}{Z_t}\right)^+ \geq \beta\right\} \geq \delta\right.\right\}$$

$$= \sup_{\beta \geq 0} \left\{ \beta \left| 1 - M \left\{ \sup_{0 \leq t \leq T} \exp(-r_f t) \left(1 - \frac{K}{Z_t} \right)^+ < \beta \right\} \geq \delta \right. \right\}$$

$$= \sup_{\beta \geq 0} \left\{ \beta | 1 - \Omega_T(\beta) \geq \delta \right\}$$

$$= \sup_{\beta \geq 0} \left\{ \beta | \Omega_T(\beta) \leq 1 - \delta \right\}$$

$$= \sup_{\beta \geq 0} \left\{ \beta | \Omega_T^{-1}(\Omega_T(\beta)) \leq \Omega_T^{-1}(1 - \delta) \right\}$$

$$= \sup_{\beta \geq 0} \left\{ \beta | \beta \leq \Omega_T^{-1}(1 - \delta) \right\}$$

$$= \Omega_T^{-1}(1 - \delta).$$

It follows from Definition 3.2 that we finally have

$$f_2 = \frac{1}{2} \left[\sup_{0 \leq t \leq T} \exp(-r_d t)(Z_t - K)^+ \right]_{\sup} (\delta) + \frac{Z_0}{2} \left[\sup_{0 \leq t \leq T} \exp(-r_f t) \left(1 - \frac{K}{Z_t} \right)^+ \right]_{\sup} (\delta)$$

$$= \frac{1}{2} \phi_T^{-1}(1 - \delta) + \frac{Z_0}{2} \Omega_T^{-1}(1 - \delta)$$

$$= \frac{1}{2} \left[\sup_{0 \leq t \leq T} \exp(-r_d t) \left(\Psi_t^{-1}(1 - \delta) - K \right)^+ \right] + \frac{Z_0}{2} \left[\sup_{0 \leq t \leq T} \exp(-r_f t) \left(1 - \frac{K}{\Psi_t^{-1}(1 - \delta)} \right)^+ \right],$$

$$\Psi_t^{-1}(1 - \delta) = \sum_{l=0}^{n-1} z_l t^l E_{p,(l+1)} \left(-bt^p \right) + \left(a + \frac{c\sqrt{3}}{\pi} \ln \frac{1 - \delta}{\delta} \right) t^p E_{p,(p+1)} \left(-bt^p \right).$$

where

Hence, the result (15) is proved. □

3.4. Asian Currency Option

Suppose an Asian currency option's striking price is K and that maturity date is T. Set f_3 in domestic currency as the contract price. At initial time 0, the investor pays f_3 for purchasing this contract and receives the maximum benefit at time T based on optimistic criterion $\left[\left(\frac{1}{T} \int_0^T Z_t dt - K \right)^+ \right]_{\sup} (\delta)$, where δ is the confidence level. Hence, the investor obtains the net return at time 0

$$-f_3 + \exp(-r_d T) \left[\left(\frac{1}{T} \int_0^T Z_t dt - K \right)^+ \right]_{\sup} (\delta).$$

On the other hand, the bank obtains profit f_3 for selling this contract at time 0, and pays $\left[\left(1 - K / \frac{1}{T} \int_0^T Z_t dt \right)^+ \right]_{\sup} (\delta)$ in foreign currency at time T. Hence, the bank can receive the net earning at time 0

$$f_3 - Z_0 \exp(-r_f T) \left[\left(1 - K / \frac{1}{T} \int_0^T Z_t dt \right)^+ \right]_{\sup} (\delta).$$

It follows from the fair price principle that the definition of an Asian currency option price is given as follows.

Definition 3.3 Assuming that an Asian currency option's striking price is K and maturity date is T. Then the option price under confidence level δ based on optimistic criterion is

$$f_3 = \frac{1}{2}\exp(-r_d T)\left[\left(\frac{1}{T}\int_0^T Z_t dt - K\right)^+\right]_{\sup}(\delta) + \frac{Z_0}{2}\exp(-r_f T)\left[\left(1 - K/\frac{1}{T}\int_0^T Z_t dt\right)^+\right]_{\sup}(\delta). \quad (16)$$

Theorem 3.3 *Suppose an Asian currency option for model (11) has a striking price* K *and that its expiration time is* T. *Then the option price under confidence level* δ *based on optimistic value criteria is*

$$f_3 = \frac{1}{2}\exp(-r_d T)\left[I_T^{-1}(1-\delta) - K\right]^+ + \frac{Z_0}{2}\exp(-r_f T)\left[1 - \frac{K}{I_T^{-1}(1-\delta)}\right]^+, \quad (17)$$

$$I_T^{-1}(1-\delta) = \sum_{l=0}^{n-1} z_l T^l E_{p,(l+2)}(-bT^p) + \left(a + \frac{c\sqrt{3}}{\pi}\ln\frac{1-\delta}{\delta}\right)T^p E_{p,(p+2)}(-bT^p).$$

where

Proof. Based on Lu and Zhu (2019), we can easily obtain that Z_t has an IUD

$$\Psi_t^{-1}(\alpha) = Z_t^\alpha$$

$$= \sum_{l=0}^{n-1} z_l t^l E_{p,(l+1)}(-bt^p) + \left(a + \frac{c\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)t^p E_{p,(p+1)}(-bt^p).$$

It follows from Theorem 2.3 that the IUD of $\frac{1}{T}\int_0^T Z_t dt$ is

$$I_T^{-1}(\alpha) = \frac{1}{T}\int_0^T Z_t^\alpha dt$$

$$= \frac{1}{T}\int_0^T\left[\sum_{l=0}^{n-1} z_l t^l E_{p,(l+1)}(-bt^p) + \left(a + \frac{c\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)t^p E_{p,(p+1)}(-bt^p)\right]dt$$

$$= \sum_{l=0}^{n-1} z_l T^l E_{p,(l+2)}(-bT^p) + \left(a + \frac{c\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)T^p E_{p,(p+2)}(-bT^p).$$

According to the definition of optimistic value for uncertain variable, we have

$$\left[\left(\frac{1}{T}\int_0^T Z_t dt - K\right)^+\right]_{\sup}(\delta) = \sup_{\tau\geq 0}\left\{\tau \left| \mathrm{M}\left\{\left(\frac{1}{T}\int_0^T Z_t dt - K\right)^+ \geq \tau\right\} \geq \delta\right.\right\}$$

$$= \sup_{\tau>0}\left\{\tau \left| \mathrm{M}\left\{\frac{1}{T}\int_0^T Z_t dt - K \geq \tau\right\} \geq \delta\right.\right\} \vee 0$$

$$= \sup_{\tau>0}\left\{\tau \left| 1 - \mathrm{M}\left\{\frac{1}{T}\int_0^T Z_t dt < K + \tau\right\} \geq \delta\right.\right\} \vee 0$$

$$= \sup_{\tau>0}\left\{\tau \left| 1 - I_T(K+\tau) \geq \delta\right.\right\} \vee 0$$

$$= \sup_{\tau>0}\left\{\tau \left| I_T(K+\tau) \leq 1 - \delta\right.\right\} \vee 0$$

$$= \sup_{\tau>0}\left\{\tau \left| I_T^{-1}(I_T(K+\tau)) \leq I_T^{-1}(1-\delta)\right.\right\} \vee 0$$

$$= \sup_{\tau>0}\left\{\tau \middle| K+\tau \le I_T^{-1}(1-\delta)\right\}\vee 0$$

$$= \sup_{\tau>0}\left\{\tau \middle| \tau \le I_T^{-1}(1-\delta)-K\right\}\vee 0$$

$$= \left[I_T^{-1}(1-\delta)-K\right]^+.$$

By use of the similar method, we can get

$$\left[\left(1-K\Big/\frac{1}{T}\int_0^T Z_t dt\right)^+\right]_{\sup}(\delta)=\sup_{\upsilon\ge 0}\left\{\upsilon\middle| M\left\{\left(1-K\Big/\frac{1}{T}\int_0^T Z_t dt\right)^+\ge\upsilon\right\}\ge\delta\right\}$$

$$= \sup_{\upsilon>0}\left\{\upsilon\middle| M\left\{1-K\Big/\frac{1}{T}\int_0^T Z_t dt\ge\upsilon\right\}\ge\delta\right\}\vee 0$$

$$= \sup_{\upsilon>0}\left\{\upsilon\middle| M\left\{(1-\upsilon)\frac{1}{T}\int_0^T Z_t dt\ge K\right\}\ge\delta\right\}\vee 0$$

$$= \sup_{0<\upsilon<1}\left\{\upsilon\middle| M\left\{\frac{1}{T}\int_0^T Z_t dt\ge\frac{K}{1-\upsilon}\right\}\ge\delta\right\}\vee 0$$

$$= \sup_{0<\upsilon<1}\left\{\upsilon\middle| 1-M\left\{\frac{1}{T}\int_0^T Z_t dt<\frac{K}{1-\upsilon}\right\}\ge\delta\right\}\vee 0$$

$$= \sup_{0<\upsilon<1}\left\{\upsilon\middle| 1-I_T\left(\frac{K}{1-\upsilon}\right)\ge\delta\right\}\vee 0$$

$$= \sup_{0<\upsilon<1}\left\{\upsilon\middle| I_T\left(\frac{K}{1-\upsilon}\right)\le 1-\delta\right\}\vee 0$$

$$= \sup_{0<\upsilon<1}\left\{\upsilon\middle| I_T^{-1}\left(I_T\left(\frac{K}{1-\upsilon}\right)\right)\le I_T^{-1}(1-\delta)\right\}\vee 0$$

$$= \sup_{0<\upsilon<1}\left\{\upsilon\middle| \frac{K}{1-\upsilon}\le I_T^{-1}(1-\delta)\right\}\vee 0$$

$$= \sup_{0<\upsilon<1}\left\{\upsilon\middle| \upsilon\le 1-\frac{K}{I_T^{-1}(1-\delta)}\right\}\vee 0$$

$$= \left[1-\frac{K}{I_T^{-1}(1-\delta)}\right]^+.$$

It follows from Definition 3.3 that we finally have

$$f_3 = \frac{1}{2}\exp(-r_d T)\left[\left(\frac{1}{T}\int_0^T Z_t dt-K\right)^+\right]_{\sup}(\delta)+\frac{Z_0}{2}\exp(-r_f T)\left[\left(1-K\Big/\frac{1}{T}\int_0^T Z_t dt\right)^+\right]_{\sup}(\delta)$$

$$= \frac{1}{2}\exp(-r_d T)\left[I_T^{-1}(1-\delta)-K\right]^+ + \frac{Z_0}{2}\exp(-r_f T)\left[1-\frac{K}{I_T^{-1}(1-\delta)}\right]^+,$$

$$I_T^{-1}(1-\delta)=\sum_{l=0}^{n-1}z_l T^l E_{p,(l+2)}\left(-bT^p\right)+\left(a+\frac{c\sqrt{3}}{\pi}\ln\frac{1-\delta}{\delta}\right)T^p E_{p,(p+2)}\left(-bT^p\right).$$

where

Hence, the result (17) is proved. □

4. Numerical Experiments

This section provides numerical examples to further explain our theoretical results.

Example 4.1 Assume $r_d = 0.05, r_f = 0.04, a = 6, b = 1, c = 0.1, z_0 = 4, z_1 = 0.5, K = 5, \quad T = 1$. By use of the pricing formula (13), we calculate the price of European currency option with different order p $(0 < p \leq 2)$ and δ under optimistic criterion, as presented in Table 1.

From Table 1, we first discover when confidence level δ remains unchanged, the price f_1 is increasing about p when $p \in (0, 1.2]$, but it is decreasing when $p \in (1.2, 2]$. The price jumps when p converts 1 into 1.1, and this is because instantaneous growth rate of exchange rate z_1 works at this time. Meanwhile, we find the price f_1 decreases with the increase of confidence level δ when p is fixed, which is in accord with the real financial markets: the more confident investors are in the future change of exchange rate, the less willing they are to spend more to diminish latent risks.

Table 1. The price f_1 with different confidence level δ and order p.

Confidence level	The price of a European currency option							
	$p = 0.5$	$p = 0.8$	$p = 1.0$	$p = 1.1$	$p = 1.2$	$p = 1.5$	$p = 1.8$	$p = 2.0$
$\delta = 0.1$	0.1808	0.2518	0.2847	0.5504	0.5580	0.5291	0.4256	0.3292
$\delta = 0.2$	0.1595	0.2293	0.2617	0.5283	0.5360	0.5081	0.4070	0.3126
$\delta = 0.3$	0.1454	0.2143	0.2464	0.5136	0.5213	0.4941	0.3945	0.3016
$\delta = 0.4$	0.1337	0.2020	0.2337	0.5015	0.5092	0.4826	0.3843	0.2925
$\delta = 0.5$	0.1230	0.1907	0.2221	0.4904	0.4981	0.4720	0.3750	0.2842
$\delta = 0.6$	0.1122	0.1793	0.2105	0.4792	0.4870	0.4615	0.3656	0.2758
$\delta = 0.7$	0.1005	0.1669	0.1978	0.4671	0.4749	0.4499	0.3553	0.2667
$\delta = 0.8$	0.0862	0.1518	0.1822	0.4522	0.4600	0.4357	0.3428	0.2555
$\delta = 0.9$	0.0645	0.1289	0.1588	0.4297	0.4376	0.4144	0.3238	0.2387

Furthermore, we discuss the properties of the option price f_1 with regard to some parameters (K, z_0, c, r_d, r_f). It is worth noting that when we analyze the sensitivity of one parameter on the price f_1, other parameters keep unchanged. According to Figure 1, we notice that f_1 is increasing monotonically concerning z_0, while it is decreasing with respect to K, c, r_d, r_f, respectively. These results coincide with the realities in financial market, which proves the accuracy of our results.
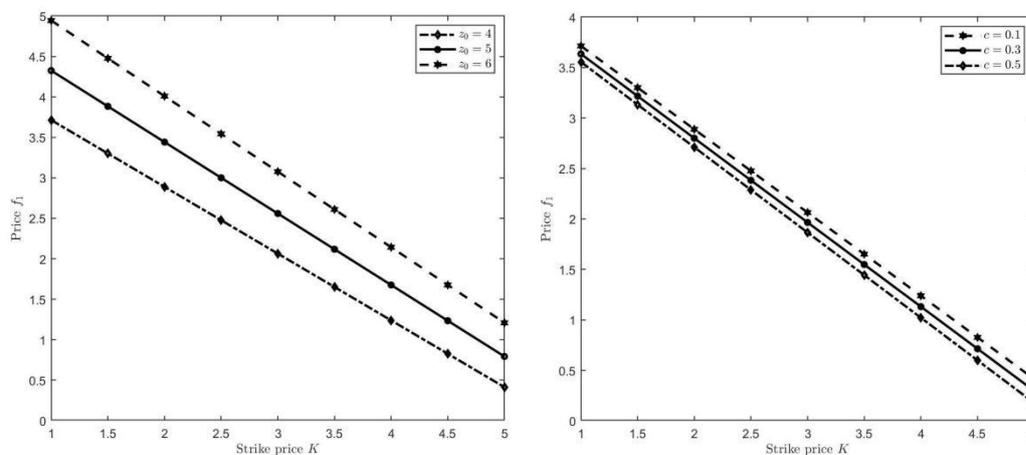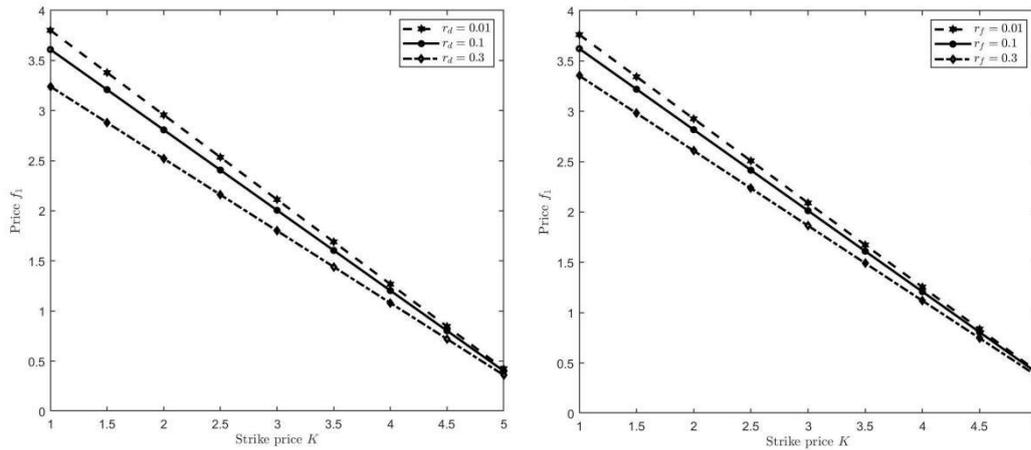
Figure 1. The price f_1 with different K on various other parameters while $p = 1.5, \delta = 0.9$.

Example 4.2 Assume $r_d = 0.05, r_f = 0.04, a = 6, b = 0.8, c = 0.1, z_0 = 5, z_1 = 0.5, K = 6, T = 1$. By use of the pricing formula (15), we calculate the price of American currency option with different order p $(0 < p \le 2)$ and δ under optimistic criterion, as presented in Table 2.

From Table 2, we first discover that when confidence level δ remains unchanged, the price f_2 is monotonously increasing about p when $p \in (0,1]$, but it is monotonously decreasing when $p \in (1,2]$. The price jumps when p converts 1 into 1.1, and this is because instantaneous growth rate of exchange rate z_1 works at this time. Meanwhile, we find the price f_2 decreases with the increase of confidence level δ when p is fixed, which is in accord with the real financial markets: the more confident investors are in the future change of exchange rate, the less willing they are to spend more to diminish latent risks.

Table 2. The price f_2 with different confidence level δ and order p.

Confidence level	The price of an American currency option							
	$p = 0.5$	$p = 0.8$	$p = 1.0$	$p = 1.1$	$p = 1.2$	$p = 1.5$	$p = 1.8$	$p = 2.0$
$\delta = 0.1$	0.3027	0.3695	0.3899	0.6724	0.6699	0.6073	0.4744	0.3632
$\delta = 0.2$	0.2789	0.3445	0.3646	0.6482	0.6458	0.5849	0.4549	0.3460
$\delta = 0.3$	0.2630	0.3278	0.3477	0.6320	0.6298	0.5699	0.4418	0.3345
$\delta = 0.4$	0.2500	0.3142	0.3338	0.6188	0.6167	0.5577	0.4311	0.3250
$\delta = 0.5$	0.2380	0.3016	0.3210	0.6066	0.6046	0.5464	0.4213	0.3164
$\delta = 0.6$	0.2259	0.2890	0.3083	0.5944	0.5925	0.5351	0.4114	0.3077
$\delta = 0.7$	0.2128	0.2752	0.2943	0.5810	0.5793	0.5227	0.4007	0.2982
$\delta = 0.8$	0.1967	0.2584	0.2772	0.5647	0.5631	0.5077	0.3875	0.2866
$\delta = 0.9$	0.1725	0.2330	0.2514	0.5401	0.5387	0.4849	0.3677	0.2691

Furthermore, we discuss the properties of the option price f_2 with regard to some parameters (K, z_0, c, r_d, r_f). It is worth noting that when we analyze the sensitivity of one parameter on the price f_2, other parameters keep unchanged. According to Figure 2, we notice that f_2 is increasing monotonically concerning z_0, while it is decreasing with respect to K, c, r_d, r_f, respectively. These results coincide with the realities in financial market, which again proves the accuracy and effectiveness of our results.
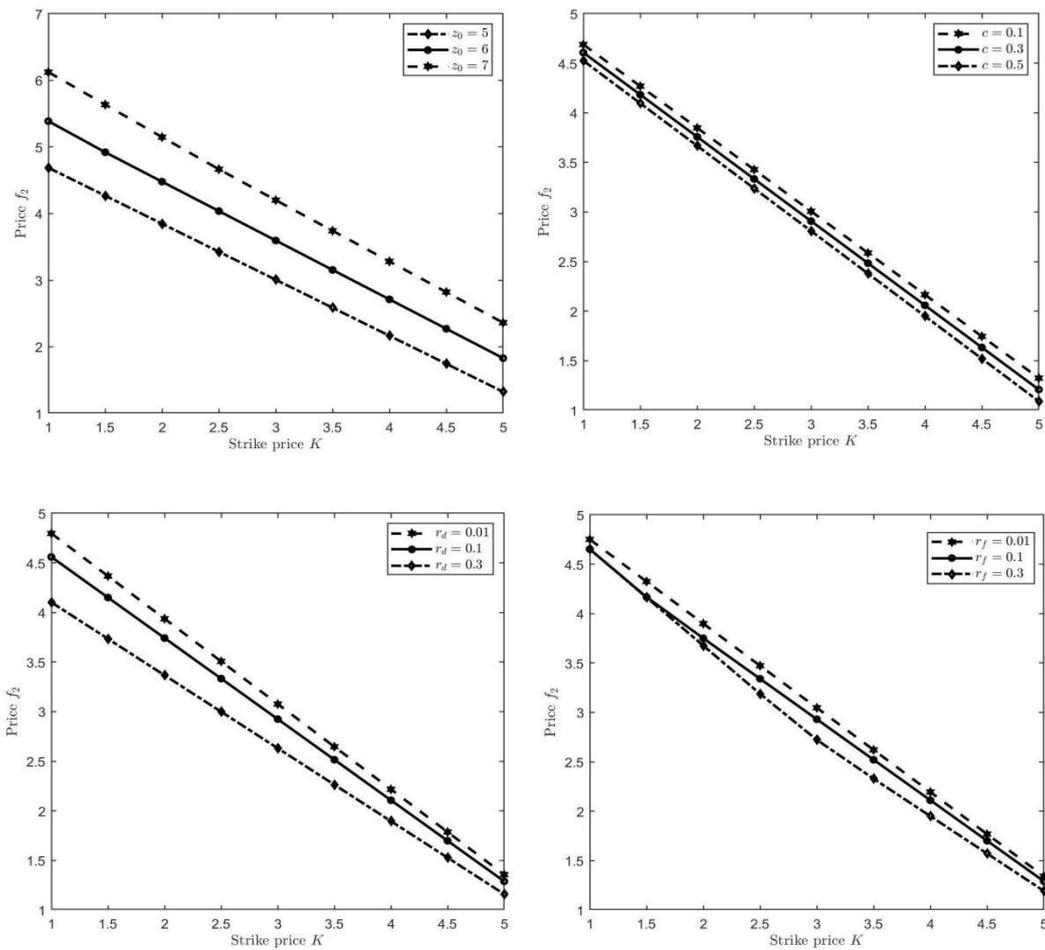
Figure 2. The price f_2 with different K on various other parameters while $p = 1.5, \delta = 0.9$.

Example 4.3 Assume $r_d = 0.05, r_f = 0.04, a = 6, b = 0.8, c = 0.2, z_0 = 5, z_1 = 0.5, K = 5, T = 1$. By use of the pricing formula (17), we calculate the price of Asian currency option with different order $p \ (0 < p \le 2)$ and δ under optimistic criterion, as presented in Table 3.

Table 3. The price f_3 with different confidence level δ and order p.

Confidence level	The price of an Asian currency option							
	$p = 0.5$	$p = 0.8$	$p = 1.0$	$p = 1.1$	$p = 1.2$	$p = 1.5$	$p = 1.8$	$p = 2.0$
$\delta = 0.1$	0.9459	0.8497	0.7727	0.8940	0.8550	0.7308	0.6092	0.5366
$\delta = 0.2$	0.9111	0.8182	0.7439	0.8675	0.8300	0.7108	0.5940	0.5242
$\delta = 0.3$	0.8879	0.7972	0.7247	0.8498	0.8134	0.6974	0.5839	0.5160
$\delta = 0.4$	0.8688	0.7800	0.7089	0.8353	0.7997	0.6864	0.5755	0.5093
$\delta = 0.5$	0.8512	0.7641	0.6944	0.8219	0.7871	0.6763	0.5679	0.5031
$\delta = 0.6$	0.8336	0.7481	0.6798	0.8085	0.7745	0.6662	0.5602	0.4969
$\delta = 0.7$	0.8143	0.7307	0.6639	0.7938	0.7607	0.6552	0.5518	0.4901
$\delta = 0.8$	0.7907	0.7094	0.6445	0.7759	0.7439	0.6417	0.5416	0.4819
$\delta = 0.9$	0.7550	0.6772	0.6150	0.7489	0.7184	0.6213	0.5262	0.4694

From Table 3, we first discover that when confidence level δ remains unchanged, the price f_3 decreases as the order p increases in $(0,1]$ and $(1,2]$, respectively. The price jumps when p converts 1 into 1.1, and this is because instantaneous growth rate of exchange rate z_1 works at this time. Meanwhile, we find the price f_3 decreases with the increase of confidence level δ when p is fixed, which is in accord with the real financial markets: the more confident investors are in the future change of exchange rate, the less willing they are to spend more to diminish latent risks.

Furthermore, we discuss the properties of the option price f_3 with regard to some parameters (K, z_0, c, r_d, r_f). It is worth noting that when we analyze the sensitivity of one parameter on the price f_3, other parameters keep unchanged. According to Figure 3, we notice that f_3 is increasing monotonically concerning z_0, while it is decreasing with respect to K, c, r_d, r_f, respectively. These results coincide with the realities in financial market, which again proves the accuracy and effectiveness of our results.
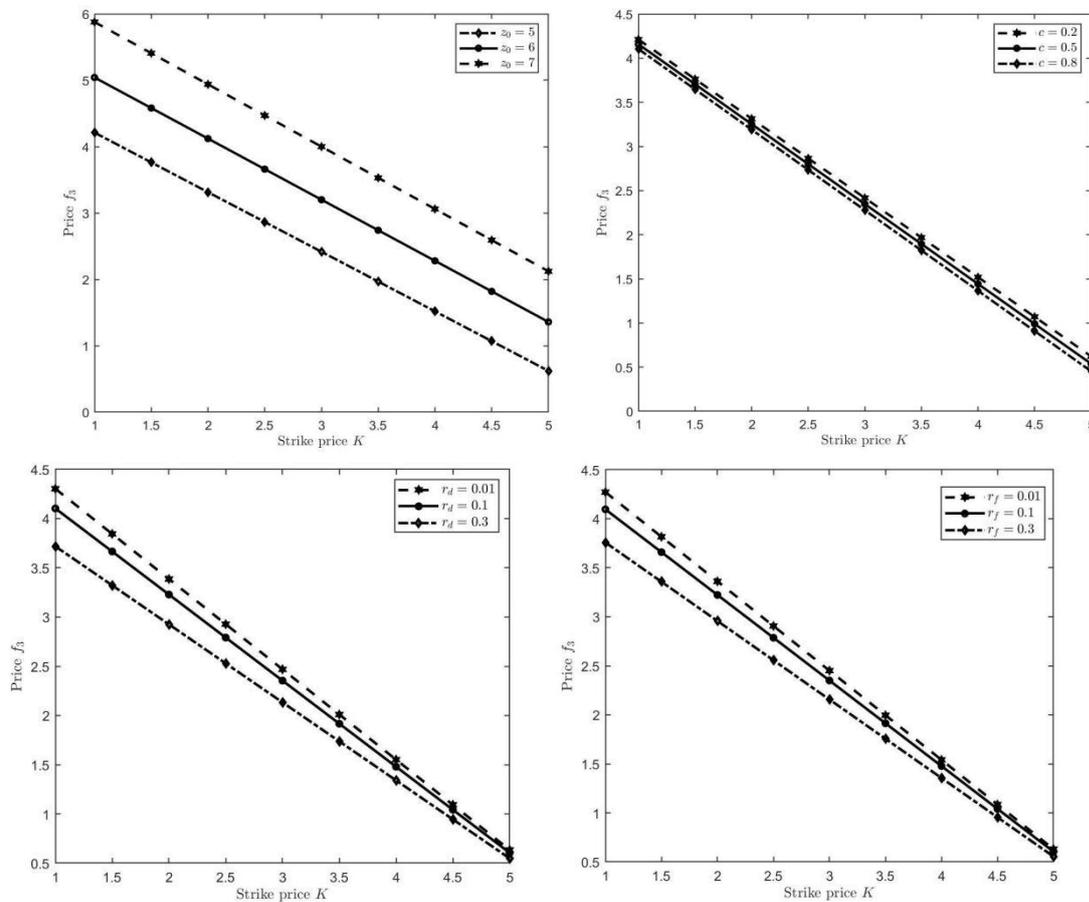

Figure 3. The price f_3 with different K on various other parameters while $p = 1.5, \delta = 0.9$.

5. Conclusions

This paper analyzed the valuation issues of currency option under optimistic value criterion. An uncertain fractional currency model with mean-reverting process based on UFDE was proposed. After that, by means of rigorous derivations, we verified the pricing formulae of currency option for the presented model. Finally, the sensitivity analyses of different parameters on the option prices were provided.

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