Article

Not peer-reviewed version

Physics as the Solution to an Optimization Problem on Entropy

Alexandre Harvey-Tremblay *

Posted Date: 17 July 2024

doi: 10.20944/preprints202404.1009.v5

Keywords: foundations of quantum physics

 doi:10.20944/preprints202404.1009.v5

Article

Physics as the Solution to an Optimization Problem on Entropy

Alexandre Harvey-Tremblay

Independent Scientist; aht@protonmail.ch

Abstract: We propose a novel approach to quantum theory construction that involves solving a maximization problem on the Shannon entropy of all possible measurements of a system, relative to its initial preparation. This maximization problem is additionally constrained by a phase condition that vanishes under measurements. Specifically, enforcing a vanishing U(1)-valued phase constraint leads to standard quantum mechanics, while a vanishing $\text{Spin}^c(3,1)$-valued phase constraint extends the theory to relativistic quantum mechanics and to quantum gravity. The latter scenario derives the metric tensor as an operator via a double-copy mechanism applied to the Dirac current. Significantly, this solution is consistent exclusively in 3+1-dimensions as all other dimensional configurations lead to fundamental obstructions. Finally, the solution uniquely incorporates the SU(3) × SU(2) × U(1) symmetries of the Standard Model. This framework seamlessly integrates fundamental concepts from quantum mechanics, relativistic quantum mechanics, quantum gravity, the dimensional specificity of spacetime, and particle physics gauge symmetries as the solution to a simple entropy optimization problem.

Keywords: foundations of quantum physics

---

1. Introduction

The canonical formalism of quantum mechanics (QM) is based on five principal axioms[1,2]:

QM Axiom 1 of 5 — **State Space:** Each physical system corresponds to a complex Hilbert space, with the system's state represented by a ray in this space.

QM Axiom 2 of 5 — **Observables:** Physical observables correspond to Hermitian operators within the Hilbert space.

QM Axiom 3 of 5 — **Dynamics:** The time evolution of a quantum system is dictated by the Schrödinger equation, where the Hamiltonian operator signifies the system's total energy.

QM Axiom 4 of 5 — **Measurement:** The act of measuring an observable results in the system's transition to an eigenstate of the associated operator, with the measurement value being one of the eigenvalues.

QM Axiom 5 of 5 — **Probability Interpretation:** The likelihood of a specific measurement outcome is determined by the squared magnitude of the state vector's projection onto the relevant eigenstate.

Contrastingly, statistical mechanics (SM), the other statistical pillar of physics, derives its probability measure through entropy maximization, constrained by the following expression:

SM Constraint 1 of 1: **Average Energy Constraint:** The average of energy measurements of a system at thermodynamic equilibrium converge to a specific value ($\overline{E}$):

$$\overline{E} = \sum_i \rho_i E_i \tag{1}$$

To maximize entropy while satisfying this constraint, the theory uses a Lagrange multiplier approach.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 July 2024 doi:10.20944/preprints202404.1009.v5

Definition 1 (Fundamental Lagrange Multiplier Equation of SM)**.**

$$\mathcal{L} = \underbrace{-k_B \sum_i \rho_i \ln \rho_i}_{\text{Boltzmann entropy}} + \underbrace{\lambda \left(1 - \sum_i \rho_i\right)}_{\text{Normalization Constraint}} + \underbrace{\beta \left(\overline{E} - \sum_i \rho_i E_i\right)}_{\text{Average Energy Constraint}} \tag{2}$$

where λ and β are the Lagrange multipliers.

Theorem 1 (Gibbs Measure)**.** *The solution to the Lagrange multiplier equation of SM is the Gibbs measure.*

$$\rho_i = \frac{1}{\underbrace{\sum_i \exp(-\beta E_i)}_{\text{Microcanonical Ensemble}}} \exp(-\beta E_i) \tag{3}$$

Proof. This is an well-known result by E. T. Jaynes [3,4]. As a convenience, we replicate the proof in Appendix A. □

As evident from E. T. Jaynes' methodological innovation, SM relies on a single constraint related to the nature of the measurements under consideration, which allows the formulation of an optimization problem sufficient to derive the relevant probability measure. This is an exceptionally parsimonious formulation of a physical theory.

We propose a generalization of E. T. Jaynes' approach to the realms of Quantum Mechanics (QM), relativistic Quantum Mechanics (RQM) and Quantum Gravity (QG). For each domain, we will introduce a single constraint related to measurements, formulate a corresponding entropy maximization problem, and present a main theorem that encapsulates the theory. This formulation reduces fundamental physics to its most parsimonious expression, deriving the core theories as optimal solutions to a well-defined entropy maximization problem.

1.1. Quantum Mechanics

To reformulate QM as the solution to an entropy maximization problem, we propose the following constraint:

QM Constraint 1 of 1 **Vanishing Complex-Phase:** Quantum measurements admit a vanishing complex phase. The constraint is:

$$0 = \operatorname{tr} \sum_i \rho_i \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix} \tag{4}$$

Here, the matrix representation engenders the complex phase, and the trace will cause it to vanish under measurement.

It associates to the follow equation:

Definition 2 (Fundamental Lagrange Multiplier Equation of QM)**.**

$$\mathcal{L} = \underbrace{-\sum_i \rho_i \ln \frac{\rho_i}{p_i}}_{\text{Relative Shannon Entropy}} + \underbrace{\lambda \left(1 - \sum_i \rho_i\right)}_{\text{Normalization Constraint}} + \underbrace{\tau \left(-\operatorname{tr} \sum_i \rho_i \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}\right)}_{\text{Vanishing Complex-Phase}} \tag{5}$$

where λ and τ are the Lagrange multipliers.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 July 2024 doi:10.20944/preprints202404.1009.v5

The *relative* Shannon entropy[5,6] is utilized because we are solving for the least biased theory that connects an initial preparation p to its final measurement ρ.

Theorem 2. *The least biased probability measure that connects an initial preparation p to its final measurement ρ, under the constraint of the vanishing complex-phase, is:*

$$\rho_i = \underbrace{\frac{1}{\sum_i p_i \|\exp(-itE_i/\hbar)\|}}_{\text{Unitarily Invariant Ensemble}} \underbrace{\|\exp(-itE_i/\hbar)\|}_{\text{Born Rule}} \underbrace{p_i}_{\text{Initial Preparation}} \tag{6}$$

where we have defined $\tau = t/\hbar$ (analogous to $\beta = 1/(k_B T)$ in SM).

The proof of this theorem will be presented in the results section. We will show that this solution entails the five axioms of QM, which are now promoted to theorems, yielding a parsimonious formulation of QM.

1.2. Relativistic Quantum Mechanics

Before we can discuss RQM, we first need to introduce a notation. Let $\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b}$, where a is a scalar, $\mathbf{x}$ is a vector, $\mathbf{f}$ is a bivector, $\mathbf{v}$ is a pseudo-vector and $\mathbf{b}$ is a pseudo-scalar, be a multivector of the geometric algebra $\text{GA}(3,1)$, and let $\mathbf{M}$ be its matrix representation. Then, the fundamental constraint is:

QG Constraint 1 of 1 **Vanishing Relativistic Phase:** Our formulation of RQM is based around a vanishing phase spanning the $\text{Spin}^c(3,1)$ group. The constraint is:

$$0 = \text{tr}\frac{1}{2}\sum_i \rho_i \mathbf{M}_i \tag{7}$$

where $\mathbf{M}_i$ is the matrix representation of the multivector $\mathbf{u}_i = \mathbf{f}_i + \mathbf{b}_i$ of $\text{GA}(3,1)$. Using the real Majorana representation of the gamma matrices, the representation is as follows:

$$\mathbf{M} = \begin{bmatrix} f_{02} & b - f_{13} & -f_{01} + f_{12} & f_{03} + f_{23} \\ -b + f_{13} & f_{02} & f_{03} + f_{23} & f_{01} - f_{12} \\ -f_{01} - f_{12} & f_{03} - f_{23} & -f_{02} & -b - f_{13} \\ f_{03} - f_{23} & f_{01} + f_{12} & b + f_{13} & -f_{02} \end{bmatrix} \tag{8}$$

$$\cong f_{01}\gamma_0\gamma_1 + f_{02}\gamma_0\gamma_2 + f_{03}\gamma_0\gamma_3 + f_{12}\gamma_1\gamma_2 + f_{13}\gamma_1\gamma_3 + f_{23}\gamma_2\gamma_3 + b\gamma_0\gamma_1\gamma_2\gamma_3 \tag{9}$$

Similarly to the QM case, here the matrix representation engenders a $\text{Spin}^c(3,1)$-phase and the trace will cause it to vanish under measurement.

The Lagrange multiplier equation is as follows:

Definition 3 (Fundamental Lagrange Multiplier Equation of RQM)**.**

$$\mathcal{L} = \underbrace{-\sum_i \rho_i \ln\frac{\rho_i}{p_i}}_{\text{Relative Shannon Entropy}} + \underbrace{\lambda\left(1 - \sum_i \rho_i\right)}_{\text{Normalization Constraint}} + \underbrace{\zeta\left(-\text{tr}\frac{1}{2}\sum_i \rho_i \mathbf{M}_i\right)}_{\text{Vanishing Spin}^c(3,1)\text{ Phase}} \tag{10}$$

where λ and ζ are the Lagrange multipliers.

Theorem 3. *The least biased probability measure that connects an initial preparation p_i to its final measurement ρ_i, under the constraint of the vanishing relativistic phase, is:*

$$\rho(q) = \underbrace{\frac{1}{\sum_i p_i \det\exp(-\zeta\frac{1}{2}\mathbf{M}_i)}}_{\text{Spin}^c(3,1)\text{ Invariant Ensemble}} \underbrace{\det\exp(-\zeta\frac{1}{2}\mathbf{M}_i)}_{\text{Spin}^c(3,1)\text{ Born Rule}} \underbrace{p_i}_{\text{Initial Preparation}} \tag{11}$$

The proof of this theorem is presented in the results section.

In the results section, we demonstrate that the solution to this optimization problem provides a foundation for a quantum mechanical theory in 3+1-dimensional spacetime. The parameter ζ emerges as a the generator of boosts, rotations, and phase transformations. This single parameter accounts for all change-of-basis transformations an observer can perform in spacetime before measurement. Furthermore, we show that the Dirac current resulting from this formulation exhibits invariance under $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ gauge symmetries, aligning with the fundamental symmetries of the Standard Model.

1.3. Quantum Gravity

Our approach to RQM extends naturally to QG. While RQM utilizes only a portion of the solution space, QG leverages its full generality. The key innovation is the introduction of a double-copy mechanism applied to the Dirac current: the probability measure multiply two Dirac currents to obtain a metric tensor expectation value. This mechanism allows us to construct the metric tensor as an observable from basis vectors, thereby establishing a direct link between the probability measure, metric measurements, and the geometric structure of spacetime.

1.4. Dimensional Obstructions

We conclude the results section with a series of theorems demonstrating that, except for SM (no vanishing phase) and QM (vanishing $\mathrm{U}(1)$ phase), the entropy maximization technique yields a solution only in 3+1-dimensional spacetime (vanishing $\mathrm{Spin}^c(3,1)$ phase). In all other dimensional configurations, various obstructions are encountered. These findings suggest an intriguing connection between the entropy maximization approach and the specific dimensionality of our universe, the implications of which are discussed.

2. Results

2.1. Quantum Mechanics

In statistical mechanics, the founding observation is that energy measurements of a thermally equilibrated system tend towards an average value. Comparatively, in QM, the founding observation involves the interplay between the systematic elimination of complex phases in measurement outcomes and the presence of interference effects in repeated measurement outcomes. To represent this observation, we introduce the *Vanishing* $\mathrm{U}(1)$-*Phase* Anti-Constraint:

$$0 = \operatorname{tr} \sum_i \rho_i \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix} \tag{12}$$

where E_i are scalar-valued. The usage of the matrix generates a $\mathrm{U}(1)$ phase, and the trace causes it to vanish under measurements.

At first glance, this expression may seem to reduce to a tautology equating zero with zero, suggesting it imposes no restriction on energy measurements. However, this appearance is deceptive. Unlike a conventional constraint that limits the solution space, this expression serves as a formal device to expand it, allowing for the incorporation of complex phases into the probability measure.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 July 2024 doi:10.20944/preprints202404.1009.v5

The expression's role in broadening, rather than restricting, the solution space leads to its designation as an "anti-constraint."

In general, usage of anti-constraints expand classical probability measures into larger domains, such as quantum probabilities.

Its significance will become evident upon the completion of the optimization problem. For the moment, this expression can be conceptualized as an ansatz that, when incorporated as an anti-constraint within an entropy-maximization problem, resolves into the axioms of quantum mechanics.

Our next procedural step involves solving the corresponding Lagrange multiplier equation, mirroring the methodology employed in statistical mechanics by E. T. Jaynes. We utilize the relative Shannon entropy because we wish to solve for the least biased probability measure that connects an initial preparation p_i to its final measurement ρ_i. For that, we deploy the following Lagrange multiplier equation:

$$\mathcal{L} = \underbrace{-\sum_i \rho_i \ln \frac{\rho_i}{p_i}}_{\text{Relative Shannon Entropy}} + \underbrace{\lambda\left(1 - \sum_i \rho_i\right)}_{\text{Normalization Constraint}} + \underbrace{\tau\left(\operatorname{tr} \sum_i \rho_i \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}\right)}_{\text{Vanishing U(1)-Phase}} \tag{13}$$

Where λ and τ are the Lagrange multipliers.

We solve the maximization problem as follows:

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}[\rho_1, \dots, \rho_n]}{\partial \rho_i} && (14)\\
&= -\ln \frac{\rho_i}{p_i} - p_i - \lambda - \tau \operatorname{tr} \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix} && (15)\\
&= \ln \frac{\rho_i}{p_i} + p_i + \lambda - \tau \operatorname{tr} \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix} && (16)\\
\implies \ln \frac{\rho_i}{p_i} &= -p_i - \lambda - \tau \operatorname{tr} \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix} && (17)\\
\implies \rho_i &= p_i \exp(-p_i - \lambda) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}\right) && (18)\\
&= \frac{1}{Z(\tau)} p_i \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}\right) && (19)
\end{aligned}$$

The partition function is obtained as follows:

$$\begin{aligned}
1 &= \sum_i p_i \exp(-p_i - \lambda) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}\right) && (20)\\
\implies (\exp(-p_i - \lambda))^{-1} &= \sum_i p_i \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}\right) && (21)\\
Z(\tau) &:= \sum_i p_i \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}\right) && (22)
\end{aligned}$$

Finally, the least biased probability measure that connects an initial preparation p_i to its final measurement ρ_i, under the constraint of the vanishing U(1) phase, is:

$$\rho_i = \frac{1}{\sum_i p_i \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}\right)} \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}\right) p_i \tag{23}$$

Though initially unfamiliar, this form effectively establishes a comprehensive formulation of quantum mechanics, as we will demonstrate.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 July 2024 doi:10.20944/preprints202404.1009.v5

Upon examination, we find that phase elimination is manifestly evident in the probability measure: since the trace evaluates to zero, the probability measure simplifies to classical probabilities, aligning precisely with the Born rule's exclusion of complex phases:

$$\rho_i = \frac{p_i}{\sum_i p_i} \tag{24}$$

However, the significance of this phase elimination extends beyond this mere simplicity. As we will soon see, the partition function Z gains unitary invariance, allowing for the emergence of interference patterns and other quantum characteristics under appropriate basis changes.

We will begin by aligning our results with the conventional quantum mechanical notation. As such, we transform the representation of complex numbers from $\left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right]$ to $a + ib$. For instance, the exponential of a complex matrix is:

$$\exp \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}, \text{ where } r = \exp a \tag{25}$$

Then, we associate the exponential trace to the complex norm using $\exp \operatorname{tr}\mathbf{M} \equiv \det \exp \mathbf{M}$:

$$\begin{aligned} \exp \operatorname{tr} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \det \exp \begin{bmatrix} a & -b \\ b & a \end{bmatrix} &= r^2 \det \begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}, \text{ where } r = \exp a && (26) \\ &= r^2(\cos^2(b) + \sin^2(b)) && (27) \\ &= \|r(\cos(b) + i\sin(b))\| && (28) \\ &= \|r\exp(ib)\| && (29) \end{aligned}$$

Finally, substituting $\tau = t/\hbar$ analogously to $\beta = 1/(k_B T)$, and applying the complex-norm representation to both the numerator and to the denominator, consolidates the Born rule, normalization, and initial prepration into :

$$\rho_i = \underbrace{\frac{1}{\sum_i p_i \|\exp(-itE_i/\hbar)\|}}_{\text{Unitarily Invariant Partition Function}} \underbrace{\|\exp(-itE_i/\hbar)\|}_{\text{Born Rule}} \underbrace{p_i}_{\text{Initial Preparation}} \tag{30}$$

We are now in a position to explore the solution space.

The wavefunction is delineated by decomposing the complex norm into a complex number and its conjugate. It is then visualized as a vector within a complex n-dimensional Hilbert space. The partition function acts as the inner product. This relationship is articulated as follows:

$$\sum_i p_i \|\exp(-itE_i/\hbar)\| = Z = \langle\psi|\psi\rangle \tag{31}$$

where

$$\begin{bmatrix} \psi_1(t) \\ \vdots \\ \psi_n(t) \end{bmatrix} = \begin{bmatrix} \exp(-itE_1/\hbar) & & \\ & \ddots & \\ & & \exp(-itE_n/\hbar) \end{bmatrix} \begin{bmatrix} \psi_1(0) \\ \vdots \\ \psi_n(0) \end{bmatrix} \tag{32}$$

We clarify that p_i represents the probability associated with the initial preparation of the wavefunction, where $p_i = \langle\psi_i(0)|\psi_i(0)\rangle$.

We also note that Z is invariant under unitary transformations.

Let us now investigate how the axioms of quantum mechanics are recovered from this result:

- The entropy maximization procedure inherently normalizes the vectors $|\psi\rangle$ with $1/Z = 1/\sqrt{\langle\psi|\psi\rangle}$. This normalization links $|\psi\rangle$ to a unit vector in Hilbert space. Furthermore, as physical states

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 July 2024 doi:10.20944/preprints202404.1009.v5

associate to the probability measure, and the probability is defined up to a phase, we conclude that physical states map to Rays within Hilbert space. This demonstrates QM Axiom 1 of 5.

- In Z, an observable must satisfy:

$$\overline{O} = \sum_i p_i O_i \| \exp(-itE_i/\hbar) \| \tag{33}$$

Since $Z = \langle \psi | \psi \rangle$, then any self-adjoint operator satisfying the condition $\langle \mathbf{O}\psi | \phi \rangle = \langle \psi | \mathbf{O}\phi \rangle$ will equate the above equation, simply because $\langle \mathbf{O} \rangle = \langle \psi | \mathbf{O} | \psi \rangle$. This demonstrates QM Axiom 2 of 5.

- Upon transforming Equation 32 out of its eigenbasis through unitary operations, we find that the energy, E_i, typically transforms in the manner of a Hamiltonian operator:

$$|\psi(t)\rangle = \exp(-it\mathbf{H}/\hbar)|\psi(0)\rangle \tag{34}$$

The system's dynamics emerge from differentiating the solution with respect to the Lagrange multiplier. This is manifested as:

$$\frac{\partial}{\partial t}|\psi(t)\rangle = \frac{\partial}{\partial t}(\exp(-it\mathbf{H}/\hbar)|\psi(0)\rangle) \tag{35}$$

$$= -i\mathbf{H}/\hbar \exp(-it\mathbf{H}/\hbar)|\psi(0)\rangle \tag{36}$$

$$= -i\mathbf{H}/\hbar|\psi(t)\rangle \tag{37}$$

$$\implies \mathbf{H}|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle \tag{38}$$

which is the Schrödinger equation. This demonstrates QM Axiom 3 of 5.

- From Equation 32 it follows that the possible microstates E_i of the system correspond to specific eigenvalues of $\mathbf{H}$. An observation can thus be conceptualized as sampling from ρ, with the measured state being the occupied microstate i. Consequently, when a measurement occurs, the system invariably emerges in one of these microstates, which directly corresponds to an eigenstate of $\mathbf{H}$. Measured in the eigenbasis, the probability measure is:

$$\rho_i(t) = \frac{1}{\langle \psi | \psi \rangle}(\psi_i(t))^\dagger \psi_i(t). \tag{39}$$

In scenarios where the probability measure $\rho_i(\tau)$ is expressed in a basis other than its eigenbasis, the probability $P(\lambda_i)$ of obtaining the eigenvalue λ_i is given as a projection on a eigenstate:

$$P(\lambda_i) = |\langle \lambda_i | \psi \rangle|^2 \tag{40}$$

Here, $|\langle \lambda_i | \psi \rangle|^2$ signifies the squared magnitude of the amplitude of the state $|\psi\rangle$ when projected onto the eigenstate $|\lambda_i\rangle$. As this argument hold for any observables, this demonstrates QM Axiom 4 of 5.

- Finally, since the probability measure (Equation 30) replicates the Born rule, QM Axiom 5 of 5 is also demonstrated.

Revisiting quantum mechanics with this perspective offers a coherent and unified narrative. Specifically, the vanishing U(1) phase constraint (Equation 12) is sufficient to entail the foundations of quantum mechanics (Axiom 1, 2, 3, 4 and 5) through the principle of entropy maximization. Equation 12 becomes the formulation's new singular foundation, and Axioms 1, 2, 3, 4, and 5 are now promoted to theorems.

2.2. RQM in 2D

In this section, we investigate RQM in 2D. Although all geometric configurations except 3+1D contain obstructions, which will be discussed later in this section, the 2D case provides a valuable

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 July 2024 doi:10.20944/preprints202404.1009.v5

starting point before addressing the more complex 3+1D case. In RQM 2D, the fundamental Lagrange Multiplier Equation is:

$$\mathcal{L} = \underbrace{-\sum_i \rho_i \ln \frac{\rho_i}{p_i}}_{\text{Relative Shannon Entropy}} + \underbrace{\lambda \left(1 - \sum_i \rho_i\right)}_{\text{Normalization Constraint}} + \underbrace{\theta \left(-\mathrm{tr} \frac{1}{2} \sum_i \rho_i \mathbf{M}_i\right)}_{\text{Vanishing Relativistic Phase}} \tag{41}$$

where λ and θ are the Lagrange multipliers, and where $\mathbf{M}_i$ is the matrix representation of a multivector $\mathbf{b}_i$ of $\mathrm{GA}(2)$, where $\mathbf{b}_i$ is a pseudo-scalar. In general a multivector $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$ of $\mathrm{GA}(2)$, where a is a scalar, $\mathbf{x}$ is a vector and $\mathbf{b}$ a pseudo-scalar, is represented as follows:

$$\begin{bmatrix} a+x & y-b \\ y+b & a-x \end{bmatrix} \cong a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y \tag{42}$$

The basis elements are defined as:

$$\sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_x \wedge \sigma_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{43}$$

If we take $a \to 0, \mathbf{x} \to 0$ then $\mathbf{M}$ reduces as follows:

$$\mathbf{u} = a + \mathbf{x} + \mathbf{b}|_{a\to 0, \mathbf{x}\to 0} = \mathbf{b} \implies \mathbf{M} = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \tag{44}$$

The Lagrange multiplier equation can be solved as follows:

$$0 = \frac{\partial \mathcal{L}[\rho_1, \ldots, \rho_n]}{\partial \rho_i} \tag{45}$$

$$= -\ln \frac{\rho_i}{p_i} - p_i - \lambda - \theta\,\mathrm{tr}\,\frac{1}{2}\begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix} \tag{46}$$

$$= \ln \frac{\rho_i}{p_i} + p_i + \lambda + \theta\,\mathrm{tr}\,\frac{1}{2}\begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix} \tag{47}$$

$$\implies \ln \frac{\rho_i}{p_i} = -p_i - \lambda - \theta\,\mathrm{tr}\,\frac{1}{2}\begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix} \tag{48}$$

$$\implies \rho_i = p_i \exp(-p_i - \lambda) \exp\left(-\theta\,\mathrm{tr}\,\frac{1}{2}\begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) \tag{49}$$

$$= \frac{1}{Z(\theta)} p_i \exp\left(-\theta\,\mathrm{tr}\,\frac{1}{2}\begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) \tag{50}$$

The partition function $Z(\theta)$, serving as a normalization constant, is determined as follows:

$$1 = \sum_i p_i \exp(-p_i - \lambda) \exp\left(-\theta\,\mathrm{tr}\,\frac{1}{2}\begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) \tag{51}$$

$$\implies (\exp(-p_i - \lambda))^{-1} = \sum_i p_i \exp\left(-\theta\,\mathrm{tr}\,\frac{1}{2}\begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) \tag{52}$$

$$Z(\theta) := \sum_i p_i \exp\left(-\theta\,\mathrm{tr}\,\frac{1}{2}\begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) \tag{53}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 July 2024 doi:10.20944/preprints202404.1009.v5

Consequently, the least biased probability measure that connects an initial preparation p_i to a final measurement ρ_i, under the constraint of the vanishing relativistic phase in 2D is:

$$\rho_i = \underbrace{\frac{1}{\sum_i p_i \det\exp\left(-\frac{1}{2}\theta\begin{bmatrix}0 & -b_i\\ b_i & 0\end{bmatrix}\right)}}_{\text{Spin(2) Invariant Ensemble}} \underbrace{\det\exp\left(-\frac{1}{2}\theta\begin{bmatrix}0 & -b_i\\ b_i & 0\end{bmatrix}\right)}_{\text{Spin(2) Born Rule}} \underbrace{p_i}_{\text{Initial Preparation}} \tag{54}$$

where $\det\exp M = \exp \operatorname{tr} M$.

In 2D, the Lagrange multiplier θ correspond to an angle of rotation, and in 1+1D it would correspond to the rapidity ζ:

$$\text{2D}: \quad \exp\theta\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \quad \theta \text{ is the angle of rotation} \tag{55}$$

$$1+1\text{D}: \quad \exp\zeta\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}\cosh\zeta & \sinh\zeta\\ \sinh\zeta & \cosh\zeta\end{bmatrix} \quad \zeta \text{ is the rapidity} \tag{56}$$

The 2D solution may appear equivalent to the QM case because they are related by an isomorphism $\text{Spin}(2) \cong \text{SO}(2) \cong \text{U}(1)$ and under the replacement $\theta \to \tau$. However, an isomorphism does not mean identical, and in Spin(2) we gain extra structures related to a relativistic description, which are not available in the QM case.

To investigate the solution in more detail, we introduce the multivector conjugate, also known as the Clifford conjugate, which generalizes the concept of complex conjugation to multivectors.

Definition 4 (Multivector conjugate (a.k.a Clifford conjugate))**.** *Let* $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$ *be a multi-vector of the geometric algebra over the reals in two dimensions* $\text{GA}(2)$*. The multivector conjugate is defined as:*

$$\mathbf{u}^\ddagger = a - \mathbf{x} - \mathbf{b} \tag{57}$$

The determinant of the matrix representation of a multivector can be expressed as a self-product:

Theorem 4 (Determinant as a Multivector Self-Product)**.**

$$\mathbf{u}^\ddagger\mathbf{u} = \det \mathbf{M} \tag{58}$$

Proof. Let $\mathbf{u} = a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y$, and let $\mathbf{M}$ be its matrix representation $\begin{bmatrix}a+x & y-b\\ y+b & a-x\end{bmatrix}$. Then:

$$\begin{aligned}
1:\quad & \mathbf{u}^\ddagger\mathbf{u} && (59)\\
& = (a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y)^\ddagger(a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y) && (60)\\
& = (a - x\sigma_x - y\sigma_y - b\sigma_x \wedge \sigma_y)(a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y) && (61)\\
& = a^2 - x^2 - y^2 + b^2 && (62)\\
2:\quad & \det \mathbf{M} && (63)\\
& = \det\begin{bmatrix}a+x & y-b\\ y+b & a-x\end{bmatrix} && (64)\\
& = (a+x)(a-x) - (y-b)(y+b) && (65)\\
& = a^2 - x^2 - y^2 + b^2 && (66)
\end{aligned}$$

□

Building upon the concept of the multivector conjugate, we introduce the multivector conjugate transpose, which serves as an extension of the Hermitian conjugate to the domain of multivectors.

 doi:10.20944/preprints202404.1009.v5

Definition 5 (Multivector Conjugate Transpose). *Let* $|V\rangle\rangle \in (\mathrm{GA}(2))^n$*:*

$$|V\rangle\rangle = \begin{bmatrix} a_1 + \mathbf{x}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{x}_n + \mathbf{b}_n \end{bmatrix} \tag{67}$$

The multivector conjugate transpose of $|V\rangle\rangle$ *is defined as first taking the transpose and then the element-wise multivector conjugate:*

$$\langle\langle V| = \begin{bmatrix} a_1 - \mathbf{x}_1 - \mathbf{b}_1 & \dots & a_n - \mathbf{x}_n - \mathbf{b}_n \end{bmatrix} \tag{68}$$

Definition 6 (Bilinear Form). *Let* $|V\rangle\rangle$ *and* $|W\rangle\rangle$ *be two vectors valued in* $\mathrm{GA}(2)$*. We introduce the following bilinear form:*

$$\langle\langle V|W\rangle\rangle = (a_1 - \mathbf{x}_1 - \mathbf{b}_1)(a_1 + \mathbf{x}_1 + \mathbf{b}_1) + \dots (a_n - \mathbf{x}_n - \mathbf{b}_n)(a_n + \mathbf{x}_n + \mathbf{b}_n) \tag{69}$$

Theorem 5 (Inner Product). *Restricted to the even sub-algebra of* $\mathrm{GA}(2)$*, the bilinear form is an inner product.*

Proof.

$$\langle\langle V|W\rangle\rangle_{\mathbf{x}\to 0} = (a_1 - \mathbf{b}_1)(a_1 + \mathbf{b}_1) + \dots (a_n - \mathbf{b}_n)(a_n + \mathbf{b}_n) \tag{70}$$

This is isomorphic to the inner product of a complex Hilbert space, with the identification $i \cong \sigma_x \wedge \sigma_y$. □

Definition 7 (Spin(2)-valued Wavefunction).

$$|\psi\rangle\rangle = \begin{bmatrix} e^{\frac{1}{2}(a_1 + \mathbf{b}_1)} \\ \vdots \\ e^{\frac{1}{2}(a_n + \mathbf{b}_n)} \end{bmatrix} = \begin{bmatrix} \sqrt{\rho_1} R_1 \\ \vdots \\ \sqrt{\rho_2} R_2 \end{bmatrix} \tag{71}$$

where $\sqrt{\rho_i} = e^{\frac{1}{2}a_i}$ *representing the square root of the probability and* $R_i = e^{\frac{1}{2}\mathbf{b}_i}$ *representing a rotor in 2D (or boost in 1+1D).*

The partition function of the probability measure can be expressed using the bilinear form applied to the Spin(2)-valued Wavefunction:

Theorem 6 (Partition Function). $Z = \langle\langle \psi|\psi\rangle\rangle$

Proof.

$$\langle\langle \psi|\psi\rangle\rangle = \sum_i \psi_i^\ddagger \psi_i = \sum_i \rho_i R_i^\ddagger R_i = \sum_i \rho_i = Z \tag{72}$$

□

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 July 2024 doi:10.20944/preprints202404.1009.v5

Definition 8 (Spin(2)-valued Evolution Operator)**.**

$$T = \begin{bmatrix} e^{-\frac{1}{2}\theta \mathbf{b}_1} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\theta \mathbf{b}_n} \end{bmatrix} \tag{73}$$

Theorem 7. *The partition function is invariant with respect to the Spin(2)-valued evolution operator.*

Proof. We note that:

$$\langle\langle T\mathbf{v}|T\mathbf{v}\rangle\rangle = \langle\langle \mathbf{v}|\mathbf{v}\rangle\rangle = \mathbf{v}^\ddagger T^\ddagger T\mathbf{v} \implies T^\ddagger T = I \tag{74}$$

then, since $\begin{bmatrix} e^{\frac{1}{2}\theta \mathbf{b}_1} & & \\ & \ddots & \\ & & e^{\frac{1}{2}\theta \mathbf{b}_n} \end{bmatrix} \begin{bmatrix} e^{-\frac{1}{2}\theta \mathbf{b}_1} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\theta \mathbf{b}_n} \end{bmatrix} = I$, the relation $T^\ddagger T = I$ is satisfied. □

We note that the even sub-algebra of GA(2), being closed under addition and multiplication and constituting an inner product through its bilinear form, allows for the construction of a Hilbert space. In this context, the Hilbert space is Spin(2)-valued. The primary distinction between a wavefunction in a complex Hilbert space and one in a Spin(2)-valued Hilbert space lies in the subject matter of the theory. Specifically, in the latter, the construction governs the change in orientation experienced by an observer, which in turn dictates the measurement basis used in the experiment, consistently with the rotational symmetry and freedom of the system.

The dynamics of observer orientation transformations are described by the Schrödinger equation, which is derived by taking the derivative of the wavefunction with respect to the Lagrange multiplier, θ:

Definition 9 (Spin(2)-valued Schrödinger Equation)**.**

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \begin{bmatrix} \psi_1(\theta) \\ \vdots \\ \psi_n(\theta) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\mathbf{b}_1 & & \\ & \ddots & \\ & & -\frac{1}{2}\mathbf{b}_n \end{bmatrix} \begin{bmatrix} \psi_1(\theta) \\ \vdots \\ \psi_n(\theta) \end{bmatrix} \tag{75}$$

Here, θ represents a global one-parameter evolution parameter akin to time, which is able to transform the wavefunction under the Spin(2), locally across the states of the Hilbert space. This is an extremely general equation that captures all transformations that can be done consistently with the symmetries of the wavefunction.

Definition 10 (David Hestenes' Formulation)**.** *In 3+1D, the David Hestenes' formulation [7] of the wavefunction is $\psi = \sqrt{\rho}Re^{ib/2}$, where $R = e^{\mathbf{f}/2}$ is a Lorentz boost or rotation and where $e^{ib/2}$ is a phase. In 2D, as the algebra only admits a bivector, his formulation would reduce to $\psi = \sqrt{\rho}R$, which is the form we have recovered.*

The definition of the Dirac current applicable to our wavefunction follows the formulation of David Hestenes:

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 July 2024 doi:10.20944/preprints202404.1009.v5

Definition 11 (Dirac Current). *Given the basis σ_x and σ_y, the Dirac current for the 2D theory is defined as:*

$$J_1 \equiv \psi^\ddagger \sigma_x \psi = \rho \underbrace{R^\ddagger \sigma_x R}_{\text{SO(2)}} = \rho \tilde{\sigma}_x \tag{76}$$

$$J_2 \equiv \psi^\ddagger \sigma_y \psi = \rho \underbrace{R^\ddagger \sigma_y R}_{\text{SO(2)}} = \rho \tilde{\sigma}_y \tag{77}$$

where $\tilde{\sigma}_x$ and $\tilde{\sigma}_y$ are a SO(2) rotated basis vectors.

2.2.1. Obstructions

As stated, all geometric configurations except 3+1D contain obstructions. Specifically, in 1+1D and 2D, we identify two obstructions:

1. **In 1+1D:** The 1+1D theory results in a split-complex quantum theory due to the bilinear form $(a - b\mathbf{e}_0 \wedge \mathbf{e}_1)(a + b\mathbf{e}_0 \wedge \mathbf{e}_1)$, which yields negative probabilities: $a^2 - b^2 \in \mathbb{R}$ for certain wavefunction states, in contrast to the non-negative probabilities $a^2 + b^2 \in \mathbb{R}^{\geq 0}$ obtained in the Euclidean 2D case. (This is why we had to use 2D instead of 1+1D in this two-dimensional introduction...)
2. **In 1+1D and in 2D:** The basis vectors (σ_x and σ_y in 2D, and $\mathbf{e}_0$ and $\mathbf{e}_1$ in 1+1D) are not self-adjoint. Although useable in the context of defining the Dirac current, their non-self-adjointness prevents the construction of the metric tensor as an observable. The benefits of having the basis vectors self-adjoint will become obvious in the 3+1D case, where we will be able to construct the metric tensor from basis measurements. Specifically, in 2D:

$$(\mathbf{e}_\mu \mathbf{u})^\ddagger \mathbf{u} \neq \mathbf{u}^\ddagger \mathbf{e}_\mu \mathbf{u} \tag{78}$$

because $(\mathbf{e}_\mu \mathbf{u})^\ddagger \mathbf{u} = \mathbf{u}^\ddagger \mathbf{e}_\mu^\ddagger \mathbf{u} = \mathbf{u}^\ddagger(-\mathbf{e}_\mu)\mathbf{u}$.

In the following section, we will explore the obstruction-free 3+1D case.

2.3. RQM in 3+1D

In this section, we extend the concepts and techniques developed for multivector amplitudes in 2D to the more physically relevant case of 3+1D dimensions. The Lagrange multiplier equation is as follows:

$$\mathcal{L} = \underbrace{-\sum_i \rho_i \ln \frac{\rho_i}{p_i}}_{\text{Relative Shannon Entropy}} + \lambda \underbrace{\left(1 - \sum_i \rho_i\right)}_{\text{Normalization Constraint}} + \underbrace{\zeta \left(-\mathrm{tr} \frac{1}{2} \sum_i \rho_i \mathbf{M}_i\right)}_{\text{Vanishing Spin}^c\text{(3,1)-Phase}} \tag{79}$$

The solution (proof in Appendix B) is obtained using the same step-by-step process as the 2D case, and yields:

$$\rho_i = \underbrace{\frac{1}{\sum_i p_i \det \exp(-\zeta \frac{1}{2} \mathbf{M}_i)}}_{\text{Spin}^c\text{(3,1) Invariant Ensemble}} \underbrace{\det \exp(-\zeta \frac{1}{2} \mathbf{M}_i)}_{\text{Spin}^c\text{(3,1) Born Rule}} \underbrace{p_i}_{\text{Initial Preparation}} \tag{80}$$

where ζ is a "twisted-phase" rapidity. (If the invariance group was Spin(3,1) instead of Spin^c(3,1), obtainable by posing $\mathbf{b} \to 0$, then it would simply be the rapidity).

2.3.1. Preliminaries

Our initial goal will be to express the partition function as a self-product of elements of the vector space. As such, we begin by defining a general multivector in the geometric algebra GA$(3,1)$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 July 2024 doi:10.20944/preprints202404.1009.v5

Definition 12 (Multivector). *Let* $\mathbf{u}$ *be a multivector of* $\mathrm{GA}(3,1)$*. Its general form is:*

$$\mathbf{u} = a \tag{81}$$
$$+ t\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3 \tag{82}$$
$$+ f_{01}\gamma_0 \wedge \gamma_1 + f_{02}\gamma_0 \wedge \gamma_2 + f_{03}\gamma_0 \wedge \gamma_3 + f_{12}\gamma_1 \wedge \gamma_2 + f_{13}\gamma_1 \wedge \gamma_3 + f_{23}\gamma_2 \wedge \gamma_3 \tag{83}$$
$$+ p\gamma_1 \wedge \gamma_2 \wedge \gamma_3 + q\gamma_0 \wedge \gamma_2 \wedge \gamma_3 + v\gamma_0 \wedge \gamma_1 \wedge \gamma_3 + w\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \tag{84}$$
$$+ b\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \tag{85}$$

where $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ *are the basis vectors in the real Majorana representation.*

A more compact notation for $\mathbf{u}$ *is*

$$\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b} \tag{86}$$

where a *is a scalar,* $\mathbf{x}$ *a vector,* $\mathbf{f}$ *a bivector,* $\mathbf{v}$ *is pseudo-vector and* $\mathbf{b}$ *a pseudo-scalar.*

This general multivector can be represented by a 4×4 real matrix using the real Majorana representation:

Definition 13 (Matrix Representation of $\mathbf{u}$).

$$\mathbf{M} = \begin{bmatrix} a + f_{02} - q - z & b - f_{13} + w - x & -f_{01} + f_{12} - p + v & f_{03} + f_{23} + t + y \\ -b + f_{13} + w - x & a + f_{02} + q + z & f_{03} + f_{23} - t - y & f_{01} - f_{12} - p + v \\ -f_{01} - f_{12} + p + v & f_{03} - f_{23} + t - y & a - f_{02} + q - z & -b - f_{13} - w - x \\ f_{03} - f_{23} - t + y & f_{01} + f_{12} + p + v & b + f_{13} - w - x & a - f_{02} - q + z \end{bmatrix} \tag{87}$$

To manipulate and analyze multivectors in $\mathrm{GA}(3,1)$, we introduce several important operations, such as the multivector conjugate, the 3,4 blade conjugate, and the multivector self-product.

Definition 14 (Multivector Conjugate (in 4D)).

$$\mathbf{u}^{\ddagger} = a - \mathbf{x} - \mathbf{f} + \mathbf{v} + \mathbf{b} \tag{88}$$

Definition 15 (3,4 Blade Conjugate). *The 3,4 blade conjugate of* $\mathbf{u}$ *is*

$$\lfloor \mathbf{u} \rfloor_{3,4} = a + \mathbf{x} + \mathbf{f} - \mathbf{v} - \mathbf{b} \tag{89}$$

The results of Lundholm[8], demonstrates that the multivector norms in the following definition, are the *unique* forms which carries the properties of the determinants such as $N(\mathbf{uv}) = N(\mathbf{u})N(\mathbf{v})$ to the domain of multivectors:

Definition 16. *The self-products associated with low-dimensional geometric algebras are:*

$$\mathrm{GA}(0,1): \quad \varphi^{\dagger}\varphi \tag{90}$$
$$\mathrm{GA}(2,0): \quad \varphi^{\ddagger}\varphi \tag{91}$$
$$\mathrm{GA}(3,0): \quad \lfloor \varphi^{\ddagger}\varphi \rfloor_3 \varphi^{\ddagger}\varphi \tag{92}$$
$$\mathrm{GA}(3,1): \quad \lfloor \varphi^{\ddagger}\varphi \rfloor_{3,4} \varphi^{\ddagger}\varphi \tag{93}$$
$$\mathrm{GA}(4,1): \quad (\lfloor \varphi^{\ddagger}\varphi \rfloor_{3,4} \varphi^{\ddagger}\varphi)^{\dagger}(\lfloor \varphi^{\ddagger}\varphi \rfloor_{3,4} \varphi^{\ddagger}\varphi) \tag{94}$$

We can now express the determinant of the matrix representation of a multivector via the self-product $\lfloor \varphi^{\ddagger}\varphi \rfloor_{3,4} \varphi^{\ddagger}\varphi$. Again, this choice is not arbitrary, but the unique choice with allows us to represent the determinant of the matrix representation of a multivector within $\mathrm{GA}(3,1)$:

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 July 2024 doi:10.20944/preprints202404.1009.v5

Theorem 8 (Determinant as a Multivector Self-Product).

$$\lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u} = \det \mathbf{M} \tag{95}$$

Proof. Please find a computer assisted proof of this equality in Appendix C. □

Definition 17 (GA(3,1)-valued Vector).

$$|V\rangle\rangle = \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} = \begin{bmatrix} a_1 + \mathbf{x}_1 + \mathbf{f}_1 + \mathbf{v}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{x}_n + \mathbf{f}_n + \mathbf{v}_n + \mathbf{b}_n \end{bmatrix} \tag{96}$$

These constructions allow us to express the partition function in terms of the multivector self-product:

Definition 18 (Double-Copy Product). *Instead of an inner product, we obtain what we call a double-copy product:*

$$\langle\langle V|V|V|V\rangle\rangle = \sum_i \lfloor \underbrace{\psi_i^\ddagger \psi_i}_{\text{copy 1}} \rfloor_{3,4} \underbrace{\psi_i^\ddagger \psi_i}_{\text{copy 2}} \tag{97}$$

$$= \underbrace{\lfloor \begin{bmatrix} \mathbf{u}_1^\ddagger & \dots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n \end{bmatrix} \rfloor_{3,4}}_{\text{copy 1}} \underbrace{\begin{bmatrix} \mathbf{u}_1^\ddagger & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n^\ddagger \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix}}_{\text{copy 2}} \tag{98}$$

Theorem 9 (Partition Function). $Z = \langle\langle V|V|V|V\rangle\rangle$

Proof.

$$\langle\langle V|V|V|V\rangle\rangle \tag{99}$$

$$= \lfloor \begin{bmatrix} \mathbf{u}_1^\ddagger & \dots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} \mathbf{u}_1^\ddagger & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n^\ddagger \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} \tag{100}$$

$$= \lfloor \begin{bmatrix} \mathbf{u}_1^\ddagger \mathbf{u}_1 & \dots & \mathbf{u}_n \mathbf{u}_n \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} \mathbf{u}_1^\ddagger \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n^\ddagger \mathbf{u}_n \end{bmatrix} \tag{101}$$

$$= \lfloor \mathbf{u}_1^\ddagger \mathbf{u}_1 \rfloor_{3,4} \mathbf{u}_1^\ddagger \mathbf{u}_1 + \dots + \lfloor \mathbf{u}_n^\ddagger \mathbf{u}_n \rfloor_{3,4} \mathbf{u}_n^\ddagger \mathbf{u}_n \tag{102}$$

$$= \sum_{i=1}^{n} \det \mathbf{M}_{\mathbf{u}_i} \tag{103}$$

$$= Z \tag{104}$$

□

Desirable properties for the double-copy product are introduced by reducing multivectors to its subgroups. First, non-negativity:

Theorem 10 (Non-negativity). *The double-copy product, applied to the even sub-algebra of* GA(3,1) *is always non-negative.*

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 July 2024 doi:10.20944/preprints202404.1009.v5

Proof. Let $|V\rangle\!\rangle = \begin{bmatrix} a_1 + \mathbf{f}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{f}_n + \mathbf{b}_n \end{bmatrix}$. Then,

$$\langle\!\langle V|V|V|V\rangle\!\rangle \tag{105}$$

$$= \lfloor \begin{bmatrix} (a_1 + \mathbf{f}_1 + \mathbf{b}_1)^\ddagger (a_1 + \mathbf{f}_1 + \mathbf{b}_1) & \dots \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} (a_1 + \mathbf{f}_1 + \mathbf{b}_1)^\ddagger (a_1 + \mathbf{f}_1 + \mathbf{b}_1) \\ \vdots \end{bmatrix} \tag{106}$$

$$= \lfloor \begin{bmatrix} (a_1 - \mathbf{f}_1 + \mathbf{b}_1)(a_1 + \mathbf{f}_1 + \mathbf{b}_1) & \dots \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} (a_1 - \mathbf{f}_1 + \mathbf{b}_1)(a_1 + \mathbf{f}_1 + \mathbf{b}_1) \\ \vdots \end{bmatrix} \tag{107}$$

$$= \lfloor \begin{bmatrix} a_1^2 + a_1\mathbf{f}_1 + a_1\mathbf{b}_1 - \mathbf{f}_1 a_1 - \mathbf{f}_1^2 - \mathbf{f}_1\mathbf{b}_1 + \mathbf{b}_1 a_1 + \mathbf{b}_1\mathbf{f}_1 + \mathbf{b}_1^2 & \dots \end{bmatrix} \rfloor_{3,4} \dots \tag{108}$$

$$= \lfloor \begin{bmatrix} a_1^2 - \mathbf{f}_1^2 + \mathbf{b}_1^2 & \dots \end{bmatrix} \rfloor_{3,4} \dots \tag{109}$$

We note 1) $\mathbf{b}^2 = (bI)^2 = -b^2$ and 2) $\mathbf{f}^2 = -E_1^2 - E_2^2 - E_3^2 + B_1^2 + B_2^2 + B_3^2 + 4e_0e_1e_2e_3(E_1B_1 + E_2B_2 + E_3B_3)$

$$= \lfloor \begin{bmatrix} a_1^2 - b_1^2 + E_1^2 + E_2^2 + E_3^2 - B_1^2 - B_2^2 - B_3^2 - 4e_0e_1e_2e_3(E_1B_1 + E_2B_2 + E_3B_3) & \dots \end{bmatrix} \rfloor_{3,4} \dots \tag{110}$$

We note that the terms are now complex numbers, which we rewrite as $\Re(z) = a_1^2 - b_1^2 + E_1^2 + E_2^2 + E_3^2 - B_1^2 - B_2^2 - B_3^2$ and $\Im(z) = -4(E_1B_1 + E_2B_2 + E_3B_3)$

$$= \lfloor \begin{bmatrix} z_1 & \dots & z_2 \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} z_n \\ \vdots \\ z_n \end{bmatrix} \tag{111}$$

$$= \begin{bmatrix} z_1^\dagger & \dots & z_2^\dagger \end{bmatrix} \begin{bmatrix} z_n \\ \vdots \\ z_n \end{bmatrix} \tag{112}$$

$$= z_1^\ddagger z_1 + \dots + z_n^\ddagger z_n \tag{113}$$

which is always non-negative. □

Then, positive-definiteness of the double-copy product is obtained by creating an equivalence class between the zero vector and any non-zero vector of length zero, and taking the zero vector as the representative of the class. To realize the equivalence class, we define the $\text{Spin}^c(3,1)$-valued wavefunction, which is valued in the even sub-algebra of $\text{GA}(3,1)$, as follows:

Definition 19 ($\text{Spin}^c(3,1)$-valued Wavefunction)**.**

$$|\psi\rangle\!\rangle = \begin{bmatrix} e^{\frac{1}{2}(a_1 + \mathbf{f}_1 + \mathbf{b}_1)} \\ \vdots \\ e^{\frac{1}{2}(a_n + \mathbf{f}_n + \mathbf{b}_n)} \end{bmatrix} = \begin{bmatrix} \sqrt{\rho_1} R_1 B_1 \\ \vdots \\ \sqrt{\rho_n} R_n B_n \end{bmatrix} \tag{114}$$

where $R_i = e^{\frac{1}{2}\mathbf{f}_i}$ *is a rotor,* $B_i = e^{\frac{1}{2}\mathbf{b}_i}$ *is a phase, and where* $\sqrt{\rho_i} = e^{\frac{1}{2}a_i} \geq 0$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 July 2024 doi:10.20944/preprints202404.1009.v5

Any even multivectors of GA(3,1) admits a unique exponential representation, except when $\rho_i = 0$ in which it is surjective. Consequently, in this representation the double-copy product yields 0 only for the zero vector, rendering the double-copy product positive-definite.

Now, let us turn our attention to the evolution operator, which leaves the partition function invariant:

Definition 20 (Spin$^c(3,1)$ Evolution Operator)**.**

$$T = \begin{bmatrix} e^{-\frac{1}{2}\zeta(\mathbf{f}_1+\mathbf{b}_1)} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\zeta(\mathbf{f}_n+\mathbf{b}_n)} \end{bmatrix} \tag{115}$$

In turn, this leads to a Schrödinger equation obtained by taking the derivative of the wavefunction with respect to the Lagrange multiplier ζ:

Definition 21 (Spin$^c(3,1)$-valued Schrödinger equation)**.**

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\begin{bmatrix}\psi_1(\zeta)\\ \vdots \\ \psi_n(\zeta)\end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(\mathbf{f}_1+\mathbf{b}_1) & & \\ & \ddots & \\ & & -\frac{1}{2}(\mathbf{f}_n+\mathbf{b}_n) \end{bmatrix}\begin{bmatrix}\psi_1(\zeta)\\ \vdots \\ \psi_n(\zeta)\end{bmatrix} \tag{116}$$

In this case ζ represents a one-parameter evolution parameter akin to time, which is able to transform the measurement basis under action of the Spin$^c(3,1)$ group. This is an extremely general equation that captures all transformations that can be done consistently with the symmetries of the wavefunction.

Theorem 11 (Spin^c(3,1) invariance)**.** *Let $e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}$ be a general element of Spin^c(3,1). Then, the equality:*

$$\lfloor \psi^\ddagger\psi\rfloor_{3,4}\psi^\ddagger\psi = \lfloor (e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^\ddagger e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi\rfloor_{3,4}(e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^\ddagger e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi \tag{117}$$

is always satisfied.

Proof.

$$\begin{aligned}
&\lfloor (e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^\ddagger e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi\rfloor_{3,4}(e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^\ddagger e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi && (118)\\
&= \lfloor \psi^\ddagger e^{-\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi\rfloor_{3,4}\psi^\ddagger e^{-\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi && (119)\\
&= \lfloor \psi^\ddagger e^{\mathbf{b}}\psi\rfloor_{3,4}\psi^\ddagger e^{\mathbf{b}}\psi && (120)\\
&= \lfloor \psi^\ddagger\psi\rfloor_{3,4}e^{-\mathbf{b}}e^{\mathbf{b}}\psi^\ddagger\psi && (121)\\
&= \lfloor \psi^\ddagger\psi\rfloor_{3,4}\psi^\ddagger\psi && (122)
\end{aligned}$$

□

2.3.2. RQM

Definition 22 (David Hestenes' Wavefunction)**.** *The* Spin$^c(3,1)$*-valued wavefunction we have recovered is formulated identically to David Hestenes'[7] formulation of the wavefunction within GA(3,1).*

$$\psi = e^{\frac{1}{2}(a+\mathbf{f}+\mathbf{b})} = \sqrt{\rho}Re^{-ib/2} \tag{123}$$

where $e^{\frac{1}{2}a} = \sqrt{\rho}$, $e^{\frac{1}{2}\mathbf{f}} = R$ and $e^{\frac{1}{2}\mathbf{b}} = e^{-ib/2}$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 July 2024 doi:10.20944/preprints202404.1009.v5

Before we continue the RQM investigation, let us note that the double-copy product contains two copies of a bilinear form $\psi^\ddagger\psi$:

$$\lfloor \underbrace{\psi^\ddagger\psi}_{\text{copy 1}} \rfloor_{3,4} \underbrace{\psi^\ddagger\psi}_{\text{copy 2}} \tag{124}$$

In the present section, we will investigate the properties of each copy individually, leaving the properties specific to the double-copy for the section on quantum gravity.

Taking a single copy, the Dirac current is obtained directly from the gamma matrices, as follows:

Definition 23 (Dirac Current)**.** *The definition of the Dirac current is the same as Hestenes':*

$$J \equiv \psi^\ddagger\gamma_\mu\psi = \rho R^\ddagger B^\ddagger\gamma_\mu BR = \rho R^\ddagger\gamma_\mu B^{-1}BR = \rho\underbrace{R^\ddagger\gamma_\mu R}_{SO(3,1)} = \rho\tilde{\gamma}_\mu \tag{125}$$

where $\tilde{\gamma}_\mu$ is a SO(3,1) rotated basis vector.

2.3.3. Standard Model Gauge Symmetries

We will now demonstrate that the copied bilinear form is automatically invariant with respect to the U(1), SU(2), and SU(3) symmetries and the unitary $U^\dagger U = I$ symmetry which play a fundamental role in the standard model of particle physics. These constitute the set of symmetries that stabilize the Dirac current $(T\psi)^\ddagger\gamma_0 T\psi = \psi^\ddagger\gamma_0\psi$.

Theorem 12 (U(1) Invariance)**.** *Let $e^{\frac{1}{2}\mathbf{b}}$ be a general element of U(1). Then, the equality*

$$\lfloor\psi^\ddagger\gamma_0\psi\rfloor_{3,4}\psi^\ddagger\gamma_0\psi = \lfloor\underbrace{(e^{\frac{1}{2}\mathbf{b}}\psi)^\ddagger\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi}_{\text{copy 1}}\rfloor_{3,4}\underbrace{(e^{\frac{1}{2}\mathbf{b}}\psi)^\ddagger\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi}_{\text{copy 2}} \tag{126}$$

is satisfied, yielding a U(1) symmetry for each copied bilinear form.

Proof. Equation 126 is invariant if this expression is satisfied:

$$e^{\frac{1}{2}\mathbf{b}}\gamma_0 e^{\frac{1}{2}\mathbf{b}} = \gamma_0 \tag{127}$$

This is always satisfied simply because $e^{\frac{1}{2}\mathbf{b}}\gamma_0 e^{\frac{1}{2}\mathbf{b}} = \gamma_0 e^{-\frac{1}{2}\mathbf{b}} e^{\frac{1}{2}\mathbf{b}} = \gamma_0$ □

Theorem 13 (SU(2) Invariance)**.** *Let $e^{\frac{1}{2}\mathbf{f}}$ be a general element of Spin(3,1). Then, the equality:*

$$\lfloor\psi^\ddagger\gamma_0\psi\rfloor_{3,4}\psi^\ddagger\gamma_0\psi = \lfloor\underbrace{(e^{\frac{1}{2}\mathbf{f}}\psi)^\ddagger\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi}_{\text{copy 1}}\rfloor_{3,4}\underbrace{(e^{\frac{1}{2}\mathbf{f}}\psi)^\ddagger\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi}_{\text{copy 2}} \tag{128}$$

is satisfied for if $\mathbf{f} = \theta_1\gamma_2\gamma_3 + \theta_2\gamma_1\gamma_3 + \theta_3\gamma_1\gamma_2$ (which generates SU(2)), yielding a SU(2) symmetry for each copied bilinear form.

Proof. Equation 128 is invariant if this expression is satisfied[9]:

$$e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}} = \gamma_0 \tag{129}$$

We now note that moving the left-most term to the right of the gamma matrix yields:

$$e^{-E_1\gamma_0\gamma_1-E_2\gamma_0\gamma_2-E_3\gamma_0\gamma_3-\theta_1\gamma_2\gamma_3-\theta_2\gamma_1\gamma_3-\theta_3\gamma_1\gamma_2}\gamma_0 e^{\frac{1}{2}\mathbf{f}} \tag{130}$$

$$= \gamma_0 e^{E_1\gamma_0\gamma_1+E_2\gamma_0\gamma_2+E_3\gamma_0\gamma_3-\theta_1\gamma_2\gamma_3-\theta_2\gamma_1\gamma_3-\theta_3\gamma_1\gamma_2} e^{\frac{1}{2}\mathbf{f}} \tag{131}$$

Therefore, the product $e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}}$ reduces to γ_0 if and only if $E_1 = E_2 = E_3 = 0$, leaving $\mathbf{f} = \theta_1\gamma_2\gamma_3 + \theta_2\gamma_1\gamma_3 + \theta_3\gamma_1\gamma_2$:

Finally, we note that $e^{\theta_1\gamma_2\gamma_3+\theta_2\gamma_1\gamma_3+\theta_3\gamma_1\gamma_2}$ generates SU(2). □

Theorem 14 (SU(3))**.** *The generators of SU(3) in GA(3,1) are given by Anthony Lesenby in [10] and are as follows:*

$$\hat{E}_{ij} = \hat{e}_i\hat{e}_j - \hat{f}_i\hat{f}_j \qquad \text{where } i < j \tag{132}$$

$$\hat{F}_{ij} = \hat{e}_i\hat{f}_j + \hat{e}_j\hat{f}_i \qquad \text{where } i < j \tag{133}$$

$$\hat{J} = \hat{e}_i\hat{f}_i \qquad \text{where } i = 1,2,3 \tag{134}$$

where

$$\hat{e}_i = \text{multiplication on the left by } \sigma_i\text{, so that } \hat{e}_i(F) = \sigma_i F \tag{135}$$

$$\hat{f}_i = \text{multiplication on the right by } I\sigma_i\text{, so that } \hat{f}_i(F) = I\sigma_i F \tag{136}$$

This defines the 9 generators of U(3).

With the additional restriction on $\hat{J}$

$$\alpha_1\hat{J}_1 + \alpha_2\hat{J}_2 + \alpha_3\hat{J}_3, \text{ with } \alpha_1 + \alpha_2 + \alpha_3 = 0 \tag{137}$$

the number generators is reduced to 8, consistently with SU(3).

We now must show that the following equation is satisfied for all 8 generators:

$$\lfloor \psi^\ddagger\gamma_0\psi\rfloor_{3,4}\psi^\ddagger\gamma_0\psi = \lfloor\underbrace{(e^{\theta^i\lambda_i}\psi)^\ddagger\gamma_0 e^{\theta^i\lambda_i}\psi}_{\text{copy 1}}\rfloor_{3,4}\underbrace{(e^{\theta^i\lambda_i}\psi)^\ddagger\gamma_0 e^{\theta^i\lambda_i}\psi}_{\text{copy 2}} \tag{138}$$

Proof. First, we note the following action:

$$-\mathbf{f}\gamma_0\mathbf{f} = \gamma_0 \tag{139}$$

which we can rewrite as follows:

$$-(E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3 + B_1\gamma_2\gamma_3 + B_2\gamma_1\gamma_3 + B_3\gamma_1\gamma_2)\gamma_0\mathbf{f} \tag{140}$$

The first three terms anticommute with γ_0, while the last three commute with γ_0:

$$= \gamma_0(E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3 - B_1\gamma_2\gamma_3 - B_2(q)\gamma_1\gamma_3 - B_3(q)\gamma_1\gamma_2)\mathbf{f}(q) \tag{141}$$

This can be written as:

$$\gamma_0(\mathbf{E} - \mathbf{B})(\mathbf{E} + \mathbf{B}) \tag{142}$$

$$= \gamma_0(\mathbf{E}^2 + \mathbf{EB} - \mathbf{BE} - \mathbf{B}^2) \tag{143}$$

where $\mathbf{E} = E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3$ and $\mathbf{B} = B_1\gamma_2\gamma_3 + B_2\gamma_1\gamma_3 + B_3\gamma_1\gamma_2$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 July 2024 doi:10.20944/preprints202404.1009.v5

Thus, for $-\mathbf{f}\gamma_0\mathbf{f} = \gamma_0$, we require: 1) $\mathbf{E}^2 - \mathbf{B}^2 = 1$ and 2) $\mathbf{EB} = \mathbf{BE}$. The first requirement expands as follows:

$$\mathbf{E}^2 - \mathbf{B}^2 = (E_1^2 + B_1^2) + (E_2^2 + B_2^2) + (E_3^2 + B_3^2) = 1 \tag{144}$$

which is the defining conditions for the SU(3) symmetry group.

Finally, as the SU(3) norm is a consequence of preserving the Dirac current, it follows that the SU(3) generators provided by Lasenby, acting on $\mathbf{f}$, cannot change the SU(3) norm, hence must also preserve the Dirac current. □

Theorem 15 (Unitary invariance)**.** *Let U be $n \times n$ unitary matrices. Then unitary invariance:*

$$\langle\!\langle \psi|\gamma_\mu\psi|\psi|\gamma_\nu\psi\rangle\!\rangle = \langle\!\langle U\psi|\gamma_\mu U\psi|U\psi|\gamma_\nu U\psi\rangle\!\rangle \implies U^\dagger U = I \tag{145}$$

is individually satisfied for each copied bilinear form.

Proof. Equation 145 is satisfied if $U^\ddagger\gamma_\mu U = \gamma_\mu$. Since U is valued in complex numbers, then $U^\ddagger = U^T$, and since $\gamma_\mu\gamma_0\gamma_1\gamma_2\gamma_3 = -\gamma_0\gamma_1\gamma_2\gamma_3\gamma_\mu$, it follows that:

$$\gamma_\mu U^\dagger U = \gamma_\mu \tag{146}$$

which is satisfied when $U^\dagger U = I$. □

The invariances SU(3), SU(2) and U(1) discussed above can be promoted to local symmetries using the usual gauge symmetry construction techniques, along with the Dirac equation or field Lagrangian.

In conventional QM, the Born rule naturally leads to a U(1)-valued gauge theory due to the following symmetry:

$$(e^{-i\theta(x)}\psi(x))^\dagger e^{-i\theta(x)}\psi(x) = \psi(x)^\dagger\psi(x) \tag{147}$$

However, the SU(3) and SU(2) symmetries do not emerge from the probability measure in the same way and must instead be introduced manually, justified by experimental considerations. This raises the question: why these specific symmetries and not others? In contrast, within the double-copy product framework, all three symmetry groups–U(1), SU(2), and SU(3)–as well as the Spin(3,1) and unitary symmetries, follow naturally from the invariance of the probability measure, in the same way that U(1) symmetry follows from the Born rule.

2.3.4. Quantum Gravity

In the previous section, we developed a quantum theory valued in Spin^c(3,1), which served as the arena for RQM. We then demonstrated how a single copy of this theory leads to the gauge symmetries of the standard model. The goal of this section is to extend this methodology to arbitrary basis vectors, in which the metric tensor emerges as an observable. To achieve this, we will utilize both copies.

Our formulation is reminiscent of the Bern-Carrasco-Johansson (BCJ) double-copy approach to perturbatively expanded quantum gravity [11]. However, our double-copy is applied directly at the level of the Dirac current, rather than to gauge theory amplitudes.

By applying the double-copy product to the Dirac current, we establish a connection between quantum theory and the geometrical structure of spacetime.

We recall the definition of the metric tensor in terms of basis vectors of geometric algebra, as follows:

$$g_{\mu\nu} = \frac{1}{2}(\mathbf{e}_\mu\mathbf{e}_\nu + \mathbf{e}_\nu\mathbf{e}_\mu) \tag{148}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 July 2024 doi:10.20944/preprints202404.1009.v5

Then, we note that the double-copy product acts on a pair of basis element $\mathbf{e}_\mu$ and $\mathbf{e}_\nu$, as follows:

$$\frac{1}{2}\Big(\lfloor\underbrace{\psi^\ddagger\mathbf{e}_\mu\psi}_{\text{copy 1}}\rfloor_{3,4}\underbrace{\psi^\ddagger\mathbf{e}_\nu\psi}_{\text{copy 2}}+\lfloor\underbrace{\psi^\ddagger\mathbf{e}_\nu\varphi}_{\text{copy 2}}\rfloor_{3,4}\underbrace{\psi^\ddagger\mathbf{e}_\mu\psi}_{\text{copy 1}}\Big) \tag{149}$$

$$=\frac{1}{2}\Big(\tilde{R}\underbrace{\rho e^{ib/2}e^{-ib/2}}_{\text{Born rule copy 1}}\mathbf{e}_\mu R\tilde{R}\underbrace{\rho e^{-ib/2}e^{ib/2}}_{\text{Born rule copy 2}}\mathbf{e}_\nu R+\tilde{R}\underbrace{\rho e^{ib/2}e^{-ib/2}}_{\text{Born rule copy 2}}\mathbf{e}_\nu R\tilde{R}\underbrace{\rho e^{-ib/2}e^{ib/2}}_{\text{Born rule copy 1}}\mathbf{e}_\mu R\Big) \tag{150}$$

$$=\frac{1}{2}\rho^2\big(\tilde{R}\mathbf{e}_\mu R\tilde{R}\mathbf{e}_\nu R+\tilde{R}\mathbf{e}_\nu R\tilde{R}\mathbf{e}_\nu R\big) \tag{151}$$

$$=\rho^2\underbrace{\frac{1}{2}(\tilde{\mathbf{e}}_\mu\tilde{\mathbf{e}}_\nu+\tilde{\mathbf{e}}_\nu\tilde{\mathbf{e}}_\mu)}_{\text{metric tensor}} \tag{152}$$

where $\tilde{\mathbf{e}}_\mu$ and $\tilde{\mathbf{e}}_\nu$ are SO(3,1) rotated basis vectors.

As one can swap $\mathbf{e}_\mu$ and $\mathbf{e}_\nu$ and obtain the same metric tensor, the double-copy product guarantees that $g_{\mu\nu}$ is symmetric.

Furthermore, since $\mathbf{e}_\mu^\ddagger=-\mathbf{e}_\mu$, we get:

$$\lfloor(\mathbf{e}_\mu\psi)^\ddagger\psi\rfloor_{3,4}(\mathbf{e}_\nu\psi)^\ddagger\psi \tag{153}$$

$$=\lfloor\psi^\ddagger(-1)\mathbf{e}_\mu^\ddagger\psi\rfloor_{3,4}\psi^\ddagger(-1)\mathbf{e}_\nu^\ddagger\psi \tag{154}$$

$$=\lfloor\psi^\ddagger\mathbf{e}_\mu\psi\rfloor_{3,4}\psi^\ddagger\mathbf{e}_\nu\psi \tag{155}$$

which allows us to conclude that $\mathbf{e}_\mu$ and $\mathbf{e}_\nu$ are self-adjoint within the double-copy product, entailing the interpretation of $g_{\mu\nu}$ as an observable.

In the double-copy product, the metric tensor emerges as a double copy of Dirac currents. This formulation suggests that the metric tensor encodes the probabilistic structure of a quantum theory of gravity *in the form of a rank-2 tensor*, analogous to how the Dirac current encodes the probabilistic structure of a special relativistic quantum theory *in the form of a 4-vector*.

Let us now investigate the dynamics. We recall that the evolution operator (Definition 20) is:

$$T=\begin{bmatrix}e^{-\frac{1}{2}\zeta(\mathbf{f}_1+\mathbf{b}_1)} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\zeta(\mathbf{f}_n+\mathbf{b}_n)}\end{bmatrix} \tag{156}$$

Acting on the wavefunction, the effect of this operator cascades down to the basis vectors via the double-copy product:

$$\lfloor\underbrace{\psi^\ddagger T^\ddagger\mathbf{e}_\mu T\psi}_{\text{copy 1}}\rfloor_{3,4}\underbrace{\psi^\ddagger T^\ddagger\mathbf{e}_\nu T\psi}_{\text{copy 2}} \tag{157}$$

which realizes an $\mathrm{SO}(3,1)$ transformation of the metric tensor via action of the exponential of a bivector, and a double-copy unitary invariant transformation via action of the exponential of a pseudo-scalar:

$$\underbrace{\lfloor\psi^\ddagger\underbrace{e^{\frac{1}{2}\zeta\mathbf{f}}\mathbf{e}_\mu e^{-\frac{1}{2}\zeta\mathbf{f}}}_{\text{SO(3,1) evolution}}\underbrace{e^{\frac{1}{2}\zeta\mathbf{b}}e^{-\frac{1}{2}\zeta\mathbf{b}}}_{\text{unitary evolution}}\psi\rfloor_{3,4}}_{\text{copy 1}}\underbrace{\psi^\ddagger\underbrace{e^{\frac{1}{2}\zeta\mathbf{f}}\mathbf{e}_\mu e^{-\frac{1}{2}\zeta\mathbf{f}}}_{\text{SO(3,1) evolution}}\underbrace{e^{\frac{1}{2}\zeta\mathbf{b}}e^{-\frac{1}{2}\zeta\mathbf{b}}}_{\text{unitary evolution}}\psi}_{\text{copy 2}} \tag{158}$$

In summary, this initial investigation has identified a scenario in which the metric tensor is measured using basis vectors. The evolution operator, governed by the Schrödinger equation, dynamically realizes SO(3,1) transformations on the metric tensor. Furthermore, the amplitudes associated with

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 July 2024 doi:10.20944/preprints202404.1009.v5

possible metric tensors are derived from a double-copy of unitary quantum theories acting on the basis vectors. This formulation simultaneously preserves the SO(3,1) symmetry, essential for describing spacetime structure, and the unitary symmetry, fundamental to quantum mechanics. It describes all changes of basis transformations that an observer in 3+1D spacetime can perform prior to measuring a quantum system.

2.3.5. Starting Point for a Quantum Theory of Gravity

The symmetries of interest can be approached through two distinct strategies:

1. Particle Physics Approach: We impose the condition $e^{-\frac{1}{2}(\mathbf{f}+\mathbf{b})}\gamma_0 e^{\frac{1}{2}(\mathbf{f}+\mathbf{b})}=\gamma_0$ on the double-copy Dirac current (Equation 158). This constraint leads to the symmetries of the standard model of particle physics, as detailed in Section 2.3.3.
2. Gravitational Approach: We allow the double-copy Dirac current (which is equivalent to the metric tensor) to transform freely under SO(3,1). Instead of constraining the current itself, we focus on constructing SO(3,1)-invariant quantities from it. These invariants are typically formed as specific combinations of the double-copy Dirac current and its derivatives. A key example is the Einstein tensor, which remains invariant under SO(3,1) transformations of the metric. The Einstein tensor is particularly significant because it arises from the variation of the Einstein-Hilbert action, which is simplest action leading to such an invariant.

The first strategy provides the gauge symmetries for the standard model of particle physics, while the second offers a path towards a quantum theory of gravity. In the following section, we will explore the gravitational approach in more detail.

2.3.6. Gravitons

Since the double-copy product of the Dirac current holds for any non-degenerate symmetric rank-2 tensor, it should be possible to show that gravitons can be expressed as a special case of this double-copy mechanism. As such, let us now investigate the wave equation in linearized gravity.

It is well known that the Einstein-Hilbert action:

$$S[g_{\mu\nu}]=\frac{c^4}{16\pi G}\int R\sqrt{-|g|}\mathrm{d}^4x \tag{159}$$

under the assumption of a small perturbation $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, and working in de Donder gauge $\partial^\alpha h_{\alpha\mu}-\frac{1}{2}\partial_\mu h=0$, where $h=\eta^{\mu\nu}h_{\mu\nu}$, can be reduced to its linearized form which is:

$$S^{(1)}_{EH}[h_{\mu\nu}]=\int \mathrm{d}^4x\big(\frac{1}{2}\partial_\mu h_{\rho\sigma}\partial^\mu h^{\rho\sigma}-\frac{1}{4}\partial_\mu h\partial^\mu h\big) \tag{160}$$

Furthermore, varying this action with respect to $h_{\mu\nu}$ and applying the transverse-traceless gauge, yields the wave equation as the equation of motion:

$$\frac{\delta S}{\delta h_{\mu\nu}}=0\implies \Box h_{\mu\nu}=0 \tag{161}$$

We now wish to express the wave equation for $h_{\mu\nu}$ in terms of basis vectors $\mathbf{h}_\mu$ and $\mathbf{h}_\nu$, such that $h_{\mu\nu}=\frac{1}{2}(\mathbf{h}_\mu\mathbf{h}_\nu+\mathbf{h}_\nu\mathbf{h}_\mu)$. The expression of the wave equation becomes:

$$\Box\frac{1}{2}(\mathbf{h}_\mu\mathbf{h}_\nu+\mathbf{h}_\nu\mathbf{h}_\mu)=0 \tag{162}$$

We identify the solution for $\mathbf{h}_\mu$ and $\mathbf{h}_\nu$ by an ansatz:

$$\mathbf{h}_\mu(\vec{x},t)=\mathrm{Re}\int \mathrm{d}^3k\Big(\sqrt{A_+}(\vec{k})e^+_\mu(\vec{k})+\sqrt{A_\times}(\vec{k})e^\times_\mu(\vec{k})\Big)e^{-\frac{1}{2}i(\vec{k}\cdot\vec{x}-\omega_k t)} \tag{163}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 July 2024 doi:10.20944/preprints202404.1009.v5

$$\mathbf{h}_{\nu}(\vec{x},t) = \mathrm{Re}\int \mathrm{d}^3k\Big(\sqrt{A_+}(\vec{k})e^+_{\nu}(\vec{k}) + \sqrt{A_\times}(\vec{k})e^\times_{\nu}(\vec{k})\Big)e^{-\frac{1}{2}i(\vec{k}\cdot\vec{x}-\omega_k t)} \tag{164}$$

The $\times$ and $+$ symbol designed two polarizations.
Then, we promote $\mathbf{h}_{\mu}(\vec{x},t)$ and $\mathbf{h}_{\nu}(\vec{x},t)$ to operators:

$$\hat{\mathbf{h}}_{\mu}(\vec{x},t) = \mathrm{Re}\int \mathrm{d}^3k\Big(\sqrt{\hat{A}_+}(\vec{k})e^+_{\mu}(\vec{k}) + \sqrt{\hat{A}_\times}(\vec{k})e^\times_{\mu}(\vec{k})\Big)e^{-\frac{1}{2}i(\vec{k}\cdot\vec{x}-\omega_k t)} \tag{165}$$

$$\hat{\mathbf{h}}_{\nu}(\vec{x},t) = \mathrm{Re}\int \mathrm{d}^3k\Big(\sqrt{\hat{A}_+}(\vec{k})e^+_{\nu}(\vec{k}) + \sqrt{\hat{A}_\times}(\vec{k})e^\times_{\nu}(\vec{k})\Big)e^{-i\frac{1}{2}(\vec{k}\cdot\vec{x}-\omega_k t)} \tag{166}$$

Finally, the probabilities associated to these operators, corresponding to a metric tensor expectation value, are given using the double-copy product as follows:

$$\langle \hat{h}_{\mu\nu}\rangle = \frac{1}{2}(\langle\!\langle \psi|\hat{\mathbf{h}}_{\mu}\psi|\psi|\hat{\mathbf{h}}_{\nu}\psi\rangle\!\rangle + \langle\!\langle \psi|\hat{\mathbf{h}}_{\nu}\psi|\psi|\hat{\mathbf{h}}_{\mu}\psi\rangle\!\rangle) \tag{167}$$

We note that each copy individually applies the Born rule to one of two operators. In contrast, in the conventional perturbative approach to quantum gravity, the metric tensor $\hat{h}_{\mu\nu}$ is quantized, and its expectation value is calculated using the Born rule as follows: $\langle\hat{h}_{\mu\nu}\rangle = \langle\psi|\hat{h}_{\mu\nu}|\psi\rangle$. Thus, our probability calculation differs from the conventional approach, as it involves the product of two separate Born rule applications rather than just one.

The double-copy mechanism, identified by Bern, Carrasco, and Johansson (BCJ) [11], has been shown to simplify calculations of scattering amplitudes for gravitons in many cases. However, it remains an open question whether our double-copy mechanism carries similar advantages.

While we have described gravitons in terms of perturbations $h_{\mu\nu}$ to the metric, it's important to note that these perturbations transform under SO(3,1) in a way that preserves the SO(3,1) invariance of the Einstein tensor. The Einstein tensor, constructed from $h_{\mu\nu}$ and its derivatives, remains invariant under these transformations. Thus completing (a special case of) the second strategy.

2.4. Dimensional Obstructions

In this section, we explore the dimensional obstructions that arise when attempting to resolve the entropy maximization problem for other dimensional configurations. We found that all geometric configurations except those we have explored here (e.g. $\mathrm{GA}(0) \cong \mathbb{R}$, $\mathrm{GA}(0,1) \cong \mathbb{C}$ and $\mathrm{GA}(3,1)$) are either obstructed or incomplete. By obstructed, we mean that the solution to the entropy maximization problem, ρ, does not satisfy the properties of a probability measure, and by incomplete we refer to the $\mathrm{GA}(2,0)$ case where the metric is not an observable.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 July 2024 doi:10.20944/preprints202404.1009.v5

$$
\begin{array}{lll}
\textit{Dimensions} & \textit{Obstruction} & \\
\mathrm{GA}(0) & \textbf{Unobstructed} \implies \text{statistical mechanics} & (168)\\
\mathrm{GA}(0,1) & \textbf{Unobstructed} \implies \text{quantum mechanics} & (169)\\
\mathrm{GA}(1,0) & \text{Negative probabilities in the RQM} & (170)\\
\mathrm{GA}(2,0) & \text{Incomplete / No metric measurement} & (171)\\
\mathrm{GA}(1,1) & \text{Negative probabilities in the RQM} & (172)\\
\mathrm{GA}(0,2) & \text{Not isomorphic to a real matrix algebra} & (173)\\
\mathrm{GA}(3,0) & \text{Not isomorphic to a real matrix algebra} & (174)\\
\mathrm{GA}(2,1) & \text{Not isomorphic to a real matrix algebra} & (175)\\
\mathrm{GA}(1,2) & \text{Not isomorphic to a real matrix algebra} & (176)\\
\mathrm{GA}(0,3) & \text{Not isomorphic to a real matrix algebra} & (177)\\
\mathrm{GA}(4,0) & \text{Not isomorphic to a real matrix algebra} & (178)\\
\mathrm{GA}(3,1) & \textbf{Unobstructed} \implies \text{quantum gravity} \wedge \mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1) & (179)\\
\mathrm{GA}(2,2) & \text{Negative probabilities in the RQM} & (180)\\
\mathrm{GA}(1,3) & \text{Not isomorphic to a real matrix algebra} & (181)\\
\mathrm{GA}(0,4) & \text{Not isomorphic to a real matrix algebra} & (182)\\
\mathrm{GA}(5,0) & \text{Not isomorphic to a real matrix algebra} & (183)\\
\vdots & \vdots & \\
\mathrm{GA}(6,0) & \text{No probability measure as a self-product} & (184)\\
\vdots & \vdots & \\
\infty & & (185)
\end{array}
$$

Let us now demonstrate the obstructions mentioned above.

Theorem 16 (Not isomorphic to a real matrix algebra)**.** *The determinant of the matrix representation of the geometric algebras in this category is either complex-valued or quaternion-valued, making them unsuitable as a probability.*

Proof. These geometric algebras are classified as follows:

$$\mathrm{GA}(0,2) \cong \mathbb{H} \tag{186}$$
$$\mathrm{GA}(3,0) \cong \mathbb{M}_2(\mathbb{C}) \tag{187}$$
$$\mathrm{GA}(2,1) \cong \mathbb{M}_2^2(\mathbb{R}) \tag{188}$$
$$\mathrm{GA}(1,2) \cong \mathbb{M}_2(\mathbb{C}) \tag{189}$$
$$\mathrm{GA}(0,3) \cong \mathbb{H}^2 \tag{190}$$
$$\mathrm{GA}(4,0) \cong \mathbb{M}_2(\mathbb{H}) \tag{191}$$
$$\mathrm{GA}(1,3) \cong \mathbb{M}_2(\mathbb{H}) \tag{192}$$
$$\mathrm{GA}(0,4) \cong \mathbb{M}_2(\mathbb{H}) \tag{193}$$
$$\mathrm{GA}(5,0) \cong \mathbb{M}_2^2(\mathbb{H}) \tag{194}$$

The determinant of these objects is valued in $\mathbb{C}$ or in $\mathbb{H}$, where $\mathbb{C}$ are the complex numbers, and where $\mathbb{H}$ are the quaternions. □

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 July 2024 doi:10.20944/preprints202404.1009.v5

Theorem 17 (Negative Probabilities in the RQM)**.** *The even sub-algebra, which associates to the RQM part of the theory, of these dimensional configurations allows for negative probabilities, making them unsuitable as a RQM.*

Proof. This category contains three dimensional configurations:

GA(1,0): Let $\psi = a + be_1$, then:

$$(a+be_1)^\ddagger(a+be_1) = (a-be_1)(a+be_1) = a^2 - b^2e_1e_1 = a^2 - b^2 \tag{195}$$

which is valued in $\mathbb{R}$.

GA(1,1): Let $\psi = a + be_0e_1$, then:

$$(a+be_0e_1)^\ddagger(a+be_0e_1) = (a-be_0e_1)(a+be_0e_1) = a^2 - b^2e_0e_1e_0e_1 = a^2 - b^2 \tag{196}$$

which is valued in $\mathbb{R}$.

GA(2,2): Let $\psi = a + be_0e_\varnothing e_1e_2$, where $e_0^2 = -1, e_\varnothing^2 = -1, e_1^2 = 1, e_2^2 = 1$, then:

$$\lfloor (a+\mathbf{b})^\ddagger(a+\mathbf{b})\rfloor_{3,4}(a+\mathbf{b})^\ddagger(a+\mathbf{b}) \tag{197}$$

$$= \lfloor a^2 + 2a\mathbf{b} + \mathbf{b}^2\rfloor_{3,4}(a^2 + 2a\mathbf{b} + \mathbf{b}^2) \tag{198}$$

We note that $\mathbf{b}^2 = b^2e_0e_\varnothing e_1e_2e_0e_\varnothing e_1e_2 = b^2$, therefore:

$$= (a^2 + b^2 - 2a\mathbf{b})(a^2 + b^2 + 2a\mathbf{b}) \tag{199}$$

$$= (a^2+b^2)^2 - 4a^2\mathbf{b}^2 \tag{200}$$

$$= (a^2+b^2)^2 - 4a^2b^2 \tag{201}$$

which is valued in $\mathbb{R}$.

In all of these cases the RQM probability can be negative. □

We repeat the following self-products[8] (Definition 16), which will help us demonstrate the next theorem:

$$\text{GA}(0,1): \quad \varphi^\dagger\varphi \tag{202}$$

$$\text{GA}(2,0): \quad \varphi^\ddagger\varphi \tag{203}$$

$$\text{GA}(3,0): \quad \lfloor\varphi^\ddagger\varphi\rfloor_3\varphi^\ddagger\varphi \tag{204}$$

$$\text{GA}(3,1): \quad \lfloor\varphi^\ddagger\varphi\rfloor_{3,4}\varphi^\ddagger\varphi \tag{205}$$

$$\text{GA}(4,1): \quad (\lfloor\varphi^\ddagger\varphi\rfloor_{3,4}\varphi^\ddagger\varphi)^\dagger(\lfloor\varphi^\ddagger\varphi\rfloor_{3,4}\varphi^\ddagger\varphi) \tag{206}$$

Theorem 18 (No Metric Measurements)**.** *This obstruction applies to* GA(2,0). *A probability measure of at least four self-products are required for the theory to be observationally complete with respect to its geometry.*

Proof. A metric measurement requires a probability measure of 4 self products because the metric tensor is defined using 2 self-products of the gamma matrices:

$$g_{\mu\nu} = \frac{1}{2}(\mathbf{e}_\mu\mathbf{e}_\nu + \mathbf{e}_\nu\mathbf{e}_\mu) \tag{207}$$

Each pair of wavefunction products fixes one basis elements. Thus, two pairs of wavefunction products are required to fix the geometry from the wavefunction. As probability measures of four self-products begin to appear in 3D, then the GA(2,0) cannot produce a metric measurement as a quantum observable, thus its geometry is not observationally complete with respect to its geometry. □

 doi:10.20944/preprints202404.1009.v5

Conjecture 1 (No probability measures as a self-product (in 6D))**.** *The multivector representation of the norm in 6D cannot satisfy any observables.*

Argument. In six dimensions and above, the self-product patterns found in Definition 16 collapse. The research by Acus et al.[12] in 6D geometric algebra demonstrates that the determinant, so far defined through a self-products of the multivector, fails to extend into 6D. The crux of the difficulty is evident in the reduced case of a 6D multivector containing only scalar and grade-4 elements:

$$s(B) = b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) \tag{208}$$

This equation is not a multivector self-product but a linear sum of two multivector self-products[12].

The full expression is given in the form of a system of 4 equations, which is too long to list in its entirety. A small characteristic part is shown:

$$a_0^4 - 2a_0^2 a_{47}^2 + b_2 a_0^2 a_{47}^2 p_{412} p_{422} + \langle 72 \text{ monomials} \rangle = 0 \tag{209}$$
$$b_1 a_0^3 a_{52} + 2b_2 a_0 a_{47}^2 a_{52} p_{412} p_{422} p_{432} p_{442} p_{452} + \langle 72 \text{ monomials} \rangle = 0 \tag{210}$$
$$\langle 74 \text{ monomials} \rangle = 0 \tag{211}$$
$$\langle 74 \text{ monomials} \rangle = 0 \tag{212}$$

From Equation 208, it is possible to see that no observable $\mathbf{O}$ can satisfy this equation because the linear combination does not allow one to factor it out of the equation.

$$b_1 \mathbf{O} B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) = b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 \mathbf{O} B g_5(g_4(B) g_3(g_2(B) g_1(B))) \tag{213}$$

Any equality of the above type between $b_1\mathbf{O}$ and $b_2\mathbf{O}$ is frustrated by the factors b_1 and b_2, forcing $\mathbf{O} = 1$ as the only satisfying observable. Since the obstruction occurs within grade-4, which is part of the even sub-algebra it is questionable that a satisfactory theory (with non-trivial observables) be constructible in 6D, suing this method. □

This conjecture proposes that the multivector representation of the determinant in 6D does not allow for the construction of non-trivial observables, which is a crucial requirement for a relevant quantum formalism. The linear combination of multivector self-products in the 6D expression prevents the factorization of observables, limiting their role to the identity operator.

Conjecture 2 (No probability measures as a self-product (above 6D))**.** *The norms beyond 6D are progressively more complex than the 6D case, which is already obstructed.*

These theorems and conjectures provide additional insights into the unique role of the unobstructed 3+1D signature in our proposal.

It is also interesting that our proposal is able to rule out $\text{GA}(1,3)$ even if in relativity, the signature of the metric $(+,-,-,-)$ versus $(-,-,-,+)$ does not influence the physics. However, in geometric algebra, $\text{GA}(1,3)$ represents 1 space dimension and 3 time dimensions. Therefore, it is not the signature itself that is ruled out but rather the specific arrangement of 3 time and 1 space dimensions, as this configuration yields quaternion-valued "probabilities" (i.e. $\text{GA}(1,3) \cong \mathbb{M}_2(\mathbb{H})$ and $\det \mathbb{M}_2(\mathbb{H}) \in \mathbb{H}$).

Consequently, 3+1D is the only dimensional configuration (other than the "non-geometric" configurations of $\text{GA}(0) \cong \mathbb{R}$ and $\text{GA}(0,1) \cong \mathbb{C}$) in which a 'least biased' solution to the problem of maximizing the Shannon entropy of quantum measurements relative to an initial preparation, exists. This is an extremely constraining result regarding the possible spacetime configurations of the universe, and our ability (or inability) to construct a least biased theory to investigate it.

3. Discussion

The principle of maximum entropy[3] states that the probability measure that best represents the current state of knowledge about a system is the one with the largest entropy, constrained by prior data. In QM, an experiment typically consists of three stages: an initial preparation, followed by some transformations, and concluding with a final measurement of the system, which yields the result of the experiment. Consistent with the maximum entropy principle, our aim is to derive the 'least biased' theory that connects the initial preparation p to its final measurement ρ. By formulating the theory as a solution to a maximization problem, rather than merely by axiomatic stipulation, we ensure (by mathematical proof) that the resulting framework is as unbiased as possible given the available information.

Using this methodology, fundamental physics can be formulated as the general solution to a maximization problem involving the Shannon entropy of all possible measurements of an arbitrary system relative to its initial preparation, subject to a vanishing phase constraint. The structure of the inferred theory is thus determined by the nature and generality of the employed constraint. In this paper, we have investigated three specific entropy maximization problems, each characterized by a different constraint and corresponding to a distinct level of description in physics:

Constraint	*Vanishing Phase*	*Inferred Theory*	*Wavefunction*
$\overline{E} = \sum_i \rho_i E_i$	none	SM	N.A.
$0 = \operatorname{tr} \sum_i \rho_i \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}$	$\mathrm{U}(1)$	QM	$\mathbb{C}^n$
$0 = \frac{1}{2} \operatorname{tr} \sum_i \rho_i \mathbf{M}_i$	$\mathrm{Spin}^c(3,1)$	RQM/QG	$(\mathbb{R} \times \mathrm{Spin}^c(3,1))^n$

where n represents the size of the ensemble.

Despite the differences in constraints, the three theories here-so formulated share a common logical genesis, adhere to the same principle of maximum entropy, and qualify as the least biased theory for their given constraint.

3.1. Guarantee of Epistemic Soundness

The Born rule is the least biased probability measure operating on a complex Hilbert space, as established in Theorem 2. However, when the framework is extended to include geometric considerations, the Born rule is no longer the least biased measure. Instead, as demonstrated in Theorem 3, the double-copy product emerges as the least biased probability measure in the 3+1D setting. Interestingly, no solutions exist for other geometric configurations, as demonstrated in Section 2.4.

3.2. Guarantee of Ontological Soundness

Our approach inverts the traditional theory construction paradigm: instead of postulating abstract mathematical entities like wavefunctions and Hilbert spaces, we derive them from the measurements—the theory's only constraint. Remarkably, the logical foundation of the theory is its ontology. This guarantees that the theory's foundation matches what transpired in the lab and also renders it inherently resistant to falsification by the very measurements used in its construction.

4. Conclusion

This paper presents a novel approach to quantum theory construction by solving a maximization problem on the Shannon entropy of all possible measurements of a system relative to its initial preparation, under the constraint of a vanishing phase. By selecting the appropriate group of the vanishing phase, the solution resolves to quantum mechanics, relativistic quantum mechanics, or a candidate for a theory of quantum gravity. The resulting measure is invariant under a wide range of geometric transformations, including those generated by the gauge groups of the Standard Model, and leads to the metric tensor as an operator involving a double copy of Dirac currents, without additional assumptions. Again without additional assumptions, the theory automatically fails in geometric configurations other than 3+1D. This result aligns with the observed dimensionality and gauge symmetries of the universe, suggesting a possible explanation for its specific structure. This approach may offer a promising avenue for the unification of fundamental physical theories and may provide new insights into the underlying principles governing the structure of our universe.

Statements and Declarations

- Competing Interests: The author declares that he has no competing financial or non-financial interests that are directly or indirectly related to the work submitted for publication.
- Data Availability Statement: No datasets were generated or analyzed during the current study.
- During the preparation of this manuscript, we utilized a Large Language Model (LLM), for assistance with spelling and grammar corrections, as well as for minor improvements to the text to enhance clarity and readability. This AI tool did not contribute to the conceptual development of the work, data analysis, interpretation of results, or the decision-making process in the research. Its use was limited to language editing and minor textual enhancements to ensure the manuscript met the required linguistic standards.

Appendix A. SM

Here, we solve the Lagrange multiplier equation of SM.

$$\mathcal{L} = \underbrace{-k_B \sum_i \rho_i \ln \rho_i}_{\text{Boltzmann Entropy}} + \underbrace{\lambda \left(1 - \sum_i \rho_i\right)}_{\text{Normalization Constraint}} + \underbrace{\beta \left(\overline{E} - \sum_i \rho_i E_i\right)}_{\text{Average Energy Constraint}} \tag{A1}$$

We solve the maximization problem as follows:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}(\rho_i, \ldots, \rho_n)}{\partial \rho_i} && \text{(A2)} \\ &= -\ln \rho_i - 1 - \lambda - \beta E_i && \text{(A3)} \\ &= \ln \rho_i + 1 + \lambda + \beta E_i && \text{(A4)} \\ \implies \ln \rho_i &= -1 - \lambda - \beta E_i && \text{(A5)} \\ \implies \rho_i &= \exp(-1-\lambda) \exp(-\beta E_i) && \text{(A6)} \\ &= \frac{1}{Z(\tau)} \exp(-\beta E_i) && \text{(A7)} \end{aligned}$$

The partition function, is obtained as follows:

$$\begin{aligned} 1 &= \sum_i \exp(-1-\lambda) \exp(-\beta E_i) && \text{(A8)} \\ \implies (\exp(-1-\lambda))^{-1} &= \sum_i \exp(-\beta E_i) && \text{(A9)} \\ Z(\tau) &:= \sum_i \exp(-\beta E_i) && \text{(A10)} \end{aligned}$$

Finally, the probability measure is:

$$\rho_i = \frac{1}{\sum_i \exp(-\beta E_i)} \exp(-\beta E_i) \tag{A11}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 July 2024 doi:10.20944/preprints202404.1009.v5

Appendix B. RQM in 3+1D

$$\mathcal{L} = \underbrace{-\sum_i \rho_i \ln \frac{\rho_i}{p_i}}_{\text{Relative Shannon Entropy}} + \underbrace{\lambda\left(1 - \sum_i \rho_i\right)}_{\text{Normalization Constraint}} + \underbrace{\zeta\left(-\text{tr}\frac{1}{2}\sum_i \rho_i \mathbf{M}_i\right)}_{\text{Vanishing Relativistic-Phase Anti-Constraint}} \tag{A12}$$

The solution is obtained using the same step-by-step process as the 2D case, and yields:

$$\rho_i = \underbrace{\frac{1}{\sum_i p_i \det\exp(-\zeta\frac{1}{2}\mathbf{M}_i)}}_{\text{Spin}^c(3,1)\text{ Invariant Ensemble}} \underbrace{\det\exp(-\zeta\frac{1}{2}\mathbf{M}_i)}_{\text{Spin}^c(3,1)\text{ Born Rule}} \underbrace{p_i}_{\text{Initial Preparation}} \tag{A13}$$

Proof. The Lagrange multiplier equation can be solved as follows:

$$0 = \frac{\partial \mathcal{L}(\rho_1, \ldots, \rho_n)}{\partial \rho_i} \tag{A14}$$

$$= -\ln\frac{\rho_i}{p_i} - p_i - \lambda - \zeta\,\text{tr}\,\frac{1}{2}\mathbf{M}_i \tag{A15}$$

$$= \ln\frac{\rho_i}{p_i} + p_i + \lambda + \zeta\,\text{tr}\,\frac{1}{2}\mathbf{M}_i \tag{A16}$$

$$\implies \ln\frac{\rho_i}{p_i} = -p_i - \lambda - \zeta\,\text{tr}\,\frac{1}{2}\mathbf{M}_i \tag{A17}$$

$$\implies \rho_i = p_i \exp(-p_i - \lambda)\exp\left(-\zeta\,\text{tr}\,\frac{1}{2}\mathbf{M}_i\right) \tag{A18}$$

$$= \frac{1}{Z(\zeta)} p_i \exp\left(-\zeta\,\text{tr}\,\frac{1}{2}\mathbf{M}_i\right) \tag{A19}$$

The partition function $Z(\zeta)$, serving as a normalization constant, is determined as follows:

$$1 = \sum_i p_i \exp(-p_i - \lambda)\exp\left(-\zeta\,\text{tr}\,\frac{1}{2}\mathbf{M}_i\right) \tag{A20}$$

$$\implies (\exp(-p_i - \lambda))^{-1} = \sum_i p_i \exp\left(-\zeta\,\text{tr}\,\frac{1}{2}\mathbf{M}_i\right) \tag{A21}$$

$$Z(\zeta) := \sum_i p_i \exp\left(-\zeta\,\text{tr}\,\frac{1}{2}\mathbf{M}_i\right) \tag{A22}$$

□

Appendix C. SageMath Program Showing $\lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u} = \det \mathbf{M}_\mathbf{u}$

```
from sage.algebras.clifford_algebra import CliffordAlgebra
from sage.quadratic_forms.quadratic_form import QuadraticForm
from sage.symbolic.ring import SR
from sage.matrix.constructor import Matrix

# Define the quadratic form for GA(3,1) over the Symbolic Ring
Q = QuadraticForm(SR, 4, [-1, 0, 0, 0, 1, 0, 0, 1, 0, 1])

# Initialize the GA(3,1) algebra over the Symbolic Ring
algebra = CliffordAlgebra(Q)
```

 doi:10.20944/preprints202404.1009.v5


```
# Define the basis vectors
e0, e1, e2, e3 = algebra.gens()

# Define the scalar variables for each basis element
a = var('a')
t, x, y, z = var('t x y z')
f01, f02, f03, f12, f23, f13 = var('f01 f02 f03 f12 f23 f13')
v, w, q, p = var('v w q p')
b = var('b')

# Create a general multivector
udegree0=a
udegree1=t*e0+x*e1+y*e2+z*e3
udegree2=f01*e0*e1+f02*e0*e2+f03*e0*e3+f12*e1*e2+f13*e1*e3+f23*e2*e3
udegree3=v*e0*e1*e2+w*e0*e1*e3+q*e0*e2*e3+p*e1*e2*e3
udegree4=b*e0*e1*e2*e3
u=udegree0+udegree1+udegree2+udegree3+udegree4

u2 = u.clifford_conjugate()*u

u2degree0 = sum(x for x in u2.terms() if x.degree() == 0)
u2degree1 = sum(x for x in u2.terms() if x.degree() == 1)
u2degree2 = sum(x for x in u2.terms() if x.degree() == 2)
u2degree3 = sum(x for x in u2.terms() if x.degree() == 3)
u2degree4 = sum(x for x in u2.terms() if x.degree() == 4)
u2conj34 = u2degree0+u2degree1+u2degree2-u2degree3-u2degree4


I = Matrix(SR, [[1, 0, 0, 0],
                [0, 1, 0, 0],
                [0, 0, 1, 0],
                [0, 0, 0, 1]])

#MAJORANA MATRICES
y0 = Matrix(SR, [[0, 0, 0, 1],
                 [0, 0, -1, 0],
                 [0, 1, 0, 0],
                 [-1, 0, 0, 0]])

y1 = Matrix(SR, [[0, -1, 0, 0],
                 [-1, 0, 0, 0],
                 [0, 0, 0, -1],
                 [0, 0, -1, 0]])

y2 = Matrix(SR, [[0, 0, 0, 1],
                 [0, 0, -1, 0],
                 [0, -1, 0, 0],
                 [1, 0, 0, 0]])
```

```
y3 = Matrix(SR, [[-1, 0, 0, 0],
                 [0, 1, 0, 0],
                 [0, 0, -1, 0],
                 [0, 0, 0, 1]])

mdegree0 = a
mdegree1 = t*y0+x*y1+y*y2+z*y3
mdegree2 = f01*y0*y1+f02*y0*y2+f03*y0*y3+f12*y1*y2+f13*y1*y3+f23*y2*y3
mdegree3 = v*y0*y1*y2+w*y0*y1*y3+q*y0*y2*y3+p*y1*y2*y3
mdegree4 = b*y0*y1*y2*y3
m=mdegree0+mdegree1+mdegree2+mdegree3+mdegree4

print(u2conj34*u2 == m.det())
```

The program outputs

```
True
```

showing, by computer assisted symbolic manipulations, that the determinant of the real Majorana representation of a multivector u is equal to the double-copy form: $\det \mathbf{M_u} = \lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u}$.

References

1. Dirac, P.A.M. *The principles of quantum mechanics*; Number 27, Oxford university press, 1981.
2. Von Neumann, J. *Mathematical foundations of quantum mechanics: New edition*; Vol. 53, Princeton university press, 2018.
3. Jaynes, E.T. Information theory and statistical mechanics. *Physical review* **1957**, *106*, 620.
4. Jaynes, E.T. Information theory and statistical mechanics. II. *Physical review* **1957**, *108*, 171.
5. Kullback, S.; Leibler, R.A. On information and sufficiency. *The annals of mathematical statistics* **1951**, *22*, 79–86.
6. Shannon, C.E. A mathematical theory of communication. *Bell system technical journal* **1948**, *27*, 379–423.
7. Hestenes, D. Spacetime physics with geometric algebra. *American Journal of Physics* **2003**, *71*, 691–714.
8. Lundholm, D. Geometric (Clifford) algebra and its applications. *arXiv preprint math/0605280* **2006**.
9. Hestenes, D. Space-time structure of weak and electromagnetic interactions. *Foundations of Physics* **1982**, *12*, 153–168.
10. Lasenby, A. Some recent results for $SU(3)$ and Octonions within the Geometric Algebra approach to the fundamental forces of nature. *arXiv preprint arXiv:2202.06733* **2022**.
11. Bern, Z.; Carrasco, J.J.M.; Johansson, H. Perturbative quantum gravity as a double copy of gauge theory. *Physical Review Letters* **2010**, *105*, 061602.
12. Acus, A.; Dargys, A. Inverse of multivector: Beyond p+ q= 5 threshold. *arXiv preprint arXiv:1712.05204* **2017**.