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Article

Center and Hopf Bifurcation of High-order Singularity in a Class of Three-dimensional Systems

Jingping Lu^{1,2,3}, Wentao Huang^{1,4} and Qinlong Wang^{1,*}

¹ School of Computing Science and Mathematics, Guilin University of Electronic Technology, Guilin 541004, China

² School of Computer Science and Information Security, Guilin University of Electronic Technology, Guilin 541004, China

³ School of Mathematics and Statistics, Guilin University of Technology, Guilin 541004, China

⁴ School of Mathematics and Statistics, Guangxi Normal University, Guilin 541004, China

* Correspondence: wqinlong@163.com

Abstract: This research focuses on the Hopf bifurcation occurring at a singular point with high-order degeneracy in a class of three-dimensional systems. Based on center manifold theorem, by expanding the technique for determining singular point values of degenerate singularities in two-dimensional systems, the formal series method to determine the singular point values at the high-order degenerate critical point is discussed. Furthermore, a class of specific three-dimensional differential systems with high-order degenerate singularities is explored. By computing the singular point values and determining center conditions, we proved that a minimum of 5 small-amplitude limit cycles in the vicinity of the origin. This research provides new perspectives on the center problem and limit cycle bifurcation in high-dimensional systems with high-order degenerate singular points.

Keywords: high-order degenerate singularity; singular point value; limit cycle; center condition

MSC: 34C05; 34C07

1. Introduction

Higher-order singularities typically exhibit more complex topological features compared to elementary singularities, making them challenging to analyze. Hodaie and his coauthors [11] highlighted the widespread occurrence of degenerate states in various physical contexts due to underlying symmetries. Therefore, their study revealed that higher-order degenerate singular point bifurcations can enhance the sensitivity of sensors. This paper aims to investigate the center condition and Hopf cyclicity of the high-order degenerate singularity in a class of three-dimensional (3-d) systems.

Consider a system defined by the equation

$$\dot{\mathbf{x}} = B\mathbf{x} + \mathbf{g}(\mathbf{x}) \quad (1)$$

here, $B \in \mathbb{R}^{3 \times 3}$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, and $\mathbf{g}(\mathbf{x})$ is a smooth function with $\mathbf{g}(\mathbf{0}) = \mathbf{0}$, $D\mathbf{g}(\mathbf{0}) = \mathbf{0}$. Then, it is evident that the origin is an equilibrium point of this system. When analyzing the Hopf bifurcation of this system, it is noted that if the Jacobi matrix of system (1) at the equilibrium point contains a pair of pure imaginary eigenvalues and one negative eigenvalue, the complexity of the analysis increases. While some relevant results can be found in reference [9,10,14,28], the above conclusions mainly focus on the cyclicity of non-degenerate singularities. Nevertheless, there is a lack of published references that investigate the cyclicity of Hopf singularity with high-order degeneracy in systems (1).

The bifurcation of limit cycles at a degenerate Hopf singularity for a 2-d system involves differentiating between a center or a focus. This task is generally more challenging compared to the non-degenerate case. Existing research in this area is limited, with some studies focusing on the Hopf bifurcation [15,19,21,22] and the center problem [1,7,8,23,25]. Recently, Li and his collaborators investigated the bifurcation of limit cycles and center problem in degenerate singularities for 2-d non-smooth systems [4,24]. Studies on the cyclicity of degenerate Hopf singularities in 3-d systems are scarce. In 2008, Llibre and Wu [20] conducted a study on the bifurcation for a specific type of degenerate Hopf singularity with a multiplicity of $2n - 1$ in a 3-d system using averaging theory. Wang

et al.[26] investigated the cyclicity of a nilpotent singularity in a group of 3-d systems. The research demonstrated that a minimum of four limit cycles emerge from the nilpotent singularity. More recently, Du et al.[5,6] explored the maximum number of limit cycles arising from a degenerate singularity in the following 3-d systems:

$$\begin{cases} \frac{dx}{dt} = (\delta x - y)(x^2 + y^2)^m + \sum_{i+j+k=2m+2}^{\infty} A_{ijk}x^i y^j u^k, \\ \frac{dy}{dt} = (x - \delta y)(x^2 + y^2)^m + \sum_{i+j+k=2m+2}^{\infty} B_{ijk}x^i y^j u^k, \\ \frac{du}{dt} = -du(x^2 + y^2)^m + \sum_{i+j+k=2m+2}^{\infty} d_{ijk}x^i y^j u^k. \end{cases} \quad (2)$$

Two appropriate transformations were made on system (2) to change the singularity of system (2) into a non-degenerate state. The singular values method proposed in article [27] was then applied to analyze the Hopf bifurcation of the singularity. However, research results on the center problem of the degenerate singularity in three-dimensional systems are very rare.

Specifically, it is important to highlight the research conducted by [16], where formal series approach was proposed for calculating the singular point values of the degenerate singularity in planar differential systems. This method not only allows for the investigation of higher degree systems but also enables the discovery of more limit cycles, as demonstrated in previous works [3,12]. A significant advantage of this method is its ease of implementation with computational algebra software like Mathematica. Building upon this, our goal is to extend this methodology to a 3-d system (3), thereby amplifying its practicality and significance.

In this study, we aim to explore the center problem and Hopf cyclicity for the following real 3-d systems

$$\begin{cases} \frac{dx}{dt} = (\delta x - y)(x^2 + y^2)^n + \sum_{i+j+k=2n+2}^{\infty} A_{ijk}x^i y^j u^k, \\ \frac{dy}{dt} = (x - \delta y)(x^2 + y^2)^n + \sum_{i+j+k=2n+2}^{\infty} B_{ijk}x^i y^j u^k, \\ \frac{du}{dt} = -du + \sum_{i+j+k=2}^{\infty} d_{kjl}x^i y^j u^k, \end{cases} \quad (3)$$

where $x, y, u, t, d, A_{ijk}, B_{ijk}, d_{ijk}$ are all real numbers, with d and n being positive, and i, j, k, n belonging to the set of natural numbers. The origin of system (3) exhibits characteristics of a high-order degenerate singularity. This is evident as the minimum degree of system (3) is 4, and the Jacobian matrix of system (3) at the origin contains two zero eigenvalues and one negative eigenvalue. However, it cannot be transformed into a non-degenerate singular point through two appropriate transformations.

The following structure is presented in the paper. Section 2 introduces the fundamental approach, as described in references [3,16] to compute the singular point values of the degenerate singularity in two-dimensional systems. Furthermore, we devise a calculation method utilizing formal series and center manifold theorem to determine the focal values of the high-order degenerate singular point in systems (3). In section 3, an illustrative example is studied, where the center conditions on a center manifold are derived, and the presence of five small amplitude limit cycles are determined.

2. Calculation Method for Singular Point Values

To analyze the computational approach for determining the focal values in the 3-d system (3), the first step involves applying the center manifold theorem to reduce the dimension to two. According to the center manifold theorem [2], the center manifold can be approximated as

$$u = u(x, y) = u_2(x, y) + \text{h.o.t.}, \quad (4)$$

where u_2 represents a quadratic homogeneous polynomial of the variables x and y , and h.o.t indicates a higher-order term exceeding 2. By substituting $u = u(x, y)$ into the first two equations of system (3), a two-dimensional system is derived

$$\begin{cases} \frac{dx}{dt} = (\delta x - y)(x^2 + y^2)^n + \sum_{l=2n+2}^{\infty} X_l(x, y), \\ \frac{dy}{dt} = (x - \delta y)(x^2 + y^2)^n + \sum_{l=2n+2}^{\infty} Y_l(x, y). \end{cases} \quad (5)$$

Here, $X_l(x, y)$ and $Y_l(x, y)$ represent l -th degree homogeneous polynomials. It is important to note that the origin of this system is degenerate, as it has two zero eigenvalues.

In order to better carry out our research efforts, it is imperative to review the relevant definitions and results provided in references [3,16].

Definition 1 (see [16]). For system (5) $_{\delta=0}$ with $n > 0$, if $v_1(2\pi) \neq 1$, then the origin is referred to as degenerate rough focus; if $v_1(2\pi) = 1$, and

$$v_2(2\pi) = v_3(2\pi) = \dots = v_{2k}(2\pi) = 0, \quad v_{2k+1}(2\pi) \neq 0, \quad (6)$$

then the origin is known as the degenerate weak focus of k -order, and the value of $v_{2k+1}(2\pi)$ is termed the k -th generalized focal value of the origin, where $k = 1, 2, \dots$; if $v_1(2\pi) = 1$ and $v_{2k+1}(2\pi) = 0$ for any positive integer k , then the origin is classified as a center.

By transformation

$$z = x + yi, \quad w = x - yi, \quad T = it, \quad i = \sqrt{-1} \quad (7)$$

system (5) $_{\delta=0}$ can be reformulated as

$$\begin{cases} \frac{dz}{dT} = z^{n+1}w^n + \sum_{j+k=2n+2}^{\infty} a_{jk}z^jw^k = Z(z, w), \\ \frac{dw}{dT} = -z^nw^{n+1} - \sum_{j+k=2n+2}^{\infty} b_{jk}z^jw^k = -W(z, w). \end{cases} \quad (8)$$

In system (8), z, w, T are complex variables. Furthermore, it is noted that $a_{jk} = \overline{b_{jk}}$ for any positive integers j, k . Hence, system (5) and (8) are referred to as concomitant (see [3,16]).

Lemma 1 ([3,16]). For system (8), the terms of the formal series (9) can be derived successively as follows:

$$F(z, w) = zw \left[1 + \sum_{m=1}^{\infty} \frac{f_{m(2n+3)}(z, w)}{(zw)^{m(n+1)}} \right] \quad (9)$$

in a way that satisfies

$$\frac{dF}{dT} = \frac{\partial F}{\partial z} Z - \frac{\partial F}{\partial w} W = (zw)^n \sum_{m=1}^{\infty} \mu_m (zw)^{m+1}. \quad (10)$$

Here, the function

$$f_k(z, w) = \sum_{\alpha+\beta=k} c_{\alpha\beta} z^\alpha w^\beta$$

is defined as a k -th degree homogeneous polynomial, where $c_{00} = 1, c_{kk} = 0, k = 1, 2, \dots$.

Definition 2 ([3,16]). For system (8), the μ_m is called the m -th singular point values at the degenerate critical point, $m = 1, 2, \dots$.

Lemma 2 ([17]). By setting $\delta = 0$ in system (5), it can be asserted that the relationship $v_{2m+1}(2\pi) \sim i\pi\mu_m$ holds for any positive integer m , specifically

$$v_{2m+1}(2\pi) = i\pi(\mu_m + \sum_{k=1}^{m-1} \zeta_m^{(k)} \mu_k), \quad (11)$$

where $\zeta_m^{(k)}$ ($k = 1, 2, \dots, m-1$) are polynomial functions of the coefficients of system (8). This establishes an algebraic equivalence between $v_{2m+1}(2\pi)$ and μ_m .

Remark 1. For the flow on center manifold of system (3), the values μ_m and $v_{2m+1}(2\pi)$ are also the m -th singular point value and focal value of the origin. For any positive integer $m = 2, 3, \dots$, if the conditions $\mu_1 = \mu_2 = \dots = \mu_{m-1} = 0$ and $v_1(2\pi) = v_3(2\pi) \dots = v_{2m-1}(2\pi) = 0$ are satisfied, then $v_{2m+1}(2\pi) = i\pi\mu_m$. Moreover, through computation of the singular point values, one can determine the bifurcation of the equilibrium in system (3).

3. Center Conditions and Hopf Bifurcation of a 3-d system

In this section, we will analyze the center conditions and cyclicity of the Hopf singularity for a three-dimensional system, which is a specific case of system (3).

3.1. Singular Point Values and Center Conditions

Consider the system

$$\begin{cases} \frac{dx}{dt} = (\delta x - y)(x^2 + y^2) + u(A_{30}x^3 + A_{21}x^2y + A_{12}xy^2 + A_{03}y^3), \\ \frac{dy}{dt} = (x - \delta y)(x^2 + y^2) + u(B_{30}x^3 + B_{21}x^2y + B_{12}xy^2 + B_{03}y^3), \\ \frac{du}{dt} = -u - d_1(x^2 + y^2), \end{cases} \quad (12)$$

here, $A_{30}, A_{21}, A_{12}, A_{03}, B_{30}, B_{21}, B_{12}, B_{03}, d_1$ are nine real parameters and $d_1 \neq 0$. Through transformation (7), system (12)| $_{\delta=0}$ can be converted into its concomitant form

$$\begin{cases} \frac{dz}{dT} = z^2w + u(a_{30}z^3 + a_{21}z^2w + a_{12}zw^2 + a_{03}w^3) = Z, \\ \frac{dw}{dT} = -zw^2 - u(b_{30}w^3 + b_{21}w^2z + b_{12}wz^2 + b_{03}z^3) = -W, \\ \frac{du}{dT} = iu + id_1zw = U. \end{cases} \quad (13)$$

where

$$\begin{aligned} a_{30} &= \frac{1}{8}(A_{03} + iA_{12} - A_{21} - iA_{30} + iB_{03} - B_{12} - iB_{21} + B_{30}), \\ a_{21} &= \frac{1}{8}(-3A_{03} - iA_{12} - A_{21} - 3iA_{30} - 3iB_{03} + B_{12} - iB_{21} + 3B_{30}), \\ a_{12} &= \frac{1}{8}(3A_{03} - iA_{12} + A_{21} - 3iA_{30} + 3iB_{03} + B_{12} + iB_{21} + 3B_{30}), \\ a_{03} &= \frac{1}{8}(-A_{03} + iA_{12} + A_{21} - iA_{30} - iB_{03} - B_{12} - iB_{21} + B_{30}), \\ b_{ij} &= \bar{a}_{ij}, (ij = 30, 21, 12, 03). \end{aligned}$$

Furthermore, the center manifold of system (13) can be determined in the form of expression (4): $u = u(x, y) = \tilde{u}(z, w)$ under the transformation (7), leading to the system

$$\begin{cases} \frac{dz}{dT} = z^2w + \tilde{u}(a_{30}z^3 + a_{21}z^2w + a_{12}zw^2 + a_{03}w^3) = \tilde{Z}, \\ \frac{dw}{dT} = -zw^2 - \tilde{u}(b_{30}w^3 + b_{21}w^2z + b_{12}wz^2 + b_{03}z^3) = -\tilde{W}. \end{cases} \quad (14)$$

Remark 2. For system (14), according to Lemma 1, when the corresponding $n = 1$ in (9) and (10), it can be concluded that every μ_m is only dependent on the coefficients of the first $2m + 3$ order terms in the system (14), where $m = 1, 2, \dots$. Therefore, in system (14), the determination of \tilde{u} only requires accuracy up to the $2m$ order

term. Considering the large computational complexity, we plan to calculate only the first 5 singular point values at the origin of the system (14), which means \tilde{u} will be determined up to the 10th order term

$$\tilde{u}(z, w) = \sum_{k=2}^{10} \tilde{u}_k(z, w). \quad (15)$$

Here, the \tilde{u}_k represent k -th degree homogeneous polynomials of the variables z and w , with their expressions detailed below

$$\begin{aligned} \tilde{u}_2 &= -d_1zw, \quad \tilde{u}_3 = \tilde{u}_4 = \tilde{u}_5 = 0, \\ \tilde{u}_6 &= -\mathbf{i}d_1wz(a_{03}d_1w^4 + (a_{12} - b_{30})d_1w^3z + (a_{21}d_1 - b_{21}d_1 - 8\mathbf{i}\delta^2)w^2z^2 \\ &\quad + (a_{30} - b_{12})d_1wz^3 - b_{03}d_1z^4), \\ \tilde{u}_7 &= 0, \quad \tilde{u}_8 = 4b_{03}d_1^2z^6w^2 - 2(a_{30} - b_{12})d_1^2z^5w^3 + 2(a_{12} - b_{30})d_1^2z^3w^5 + 4a_{03}d_1^2z^2w^6, \\ \tilde{u}_9 &= 0, \\ \tilde{u}_{10} &= 2a_{03}^2d_1^3w^9z + a_{03}(5a_{12} - 9b_{30})d_1^3w^8z^2 + d_1^2(16\mathbf{i}a_{03} + 3a_{12}^2d_1 + 6a_{03}a_{21}d_1 \\ &\quad - 10a_{03}b_{21}d_1 - 8a_{12}b_{30}d_1 + 5b_{30}^2d_1)w^7z^3 + d_1^2(4\mathbf{i}a_{12} - 4\mathbf{i}b_{30} + 7a_{12}a_{21}d_1 \\ &\quad + 7a_{03}a_{30}d_1 - 11a_{03}b_{12}d_1 - 9a_{12}b_{21}d_1 - 7a_{21}b_{30}d_1 + 9b_{21}b_{30}d_1)w^6z^4 \\ &\quad - 2(-2a_{21}^2 - 4a_{12}a_{30} + 6a_{03}b_{03} + 5a_{12}b_{12} + 4a_{21}b_{21} - 2b_{21}^2 + 3a_{30}b_{30} - \\ &\quad 4b_{12}b_{30})d_1^3w^5z^5 + d_1^2(4\mathbf{i}a_{30} - 4\mathbf{i}b_{12} + 9a_{21}a_{30}d_1 - 11a_{12}b_{03}d_1 - 9a_{21}b_{12}d_1 \\ &\quad - 7a_{30}b_{21}d_1 + 7b_{12}b_{21}d_1 + 7b_{03}b_{30}d_1)w^4z^6 - d_1^2(16\mathbf{i}b_{03} - 5a_{30}^2d_1 + 10a_{21}b_{03}d_1 \\ &\quad + 8a_{30}b_{12}d_1 - 3b_{12}^2d_1 - 6b_{03}b_{21}d_1)w^3z^7 - b_{03}(9a_{30} - 5b_{12})d_1^3w^2z^8 + 2b_{03}^2d_1^3wz^9. \end{aligned}$$

Therefore, \tilde{Z} and \tilde{W} within system (14) correspond to two polynomials of degree 13.

According to Lemma 1 and Remark 2 in conjunction with the software Mathematica, we are able to determine the first singular point value at the origin of system (13) when restricted to the center manifold, which is expressed as:

$$\mu_1 = d_1(b_{21} - a_{21}),$$

Since $d_1 \neq 0$, we deduce from $\mu_1 = 0$ that $a_{21} = b_{21}$. Subsequently, we determine

$$\mu_2 = d_1^2(b_{30}b_{12} - a_{30}a_{12}).$$

Further analysis is conducted for $\mu_2 = 0$, followed by the computation of subsequent singular point values.

Case 1: If $a_{30} = b_{30} = 0$, then

$$\mu_3 = \frac{1}{8}d_1^2[16\mathbf{i}(a_{03}b_{03} + a_{12}b_{12}) - 3(b_{03}a_{12}^2 - a_{03}b_{12}^2)d_1].$$

Based on $d_1 \neq 0$, subsequent singular point values will be calculated in two different cases.

Case 1.1: If $b_{03}a_{12}^2 - a_{03}b_{12}^2 = 0$, then $\mu_3 = 0$ implies $a_{03}b_{03} + a_{12}b_{12} = 0$, indicating $a_{12} = b_{12} = a_{03} = b_{03} = 0$, resulting in

$$\mu_4 = \mu_5 = 0.$$

Case 1.2: If $b_{03}a_{12}^2 - a_{03}b_{12}^2 \neq 0$, from $\mu_3 = 0$, we get

$$d_1 = \frac{16\mathbf{i}(a_{03}b_{03} + a_{12}b_{12})}{3(b_{03}a_{12}^2 - a_{03}b_{12}^2)},$$

which implies $a_{03}b_{03} + a_{12}b_{12} \neq 0$. Further computation results in

$$\mu_4 = \frac{2048(a_{03}b_{03} + a_{12}b_{12})^3}{81(b_{03}a_{12}^2 - a_{03}b_{12}^2)^3}M_1,$$

where

$$M_1 = 15(b_{03}a_{12}^2 + a_{03}b_{12}^2) - 16(a_{03}b_{03} + a_{12}b_{12})b_{21}.$$

Letting $\mu_4 = 0$, we find that $M_1 = 0$, leading to

$$b_{21} = \frac{15(b_{03}a_{12}^2 + a_{03}b_{12}^2)}{16(a_{03}b_{03} + a_{12}b_{12})}.$$

Subsequent calculation yields

$$\mu_5 = \frac{512i(a_{03}b_{03} + a_{12}b_{12})^2}{729(b_{03}a_{12}^2 - a_{03}b_{12}^2)^4} M_2,$$

where

$$M_2 = 2619a_{03}a_{12}^4b_{03}^3 - 3328a_{03}^4b_{03}^4 + 1647a_{12}^5b_{03}^2b_{12} - 320a_{03}^3a_{12}b_{03}^3b_{12} + 3846a_{03}^2a_{12}^2b_{03}^2b_{12}^2 + 1982a_{03}a_{12}^3b_{03}b_{12}^3 - 800a_{12}^4b_{12}^4 + 2619a_{03}^3b_{03}b_{12}^4 + 1647a_{03}^2a_{12}b_{12}^5.$$

Case 2: If $a_{30}b_{30} \neq 0$, we set $a_{12} = sb_{30}$, $b_{12} = sa_{30}$, then

$$\mu_3 = \frac{1}{8}d_1^2F_0,$$

where

$$F_0 = 16iF_{01} + F_{02}d_1, \\ F_{01} = a_{03}b_{03} + a_{30}b_{30}(1-s)^2, \quad F_{02} = (a_{03}a_{30}^2 - b_{03}b_{30}^2)(s-3)(1+3s).$$

Similar to Case 1, we will continue to calculate the subsequent singular point values in two different cases.

Case 2.1: If $F_{02} = 0$, then $\mu_3 = 0$ implies $F_{01} = 0$. This leads to $a_{03} = b_{03} = 0$, $s = 1$, meaning $a_{03} = b_{03} = 0$, $a_{12} = b_{30}$, $b_{12} = a_{30}$, and we find

$$\mu_4 = \mu_5 = 0.$$

Case 2.2: If $F_{02} \neq 0$, from $\mu_3 = 0$, we derive

$$d_1 = -\frac{16iF_{01}}{F_{02}},$$

which implies $F_{01} \neq 0$. Further calculations lead to

$$\mu_4 = \frac{2048F_{01}^3}{F_{02}^3(1+3s)}F_1,$$

where

$$F_1 = -(a_{03}a_{30}^2 + b_{03}b_{30}^2)(s-1)(1+3s)(3+5s) + 16b_{21}sF_{01}.$$

Setting $\mu_4 = 0$, we find that $F_1 = 0$, resulting in

$$b_{21} = \frac{(a_{03}a_{30}^2 + b_{03}b_{30}^2)(s-1)(1+3s)(3+5s)}{16sF_{01}}.$$

Finally, computing yields

$$\mu_5 = \frac{512iF_{01}^2}{3F_{02}s^2(s-3)^3(1+3s)^4}F_2,$$

where

$$F_2 = -256a_{03}^4 b_{03}^4 s^2(3 + 13s) + a_{30} b_{30}^4 (-32a_{30}^3 (s - 1)^6 s^2(3 + s)(25s^2 - 1) + 3b_{30}(s - 3)(b_{03} + b_{03}(2 - 3s)s)^2(s(4 + s(54 + s(61s - 148))) - 3)) + a_{03}^3 (-64a_{30} b_{03}^3 b_{30} s^2(27 + s(87 + 5(s - 43)s)) + 3a_{30}^4 b_{03}(s - 3)(1 + 3s)^2(s(4 + s(18 + s(97s - 244))) - 3)) + a_{03} b_{03} b_{30}^3 (3b_{30}(s - 3)(b_{03} + 3b_{03}s)^2(s(4 + s(18 + s(97s - 244))) - 3) + 2a_{30}^3 (s - 1)^2(27 + s(117 + s(363 + s(s(s(417 + 991s) - 2121) - 2599) - 267)))) + a_{03}^2 a_{30}^2 b_{30} (3a_{30}^3 (s - 3)(1 + (2 - 3s)s)^2(s(4 + s(54 + s(61s - 148))) - 3) + 2b_{03}^2 b_{30} (27 + s(117 + s(s(s(s(6605 + 1923s) - 15633) - 1023) - 3815) - 489)))).$$

For each μ_k that was calculated, we have already let $\mu_1 = \dots = \mu_{k-1} = 0$, $k = 2, 3, \dots, 5$.

In the following, we focus on the center problem of system (13). Analyzing the Case 1.1 and Case 2.1 in the calculation process of the singular point values, we establish the following lemma.

Lemma 3. For the flow on the center manifold of system (13), the simultaneous vanishing of the first five singular point values of the origin can be achieved by the following two sets of conditions:

$$K_1 : a_{21} = b_{21}, \quad a_{30} = b_{30} = a_{12} = b_{12} = a_{03} = b_{03} = 0; \quad (16)$$

$$K_2 : a_{30} b_{30} \neq 0, \quad a_{21} = b_{21}, \quad a_{03} = b_{03} = 0, \quad a_{12} = b_{30}, \quad b_{12} = a_{30}. \quad (17)$$

Obviously, conditions K_1 and K_2 can be merged into a single condition

$$K : a_{21} = b_{21}, \quad a_{03} = b_{03} = 0, \quad a_{12} = b_{30}, \quad b_{12} = a_{30}. \quad (18)$$

Furthermore, it will be shown K in (18) is a center condition of system (13) restricted to the center manifold. Subsequently, a corresponding theorem will be presented.

Theorem 1. If the condition K in equation (18) is satisfied, then the origin of system (13), i.e., the origin of its corresponding real system (12) with $\delta = 0$ is a center on the local center manifold.

Proof. When the condition K is met, the system (13) can be expressed as

$$\begin{cases} \frac{dz}{dT} = z^2 w + u(a_{30} z^3 + b_{21} z^2 w + b_{30} z w^2), \\ \frac{dw}{dT} = -z w^2 - u(b_{30} w^3 + b_{21} w^2 z + a_{30} w z^2), \\ \frac{du}{dT} = \mathbf{i}u + \mathbf{i}d_1 z w = U. \end{cases} \quad (19)$$

It can be shown that the first two equations of system (19) have a first integral

$$H = zw = c_0. \quad (20)$$

By substituting (20) into the third equation of the system (19), a general solution $u = c_1 e^{\mathbf{i}T} - c_0 d_1$ or $u = c_1 e^{-t} - c_0 d_1$ can be derived, where c_0 and c_1 are integral constants. This indicates that system (19) is integrable and its origin is a center. The proof of Theorem 1 is complete. \square

3.2. Hopf Bifurcation

In this section, we investigate the cyclicity of the Hopf singularity of system (13). According to the algebraic equivalence demonstrated in Lemma 2, the first five focal values of the origin for the flow on the center manifold of system (12)| $_{\delta=0}$ can be easily calculated as:

$$v_{2m+1} = \mathbf{i}\pi\mu_m, \quad m = 1, 2, \dots, 5.$$

To facilitate the analysis of the cyclicity of origin in the perturbed system (12), real parameters a_j and b_j are introduced as

$$a_{ij} = a_j + \mathbf{i}b_j, \quad b_{ij} = a_j - \mathbf{i}b_j, \quad i, j = 0, 1, 2, 3. \quad (21)$$

Moreover, we specifically concentrate on the Case 1.2 for in-depth analysis. This enables us to establish a new theorem.

Theorem 2. For the flow on the center manifold of the system (12)| $_{\delta=0}$, when the Case 1.2 is satisfied, the first five focal values at the origin are given by

$$\begin{aligned} v_3 &= 2\pi d_1 b_1, \\ v_5 &= 2\pi d_1^2 (a_2 b_0 + a_0 b_2), \\ v_7 &= \frac{1}{4} \pi d_1^2 S_0, \\ v_9 &= \frac{512\pi S_1^3 G_1}{81S_2^3}, \\ v_{11} &= \frac{128\pi S_1^2 G_2}{729S_2^4}, \end{aligned} \quad (22)$$

where

$$\begin{aligned} S_0 &= -8S_1 - 3S_2 d_1, \\ S_1 &= a_2^2 + a_3^2 + b_2^2 + b_3^2, \quad S_2 = -2a_2 a_3 b_2 + a_2^2 b_3 - b_2^2 b_3, \\ G_1 &= -8a_1 a_2^2 + 15a_2^2 a_3 - 8a_1 a_3^2 - 8a_1 b_2^2 - 15a_3 b_2^2 + 30a_2 b_2 b_3 - 8a_1 b_3^2, \\ G_2 &= 200a_2^8 - 1319a_2^6 a_3^2 - 2271a_2^4 a_3^4 + 80a_2^2 a_3^6 + 832a_3^8 + 800a_2^6 b_2^2 + 2631a_2^4 a_3^2 b_2^2 \\ &\quad + 5934a_2^2 a_3^4 b_2^2 + 80a_3^6 b_2^2 + 1200a_2^4 b_2^4 + 2631a_2^2 a_3^2 b_2^4 - 2271a_3^4 b_2^4 + 800a_2^2 b_2^6 \\ &\quad - 1319a_3^2 b_2^6 + 200b_2^8 - 6588a_2^5 a_3 b_2 b_3 - 10476a_2^3 a_3^3 b_2 b_3 + 10476a_2 a_3^3 b_2^3 b_3 \\ &\quad + 6588a_2 a_3 b_2^5 b_3 + 328a_2^6 b_3^2 - 1923a_2^4 a_3^2 b_3^2 + 240a_2^2 a_3^4 b_3^2 + 3328a_2^6 b_3^2 \\ &\quad - 5604a_2^4 b_2^2 b_3^2 - 3846a_2^2 a_3^2 b_2^2 b_3^2 + 240a_3^4 b_2^2 b_3^2 - 5604a_2^2 b_2^4 b_3^2 - 1923a_2^2 b_2^4 b_3^2 \\ &\quad + 328b_2^6 b_3^2 - 10476a_2^3 a_3 b_2 b_3^3 + 10476a_2 a_3 b_2^3 b_3^3 + 348a_2^4 b_3^4 + 240a_2^2 a_3^2 b_3^4 \\ &\quad + 4992a_3^4 b_3^4 - 9780a_2^2 b_2^2 b_3^4 + 240a_2^2 b_2^2 b_3^4 + 348b_2^4 b_3^4 + 80a_2^2 b_3^6 + 3328a_2^2 b_3^6 \\ &\quad + 80b_2^2 b_3^6 + 832b_3^8. \end{aligned}$$

For each v_{2i+1} that was calculated, we already imposed $v_3 = \dots = v_{2i-1} = 0, j = 3, \dots, 5$.

Next, we discuss the conditions under which the origin is a weak focus of five-order on the center manifolds for system (12)| $_{\delta=0}$. Since $d_1 \neq 0$, we can derive $b_1 = 0$ from $v_3 = 2\pi d_1 b_1$. When Case 1.2 holds, from $a_{30} = b_{30} = 0, b_{03} a_{12}^2 - a_{03} b_{12}^2 \neq 0, a_{03} b_{03} + a_{12} b_{12} \neq 0$ and based on equation (21), we obtain $a_0 = b_0 = 0, S_2 \neq 0, S_1 \neq 0$. Furthermore, by letting $S_0 = 0$, it follows that $d_1 = -\frac{8S_1}{3S_2}$. Consequently, the following outcomes can be generated.

Theorem 3. For the flow on center manifold of system (12)| $_{\delta=0}$, when the Case 1.2 holds, the origin identified as a weak focus of five-order, namely, $v_3 = v_5 = v_7 = v_9 = 0, v_{11} \neq 0$, if and only if the condition below is met:

$$b_1 = a_0 = b_0 = S_0 = G_1 = 0, \quad S_1 \neq 0, \quad S_2 \neq 0, \quad G_2 \neq 0. \quad (23)$$

In addition, letting $G_1 = 0$, we figure out

$$a_1 = \frac{15}{8S_1} (a_2^2 a_3 - a_3 b_2^2 + 2a_2 b_2 b_3).$$

Hence, there must exist a set of critical values satisfying the condition (23), for example:

$$b_1 = a_0 = b_0 = 0, \quad a_2 = b_2 = a_3 = b_3 = 1, \quad d_1 = \frac{16}{3}, \quad a_1 = \frac{15}{16}. \quad (24)$$

Next, we discuss the Hopf bifurcation of the origin in the perturbed system (12).

Theorem 4. *At least 5 limit cycles can bifurcate from the origin of system (12).*

Proof. For system (12), it can be determined that the first order focal value is $v_1 = e^{2\pi\delta} - 1 = 2\pi\delta + o(\delta)$. According to theorem 2, the Jacobian determinant of the groups of functions v_1, v_3, v_5, v_7, v_9 with respect to the variables $\delta, b_1, b_0, d_1, a_1$ can be easily calculated,

$$J = \left| \frac{\partial(v_1, v_3, v_5, v_7, v_9)}{\partial(\delta, b_1, b_0, d_1, a_1)} \right| = \frac{8192\pi^5 a_2 d_1^4 S_1^4}{81S_2^3} (16S_1 + 9S_2 d_1) + o(\delta), \quad (25)$$

By letting $\delta = 0$, and substituting the group of critical values (24) into (25), we can determine that $J = \frac{549755813888\pi^5}{6561} \neq 0$. Referring to (Theorem 4.7, [18]) or (Theorem 2, [13]), this indicates that system (12) can have a minimum of 5 small-amplitude limit cycles via a Hopf bifurcation. \square

Remark 3. *To find more limit cycles emerging from the origin of the system (12), it is necessary to add more higher-order terms to the function $\tilde{u}(z, w)$ defined in (15). This will inevitably lead to an increase in the complexity of calculations involved in the aforementioned process. Furthermore, a more in-depth investigation into the linearizability of the high-order degenerate singularity is required.*

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