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Solutions for the System of Nonlinear Mixed Variational Inequality Problems

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Abstract: In this paper, we propose a system of nonlinear mixed variational inequality problems, which consists of two elliptic mixed variational inequality problems on Banach spaces. Under suitable assumptions, using the Kakutani-Ky Fan fixed point theorem and Minty techniques, we prove the solution set to the system of nonlinear mixed variational inequality problem is nonempty, weakly compact and unique. Additionally, we suggest a stability result for the system of nonlinear mixed variational inequality problem by perturbing the duality mappings. Furthermore, we present an optimal control problem governed by the system of nonlinear mixed variational inequality problems and establish a solvability result.

Keywords: system of nonlinear mixed variational inequality problem; inverse relaxed monotonicity; existence; uniqueness; stability; optimal control problem

PACS: 47J20; 49J30; 35A15; 49N45; 47H20; 49H52

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1. Introduction

Lin [1] introduced the system of generalized quasi-variational inclusion problems. This system consists of a set of problems defined on a product set. It includes several well-known problems such as variational inequalities, equilibrium problems, vector equilibrium problems, vector quasi-equilibrium problems, variational inclusions problems and variational disclusion problems.
Undoubtedly, in the realm of engineering, sciences, technology, chemical processes, and economics, several challenging and complex problems frequently result in inequalities instead of straightforward equations. In this scenario, variational inequalities have emerged as a powerful mathematical resource. Variational inequalities essentially arise from applied models with an underlying convex foundation and have been the subject of extensive research since the 1960s, encompassing mathematical theories, numerical techniques, and practical applications, among other significant sources, *see*, [2–7].
It should be noted that the results mentioned earlier cannot be applied to coupled systems that consist of two elliptical mixed variational inequalities. However, coupled variational inequalities can be a useful mathematical tool for examining various coupled mixed boundary value problems, control problems, and similar problems. More details can be found in [8–12].

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In this paper, we introduce a system of nonlinear mixed variational inequality problems and using the Kakutani-Ky Fan fixed point theorem, the Minty techniques and inverse relaxed monotonicity to establish the existence, convergence, uniqueness, stability, and optimal control of the problems.
Before we proceed, let us first define the problem that will play a crucial role in this paper. Consider two reflexive Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, each with its dual spaces $(X^*, \|\cdot\|_{X^*})$ and $(Y^*, \|\cdot\|_{Y^*})$, *respectively*. We use $\langle\cdot,\cdot\rangle_X$ (resp., $\langle\cdot,\cdot\rangle_Y$) to denote the duality pairing between X^* and X (resp., the duality pairing between Y^* and Y). We use the symbols $\xrightarrow{w}$ and $\longrightarrow$ to denote the weak and the strong convergence in the space X and employ the notation X_w for the space X equipped with the weak topology. The limits, lower limits and upper limits are considered as $n \longrightarrow \infty$, even if we do not mention it explicitly. We can formulate a system of nonlinear mixed variational inequality problems on Banach spaces:

Problem 1. *Find* $(x,y) \in \Omega \times \mho$ *such that*

$$\langle \mathscr{A}(y,x), v-x\rangle_X + \varphi(x,v) - \varphi(x,x) \geq \langle \gamma, v-x\rangle_X,\ \forall v \in \Omega \tag{1}$$

and

$$\langle \mathscr{B}(x,y), w-y\rangle_Y + \phi(y,w) - \phi(y,y) \geq \langle \zeta, w-y\rangle_Y,\ \forall w \in \mho. \tag{2}$$

We note that if $\varphi(x,x) = \varphi(x)$ and $\phi(x,x) = \phi(x)$, then Problem 1 reduces to the coupled system of variational inequality problems for finding $(x,y) \in \Omega \times \mho$ such that

$$\begin{cases} \langle \mathscr{A}(y,x), v-x\rangle_X + \varphi(v) - \varphi(x) \geq \langle \gamma, v-x\rangle_X,\ \forall v \in \Omega \\ \langle \mathscr{B}(x,y), w-y\rangle_Y + \phi(w) - \phi(y) \geq \langle \zeta, w-y\rangle_Y,\ \forall w \in \mho \end{cases} \tag{3}$$

studied in [13].

Definition 2. *Let* Ω *be a nonempty subset of a Banach space* X. *Let* $\varphi\colon \Omega \longrightarrow \overline{\mathbb{R}}$ *be a proper convex and lower semicontinuous function, and* $\mathscr{Q}\colon \Omega \longrightarrow X^*$. *Then* $\mathscr{Q}$ *is called*

(i) monotone, if it holds

$$\langle \mathscr{Q}u - \mathscr{Q}v, u-v\rangle_X \geq 0,\ \forall u,v \in \Omega;$$

(ii) strictly monotone, if it holds

$$\langle \mathscr{Q}u - \mathscr{Q}v, u-v\rangle_X > 0,\ \forall u,v \in \Omega \text{ and } u \neq v;$$

(iii) strongly monotone with constant $\alpha_{\mathscr{Q}} > 0$, *if it holds*

$$\langle \mathscr{Q}u - \mathscr{Q}v, u-v\rangle_X \geq \alpha_{\mathscr{Q}}\|u-v\|_X^2,\ \forall u,v \in \Omega;$$

(iv) relaxed strongly monotone with constant $\alpha_{\mathscr{Q}} > 0$, *if it holds*

$$\langle \mathscr{Q}u - \mathscr{Q}v, u-v\rangle_X \geq -\alpha_{\mathscr{Q}}\|u-v\|_X^2,\ \forall u,v \in \Omega;$$

(v) inverse relaxed monotone with constant $\alpha_{\mathscr{Q}} > 0$, *if it holds*

$$\langle \mathscr{Q}u - \mathscr{Q}v, u-v\rangle_X \geq -\alpha_{\mathscr{Q}}\|Qu - Qv\|_X^2,\ \forall u,v \in \Omega;$$

(vi) *Lipschitz continuous with constant $\beta_{\mathscr{Q}} > 0$, if it holds*

$$\|\mathscr{Q}u - \mathscr{Q}v\|_X \leq \beta_{\mathscr{Q}}\|u - v\|_X^2, \ \forall u, v \in \Omega;$$

(vii) *pseudomonotone, if for any $u, v \in \Omega$ we have*

$$\langle \mathscr{Q}u, v - u\rangle_X \geq 0$$

then it entails that

$$\langle \mathscr{Q}v, v - u\rangle_X \geq 0;$$

(viii) *stable pseudo monotone with respect to the set $\mathscr{W} \subset X^*$, if $\mathscr{Q}$ and $u \rightarrowtail \mathscr{Q}u - w$ are pseudo monotone for all $w \in \mathscr{W}$;*

(ix) *φ-pseudomonotone, if for any $u, v \in \Omega$ we have*

$$\langle \mathscr{Q}u, v - u\rangle_X + \varphi(v) - \varphi(u) \geq 0$$

then it entails that

$$\langle \mathscr{Q}v, v - u\rangle_X + \varphi(v) - \varphi(u) \geq 0;$$

(x) *stable φ-pseudomonotone with respect to the set $\mathscr{W} \subset X^*$, if $\mathscr{Q}$ and $u \rightarrowtail \mathscr{Q}u - w$ are φ-pseudomonotone for each $w \in \mathscr{W}$.*

Definition 3. *[14] Let $\varphi\colon X \longrightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. The mapping $\partial_c\varphi\colon X \longrightarrow 2^{X^*}$ defined by*

$$\partial_c\varphi(u) = \{u^* \in X^* | \langle u^*, v - u\rangle_X \leq \varphi(v) - \varphi(u), \ \forall v \in X\}$$

for $u \in X$, is called the subdifferential of φ. An element $x^* \in \partial_c\varphi(x)$ is called a subgradient of φ in u.

Definition 4. *[14] Given a locally Lipschitz function $\varphi\colon X \longrightarrow \mathbb{R}$, we denote by $\varphi^0(u; v)$ the Clarke generalized directional derivative of φ at the point $u \in X$ in the direction $v \in X$ defined by*

$$\varphi^0(u; v) = \limsup_{\lambda \longrightarrow 0^+, \eta \longrightarrow u} \frac{\varphi(\eta + \lambda v) - \varphi(\eta)}{\lambda}.$$

The Clarke subdifferential or the generalized gradient of φ at $u \in X$, denoted by $\partial\varphi(u)$, is a subset of X^* given by

$$\partial\varphi(u) = \{u^* \in X^* | \varphi^0(u; v) \geq \langle u^*, v\rangle_X, \forall v \in X\}.$$

Let Z and Y be topological spaces and V be a nonempty subset of Z. We denote by 2^V the collection of subsets of V. Given a set-valued mapping $\mathscr{B}\colon Z \longrightarrow 2^Y$, we use $Gr(\mathscr{B})$ to represent the graph of $\mathscr{B}$. In other word, $Gr(\mathscr{B})$ is defined as follows:

$$Gr(\mathscr{B}) = \{(x, y) \in Z \times Y | y \in \mathscr{B}(x)\} \subset Z \times Y.$$

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We say that the graph of $\mathscr{B}$ is sequentially closed in $Z \times Y$, or $\mathscr{B}$ is sequentially closed, if for any sequence $\{(x_n, y_n)\} \subset Gr(\mathscr{B})$. If this sequence converges to $(x, y) \in Z \times Y$ as $n \longrightarrow \infty$, then we have

$$(x, y) \in Gr(\mathscr{B}) \ \ (i.e., \ \ y \in \mathscr{B}(x)).$$

Theorem 1. *[15], Let Y be a reflexive Banach space and D be a nonempty, bounded, closed and convex set subset of Y. Let $\Lambda\colon D \longrightarrow 2^D$ be a set-valued map with nonempty, closed and convex values such that its graph is sequentially closed in $Y_w \times Y_w$ topology. Then, Λ has a fixed point.*

2. Main Results

In this section, we focus on the uniqueness of solutions and their existence to Problem 1. We use the Minty methodology, Theorem 1, and the Kakutani-Ky Fan fixed point theorem to establish the existence theorem for the solutions for the system of nonlinear mixed variational inequality problems under given modest assumptions. Additionally, we deploy the inverse relaxed monotonicity and Lipschitz continuity argument to present two uniqueness results for Problem 1.

Furthermore, we propose the set-valued mappings $\mathscr{S}\colon \mho \longrightarrow 2^{\Omega}$ and $\mathscr{T} : \Omega \longrightarrow 2^{\mho}$, described by

$$\mathscr{S}(y) = \{x \in \Omega |\ x \text{ is a solution of (1) corresponding to } y\}, \forall y \in \mho,$$

and

$$\mathscr{T}(x) = \{y \in \mho |\ y \text{ is a solution of (2) corresponding to } x\}, \forall x \in \Omega,$$

respectively.

To investigate the existence of solutions to Problem 1, we now suggest the following appropriate assumptions.

(A): $\Omega \subset X$ and $\mho \subset Y$ are both nonempty, closed and convex.
(B): $\gamma \in X^*$ and $\zeta \in Y^*$.
(C): $\varphi\colon X \times X \longrightarrow \overline{\mathbb{R}}$ is such that

(i) $\varphi(\eta, \cdot)\colon X \longrightarrow \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function,
(ii) there exists $\varrho_\varphi \geq 0$ such that

$$\varphi(\eta_1, y_2) - \varphi(\eta_1, y_1) + \varphi(\eta_2, y_1) - \varphi(\eta_2, y_2) \leq \varrho_\varphi \|\eta_1 - \eta_2\|_X \|y_1 - y_2\|_X, \forall \eta_1, \eta_2, y_1, y_2 \in X,$$

(iii) for each $\eta \in X$, there exists $\varrho_\varphi(\eta) > 0$ such that

$$\varphi(\eta, y_1) - \varphi(\eta, y_2) \leq \varrho_\varphi(\eta) \|y_1 - y_2\|_X, \forall y_1, y_2 \in X, \textit{see } [16].$$

(D): $\mathscr{A} : Y \times X \longrightarrow X^*$ is such that

(i) for each $y \in Y, x \rightarrowtail \mathscr{A}(y, x)$ is stable φ-pseudomonotone with $\{\gamma\}$ and fulfills

$$\lim_{\lambda \longrightarrow 0} \sup \langle \mathscr{A}(y, \lambda v + (1 - \lambda)x), v - x\rangle_X \leq \langle \mathscr{A}(y, x), v - x\rangle_X, \forall y \in Y \text{ and } x, v \in X;$$

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(ii) it holds

$$\lim_{n\longrightarrow\infty}\sup\langle\mathscr{A}(y_n,v),v-x_n\rangle_X\leq\langle\mathscr{A}(y,v),v-x\rangle_X,$$

whenever $v\in X,(x,y)\in X\times Y,\{y_n\}\subset Y$ and $\{x_n\}\subset X$ are such that

$$y_n\xrightarrow{w}y\in Y\ \text{ and }\ x_n\xrightarrow{w}x\in X\text{ as }n\longrightarrow\infty;$$

(iii) there exists a function $r\colon\mathbb{R}^+\times\mathbb{R}^+\longrightarrow\mathbb{R}$ such that

$$\langle\mathscr{A}(y,x),x\rangle_X\geq r(\|x\|_X,\|y\|_Y)\|x\|_X,\forall x\in X\text{ and }y\in Y,$$

and

♠ every nonempty and bounded set $D\subset\mathbb{R}^+$, we have

$$r(t,s)\longrightarrow+\infty\text{ as }t\longrightarrow+\infty,\forall s\in D,$$

♠ for any constants $\varrho_1,\varrho_2\geq 0$, it holds $r(t,\varrho_1 t+\varrho_2)\longrightarrow+\infty$ as $t\longrightarrow+\infty$.

(iv) there exists a constant $\varrho_{\mathscr{A}}>0$ such that

$$\|\mathscr{A}(y,x)\|_{X^*}\leq\varrho_{\mathscr{A}}(1+\|x\|_X+\|y\|_Y),\forall(x,y)\in X\times Y.$$

(E): $\phi\colon Y\times Y\longrightarrow\overline{\mathbb{R}}$ is such that

(i) $\phi(\eta,\cdot)\colon Y\longrightarrow\overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function,

(ii) there exists $\varrho_\phi\geq 0$ such that

$$\phi(\eta_1,y_2)-\phi(\eta_1,y_1)+\phi(\eta_2,y_1)-\phi(\eta_2,y_2)\leq\varrho_\phi\|\eta_1-\eta_2\|_Y\|y_1-y_2\|_Y,\forall\eta_1,\eta_2,y_1,y_2\in Y,$$

(iii) for each $\eta\in X$, there exists $\varrho_\phi(\eta)>0$ such that

$$\phi(\eta,y_1)-\phi(\eta,y_2)\leq\varrho_\phi(\eta)\|y_1-y_2\|_X,\forall y_1,y_2\in X,\textit{see } [16].$$

(F): $\mathscr{B}\colon X\times Y\longrightarrow Y^*$ is such that

(i) for each $x\in X,y\rightarrowtail\mathscr{B}(x,y)$ is stable ϕ-pseudomonotone with respect to $\{\zeta\}$ and satisfies

$$\lim_{\lambda\longrightarrow 0}\sup\langle\mathscr{B}(x,\lambda w+(1-\lambda)y),w-y\rangle_Y\leq\langle\mathscr{B}(x,y),w-y\rangle_Y,\forall w,y\in Y\text{ and }x\in X;$$

(ii) it holds

$$\lim_{n\longrightarrow\infty}\sup\langle\mathscr{B}(x_n,w),w-y_n\rangle_Y\leq\langle\mathscr{B}(x,w),w-y\rangle_Y,$$

whenever $w\in Y,(x,y)\in X\times Y,\{y_n\}\subset Y$ and $\{x_n\}\subset X$ are such that

$$y_n\xrightarrow{w}y\in Y\ \text{ and }x_n\xrightarrow{w}x\in X\ \text{ as }n\longrightarrow\infty;$$

(iii) there exists a function $\ell:\mathbb{R}^+\times\mathbb{R}^+\longrightarrow\mathbb{R}$ such that

$$\langle\mathscr{B}(x,y),y\rangle_Y\geq\ell(\|y\|_Y,\|x\|_X)\|y\|_Y,\forall x\in X\text{ and }y\in Y,$$

and

♠ every nonempty and bounded set $D \subset \mathbb{R}^+$, we have

$$\ell(t,s) \longrightarrow +\infty \text{ as } t \longrightarrow +\infty, \text{ for all } s \in D,$$

♠ for any constants $\varrho_1, \varrho_2 \geq 0$, it holds $\ell(t, \varrho_1 t + \varrho_2) \longrightarrow +\infty$ as $t \longrightarrow +\infty$.

(iv) there exists a constant $\varrho_{\mathscr{B}} > 0$ such that

$$\|\mathscr{B}(x,y)\|_{Y^*} \leq \varrho_{\mathscr{B}}(1 + \|x\|_X + \|y\|_Y), \forall (x,y) \in X \times Y.$$

Remark 1. *If function r given in **(D)**(iii) (resp. ℓ given in **(F)**(iii)) is independent of its second variable, then condition **(D)**(iii) (resp. **(F)**(iii)) reduces to the following uniformly coercive condition:*

***(D)**(iii)′: there exists a function $r\colon \mathbb{R}^+ \longrightarrow \mathbb{R}$ with $r(s) \longrightarrow +\infty$ as $s \longrightarrow +\infty$ such that*

$$\langle \mathscr{A}(y,x), x\rangle_X \geq r(\|x\|_X)\|x\|_X, \forall x \in X \text{ and } y \in Y$$

*(resp. **(F)**(iii)′: there exists a function $\ell\colon \mathbb{R}^+ \longrightarrow \mathbb{R}$ with $\ell(s) \longrightarrow +\infty$ as $s \longrightarrow +\infty$ such that*

$$\langle \mathscr{B}(x,y), y\rangle_Y \geq \ell(\|y\|_Y)\|y\|_Y, \forall x \in X, \text{ and } y \in Y).$$

The main result of this paper, related to the existence of solutions to Problem 1, is as follows.

Theorem 2. *Suppose that **(A), (B), (C), (D), (E)** and **(F)** are held. Then, the solution set denoted by $\Gamma(\gamma,\zeta)$ of Problem 1 corresponding to $(\gamma,\zeta) \in X^* \times Y^*$ is nonempty and weakly compact in $X \times Y$.*

We require the following lemmas to prove this theorem.

Lemma 1. *Assume that **(A), (B), (C)** and **(D)** are satisfied. Then, the following statements hold*

(i) for each fix $y \in Y$, $x \in \Omega$ is a solution of (1), if and only if, x solves the following Minty inequality for finding $x \in \Omega$ such that

$$\langle \mathscr{A}(y,v), v - x\rangle_X + \varphi(x,v) - \varphi(x,x) \geq \langle \gamma, v - x\rangle_X, \forall v \in \Omega; \quad (4)$$

(ii) for each fix $y \in Y$, the solution set denoted by $\mathscr{S}(y)$ of (1) is nonempty, bounded, closed and convex;

(iii) the graph of the set-valued mapping $\mathscr{S}\colon \mho \longrightarrow 2^\Omega$ is sequentially closed in $Y_w \times X_w$, i.e., $\mathscr{S}$ is sequentially closed from Y endowed with the weak topology into the subsets of X with the weak topology;

(iv) for each fix $y \in Y$, if the mapping $x \mapsto \mathscr{A}(y,x)$ is strictly monotone, then $\mathscr{S}$ is a single-valued mapping and weakly continuous.

Proof. The assumptions (i) and (ii) are the straightforward consequences of [[17], Theorem 3.3] and [[18], Lemma 3.3]. Next, we present the conclusion (iii).

Let $\{(y_n, x_n)\} \subset Gr(\mathscr{S})$ be such that

$$y_n \xrightarrow{w} y \in Y \text{ and } x_n \xrightarrow{w} x \in X \text{ as } n \longrightarrow \infty \text{ for } (x,y) \in X \times Y. \quad (5)$$

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Then, for each $n \in \mathbb{N}$, we have $x_n \in \mathscr{S}(y_n)$, *i.e.*,

$$\langle \mathscr{A}(y_n, x_n), v - x_n \rangle_X + \varphi(x_n, v) - \varphi(x_n, x_n) \geq \langle \gamma, v - x_n \rangle_X, \forall v \in \Omega.$$

The assertion (i) asserts that

$$\langle \mathscr{A}(y_n, v), v - x_n \rangle_X + \varphi(x_n, v) - \varphi(x_n, x_n) \geq \langle \gamma, v - x_n \rangle_X, \forall v \in \Omega. \tag{6}$$

To establish the upper limit as $n \longrightarrow \infty$, use assumption **(D)**(ii) and weak lower semicontinuity of φ (due to the convexity and lower semicontinuity of φ) to determine

$$\begin{aligned}\langle \mathscr{A}(y, v), v - x \rangle_X + \varphi(x, v) - \varphi(x, x) &\geq \lim_{n \longrightarrow \infty} \sup \langle \mathscr{A}(y_n, v), v - x_n \rangle_X \\ &+ \lim_{n \longrightarrow \infty} \inf[\varphi(x_n, v)] - \lim_{n \longrightarrow \infty} \inf[\varphi(x_n, x_n)] \\ &\geq \lim_{n \longrightarrow \infty} \sup[\langle \mathscr{A}(y_n, v), v - x_n \rangle_X + \varphi(x_n, v) - \varphi(x_n, x_n)] \\ &\geq \lim_{n \longrightarrow \infty} \sup \langle \gamma, v - x_n \rangle_X \\ &= \langle \gamma, v - x \rangle_X, \forall v \in \Omega.\end{aligned}$$

Using the assumption (i) again, we obtain

$$x \in \mathscr{S}(y).$$

Consequently, $(y, x) \in Gr(\mathscr{S})$, *i.e.*, the graph of the set-valued mapping $\mathscr{S} : \mho \in 2^{\Omega}$ is sequentially closed in $Y_w \times X_w$.

Additionally, assume that $x \rightarrowtail \mathscr{A}(y, x)$ is strictly monotone. Let us consider $x_1, x_2 \in \Omega$ be two solutions to (1). Then, we have

$$\langle \mathscr{A}(y, x_1), v - x_1 \rangle_X + \varphi(x_1, v) - \varphi(x_1, x_1) \geq \langle \gamma, v - x_1 \rangle_X, \forall v \in \Omega \tag{7}$$

and

$$\langle \mathscr{A}(y, x_2), v - x_2 \rangle_X + \varphi(x_2, v) - \varphi(x_2, x_2) \geq \langle \gamma, v - x_2 \rangle_X, \forall v \in \Omega. \tag{8}$$

Putting $v = x_2$ in (7) and $v = x_1$ in (8), we have

$$\langle \mathscr{A}(y, x_1), x_2 - x_1 \rangle_X + \varphi(x_1, x_2) - \varphi(x_1, x_1) \geq \langle \gamma, x_2 - x_1 \rangle_X, \tag{9}$$

and

$$\langle \mathscr{A}(y, x_2), x_1 - x_2 \rangle_X + \varphi(x_2, x_1) - \varphi(x_2, x_2) \geq \langle \gamma, x_1 - x_2 \rangle_X. \tag{10}$$

Adding (9) and (10), we have

$$\langle \mathscr{A}(y, x_1) - \mathscr{A}(y, x_2), x_1 - x_2 \rangle_X - \varphi(x_1, x_2) + \varphi(x_1, x_1) - \varphi(x_2, x_1) + \varphi(x_2, x_2) \leq 0. \tag{11}$$

Hence, from the assumption **(C)** and the strict monotonicity of $x \rightarrowtail \mathscr{A}(y, x)$ guarantees that $x_1 = x_2$. Therefore, $\mathscr{S}$ is a single-valued mapping. But, by virtue of assumption (iii), we can observe that $\mathscr{S}$ is weakly continuous. □

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Similarly, for problem (2), we have the following lemma.

Lemma 2. *Assume that **(A), (B), (E)** and **(F)** are satisfied. Then, the following statements hold*

(i) for each fix $x \in X$, $y \in \mho$ is a solution of (2), if and only if y solves the following Minty inequality for finding $y \in \mho$ such that

$$\langle \mathscr{B}(x,w), w-y\rangle_Y + \phi(y,w) - \phi(y,y) \geq \langle \zeta, w-y\rangle_Y, \forall w \in \mho; \tag{12}$$

(ii) for each fix $x \in X$, the solution set $\mathscr{T}(x)$ namely, of (2) is nonempty, bounded, closed and convex;

(iii) the graph of the set-valued mapping $\mathscr{T} : \Omega \longrightarrow 2^{\mho}$ is sequentially closed in $X_w \times Y_w$;

(iv) for each fix $x \in X$, if the mapping $y \rightarrowtail \mathscr{B}(x,y)$ is strictly monotone, then $\mathscr{T}$ is a single-valued mapping and weakly continuous.

Furthermore, we focus on providing a priori estimates for the solutions of Problem 1.

Lemma 3. *Assume that **(A), (B), (C), (D), (E)** and **(F)** are satisfied. If the solution set $\Gamma(\gamma,\zeta)$ namely, of Problem 1 is nonempty, then there exists a constant $M > 0$ such that*

$$\|x\|_X \leq M \text{ and } \|y\|_Y \leq M, \ \forall (x,y) \in \Gamma(\gamma,\zeta). \tag{13}$$

Proof. Suppose that $\Gamma(\gamma,\zeta) \neq \emptyset$. Let $(x,y) \in \Gamma(\gamma,\zeta)$ be arbitrary and $(x_0,y_0) \in (D(\varphi)\cap\Omega) \times (D(\phi)\cap\mho)$. By swapping $v = x_0$ and $w = y_0$ into (1) and (2), *respectively*, we obtain

$$\langle \mathscr{A}(y,x), x\rangle_X \leq \langle \mathscr{A}(y,x), x_0\rangle_X + \varphi(x,x_0) - \varphi(x,x) + \langle \gamma, x_0 - x\rangle_X \tag{14}$$

and

$$\langle \mathscr{B}(x,y), y\rangle_Y \leq \langle \mathscr{B}(x,y), y_0\rangle_Y + \phi(y,y_0) - \phi(y,y) + \langle \zeta, y_0 - y\rangle_Y. \tag{15}$$

Taking account of (14), we use hypotheses **(C)**(i),(iii) and **(D)**(iii)–(iv) to obtain

$$\begin{aligned} r(\|x\|_X, \|y\|_Y)\|x\|_X &\leq \langle \mathscr{A}(y,x), x\rangle_X \\ &\leq \langle \mathscr{A}(y,x), x_0\rangle_X + \varphi(x,x_0) - \varphi(x,x) + \langle \gamma, x_0 - x\rangle_X \\ &\leq \|\mathscr{A}(y,x)\|_{X^*}\|x_0\|_X + \varphi(x,x_0) - \varphi(x,x) + \|\gamma\|_{X^*}(\|x_0\|_X + \|x\|_X) \\ &\leq \varrho_{\mathscr{A}}(1 + \|x\|_X + \|y\|_Y)\|x_0\|_X + \varrho_\varphi(x)(\|x_0\|_X + \|x\|_X) + \|\gamma\|_{X^*}(\|x_0\|_X + \|x\|_X). \end{aligned}$$

This implies that

$$r(\|x\|_X, \|y\|_Y) \leq \frac{\varrho_{\mathscr{A}}(1 + \|x\|_X + \|y\|_Y)\|x_0\|_X}{\|x\|_X} + \frac{(\varrho_\varphi(x) + \|\gamma\|_{X^*})\|x_0\|_X}{\|x\|_X} + \varrho_\varphi(x) + \|\gamma\|_{X^*}. \tag{16}$$

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Similarly, taking account of (15), we use hypotheses **(E)**(i), (iii) and **(F)**(iii)–(iv) to obtain

$$\begin{aligned}
&\ell(\|y\|_Y, \|x\|_X)\|y\|_Y \leq \varrho_{\mathscr{B}}(1 + \|x\|_X + \|y\|_Y)\|y_0\|_Y + \varrho_\phi(y)(\|y_0\|_Y + \|y\|_Y) + \|\zeta\|_{Y^*}(\|y_0\|_Y + \|y\|_Y) \\
&\Longrightarrow \\
&\ell(\|y\|_Y, \|x\|_X) \leq \frac{\varrho_{\mathscr{B}}(1 + \|x\|_X + \|y\|_Y)\|y_0\|_Y}{\|y\|_Y} + \frac{(\varrho_\phi(y) + \|\zeta\|_{Y^*})\|y_0\|_Y}{\|y\|_Y} \\
&\qquad + \varrho_\phi(y) + \|\zeta\|_{Y^*}.
\end{aligned} \tag{17}$$

Contrary, suppose $\Gamma(\gamma, \zeta)$ is unbounded. Then, taking a subsequence if necessary, it is possible to find a sequence $\{(x_n, y_n)\} \subset \Omega \times \mho$ such that it holds

$$\|x_n\|_X \uparrow +\infty \text{ as } n \longrightarrow \infty, \tag{18}$$

or

$$\|y_n\|_Y \uparrow +\infty \text{ as } n \longrightarrow \infty. \tag{19}$$

Let us segregate the subsequent cases:

ⓐ Assuming that (18) is fulfilled and the sequence $\{y_n\}$ is bounded in the space Y,

ⓑ Assuming that (19) is fulfilled and the sequence $\{x_n\}$ is bounded in the space X,

ⓒ Assuming that both (18) and (19) are fulfilled.

Assume that ⓐ is valid, then we put $x = x_n$ and $y = y_n$ in (16) to get

$$\begin{aligned}
r(\|x_n\|_X, \|y_n\|_Y) \leq \frac{\varrho_{\mathscr{A}}(1 + \|x_n\|_X + \|y_n\|_Y)\|x_0\|_X}{\|x_n\|_X} + \frac{(\varrho_\varphi(x) + \|\gamma\|_{X^*})\|x_0\|_X}{\|x_n\|_X} \\
+ \varrho_\varphi(x) + \|\gamma\|_{X^*}.
\end{aligned} \tag{20}$$

When we let n approach infinity in the inequality (20) and make use of (18) along with property **(D)**(iii), we obtain the following:

$$\begin{aligned}
+\infty &= \lim_{n\longrightarrow\infty} r(\|x_n\|_X, \|y_n\|_Y) \\
&\leq \lim_{n\longrightarrow\infty} \left[\frac{\varrho_{\mathscr{A}}(1 + \|x_n\|_X + \|y_n\|_Y)\|x_0\|_X}{\|x_n\|_X} + \frac{(\varrho_\varphi(x) + \|\gamma\|_{X^*})\|x_0\|_X}{\|x_n\|_X} + \varrho_\varphi(x) + \|\gamma\|_{X^*}\right] \\
&= \varrho_{\mathscr{A}}\|x_0\|_X + \varrho_\varphi(x) + \|\gamma\|_{X^*}.
\end{aligned} \tag{21}$$

Hence, (21) generates a contradiction. Similarly, for the case ⓑ we could also use (17) to get a contradiction. However, we assume that ⓒ hold, we will proceed to discuss two additional situations:

(1) $\dfrac{\|y_n\|_Y}{\|x_n\|_X} \longrightarrow +\infty$ as $n \longrightarrow \infty$;

(2) there exist $n_0 \in \mathbb{N}$ and $\widehat{\varrho_0} > 0$ such that

$$\frac{\|y_n\|_Y}{\|x_n\|_X} \leq \widehat{\varrho_0}, \ \forall n \geq n_0.$$

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If the item (1) is true, then we put $x = x_n$ and $y = y_n$ into (17) to yield

$$\ell(\|y_n\|_Y, \|x_n\|_X) \le \frac{\varrho_{\mathscr{B}}(1 + \|x_n\|_X + \|y_n\|_Y)\|y_0\|_Y}{\|y_n\|_Y} + \frac{(\varrho_\phi(y) + \|\zeta\|_{Y^*})\|y_0\|_Y}{\|y_n\|_Y} + \varrho_\phi(y) + \|\zeta\|_{Y^*}.$$

Taking the limit as n approaches infinity for the inequality mentioned above yields:

$$\begin{aligned} +\infty &= \lim_{n\longrightarrow\infty} \ell(\|y_n\|_Y, \|x_n\|_X) \\ &\le \lim_{n\longrightarrow\infty}\left[\frac{\varrho_{\mathscr{B}}(1 + \|x_n\|_X + \|y_n\|_Y)\|y_0\|_Y}{\|y_n\|_Y} + \frac{(\varrho_\phi(y) + \|\zeta\|_{Y^*})\|y_0\|_Y}{\|y_n\|_Y} + \varrho_\phi(y) + \|\zeta\|_{Y^*}\right] \\ &= \varrho_{\mathscr{B}}\|y_0\|_Y + \varrho_\phi(y) + \|\zeta\|_{Y^*}. \end{aligned} \tag{22}$$

It is obviously impossible, however for a situation (2), we can deduce from (16) that

$$\begin{aligned} +\infty \leftarrow\ & r(\|x_n\|_X, \|y_n\|_Y) \quad (\text{as } n \longrightarrow \infty) \\ &\le \frac{\varrho_{\mathscr{A}}(1 + \|x_n\|_X + \|y_n\|_Y)\|x_0\|_X}{\|x_n\|_X} + \frac{(\varrho_\varphi(x) + \|\gamma\|_{X^*})\|x_0\|_X}{\|x_n\|_X} + \varrho_\varphi(x) + \|\gamma\|_{X^*} \\ &= \varrho_{\mathscr{A}}(2 + \widehat{\varrho}_0)\|x_0\|_X + \varrho_\varphi(x) + \|\gamma\|_{X^*}\|x_0\|_X + \varrho_\varphi(x) + \|\gamma\|_{X^*}, \text{ for } n \ge n_1, \end{aligned} \tag{23}$$

where $n_1 \ge n_0$ is such that

$$\|x_{n_1}\|_X > 1.$$

This leads to a contradiction. Thus, we conclude that $\Gamma(\gamma, \zeta)$ is bounded in $X \times Y$, allowing us to find a constant $M > 0$ satisfying (13). □

Consider the set-valued mapping $\Lambda : \Omega \times \mho \longrightarrow 2^{\Omega\times\mho}$ given by

$$\Lambda(x, y) = (\mathscr{S}(y), \mathscr{T}(x)), \forall (x, y) \in \Omega \times \mho. \tag{24}$$

By invoking Lemma 1 and Lemma 2, it can be seen that Λ is well-defined. Additionally, there exists a bounded, closed, and convex set $\mathfrak{D}$ in $\Omega \times \mho$ such that Λ maps $\mathfrak{D}$ into itself.

Lemma 4. *Assume that conditions* **(A), (B), (C), (D), (E)** *and* **(F)** *are satisfied. Then, there exists a constant $\widehat{M} > 0$ satisfying $\Lambda(\overline{\mathscr{B}(0, \widehat{M})}) \subset \overline{\mathscr{B}(0, \widehat{M})}$, where $\overline{\mathscr{B}(0, \widehat{M})} \subset X \times Y$ is defined by*

$$\overline{\mathscr{B}(0, \widehat{M})} = \{(x, y) \in \Omega \times \mho \mid \ \|x\|_X \le \widehat{M} \ \text{ and } \ \|y\|_Y \le \widehat{M}\}.$$

Proof. Our proof will be based on contradiction. Suppose for each $n \in \mathbb{N}$, it holds $\Gamma(\overline{\mathscr{B}(0, n)}) \not\subset \overline{\mathscr{B}(0, n)}$. Then, for every $n \in \mathbb{N}$, we can find $(x_n, y_n) \in \overline{\mathscr{B}(0, n)}$ and $(z_n, w_n) \in \Gamma(x_n, y_n)$ (i.e., $z_n \in \mathscr{S}(y_n)$ and $w_n \in \mathscr{T}(x_n)$) such that

$$\|z_n\|_X > n \quad \text{or} \quad \|w_n\|_Y > n. \tag{25}$$

Thus, passing to a relabeled subsequence if needed, we can assume that $\|z_n\|_X > n$ for each $n \in \mathbb{N}$ (since the proof for the case that $\|w_n\|_Y > n$ for each $n \in \mathbb{N}$ is similar). Employing (16), it determines

$$r(\|z_n\|_X, \|y_n\|_Y) \le \frac{\varrho_{\mathscr{A}}(1 + \|z_n\|_X + \|y_n\|_Y)\|x_0\|_X}{\|z_n\|_X} + \frac{(\varrho_\varphi(x) + \|\gamma\|_{X^*})\|x_0\|_X}{\|z_n\|_X} + \varrho_\varphi(x) + \|\gamma\|_{X^*}.$$

Since

$$\|y_n\|_Y \leq n < \|z_n\|_X.$$

Therefore, passing to the limit as $n \longrightarrow \infty$ for the inequality above, it gives

$$\begin{aligned} +\infty &= \lim_{n\longrightarrow\infty} r(\|z_n\|_X, \|y_n\|_Y) \\ &\leq \lim_{n\longrightarrow\infty}\left[\frac{\varrho_{\mathscr{A}}(1+\|z_n\|_X+\|y_n\|_Y)\|x_0\|_X}{\|z_n\|_X} + \frac{(\varrho_\varphi(x)+\|\gamma\|_{X^*})\|x_0\|_X}{\|z_n\|_X} + \varrho_\varphi(x) + \|\gamma\|_{X^*}\right] \\ &\leq 2\varrho_{\mathscr{A}}\|x_0\|_X + \varrho_\varphi(x) + \|\gamma\|_{X^*}. \end{aligned}$$

This leads to a contradiction. Thus, there exists a constant $\widehat{M} > 0$ satisfying

$$\Lambda(\overline{\mathscr{B}(0,\widehat{M})}) \subset \overline{\mathscr{B}(0,\widehat{M})}.$$

□

Proof. (Proof of Theorem 2) Let us see that if (x^*, y^*) is a fixed point of Λ, then we have $x^* \in \mathscr{S}(y^*)$ and $y^* \in \mathscr{T}(x^*)$. Using the definitions of $\mathscr{S}$ and $\mathscr{T}$, it provides

$$\langle \mathscr{A}(y^*, x^*), v - x^*\rangle_X + \varphi(x^*, v) - \varphi(x^*, x^*) \geq \langle \gamma, v - x^*\rangle_X, \forall v \in \Omega,$$

and

$$\langle \mathscr{B}(x^*, y^*), w - y^*\rangle_Y + \phi(y^*, w) - \phi(y^*, y^*) \geq \langle \zeta, w - y^*\rangle_Y, \forall w \in \mho.$$

Consequently, it is clear that (x^*, y^*) is also a solution to Problem 1. We will apply Theorem 1, the Kakutani-Ky Fan fixed point theorem, to determine the existence of a fixed point for Λ.

Moreover, Lemma 1, Lemma 2, and Lemma 4, in fact, infer that $\Lambda\colon \overline{\mathscr{B}(0,\widehat{M})} \longrightarrow 2^{\overline{\mathscr{B}(0,\widehat{M})}}$ has nonempty, closed, and convex values, and that the graph of Λ is sequentially closed in $(X\times Y)_w \times (X\times Y)_w$. The conditions stated in Theorem 1 have been verified. By using this theorem, there exists a solution $(x^*, y^*) \in \Omega \times \mho$ to Problem 1 such that

$$(x^*, y^*) \in \Lambda(x^*, y^*)$$

and hence,

$$\Gamma(\gamma, \zeta) \neq \emptyset.$$

It is evident from Lemma 3 that $\Gamma(\gamma, \zeta)$ is bounded in $X \times Y$. Next, we will demonstrate that $\Gamma(\gamma, \zeta)$ is weakly closed. Let $\{(x_n, y_n)\} \subset \Gamma(\gamma, \zeta)$ be such that

$$(x_n, y_n) \xrightarrow{w} (x, y) \in X \times Y \text{ as } n \longrightarrow \infty, \text{ for some } (x, y) \in \Omega \times \mho. \tag{26}$$

It is easy to see that for each natural number n, the pair $(x_n, y_n) \in \Lambda(x_n, y_n)$. Since Λ is sequentially closed from $(X \times Y)_w$ to $(X \times Y)_w$ (*see* Lemma 1 and Lemma 2), we can conclude that

$$(x, y) \in \Lambda(x, y).$$

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This implies that

$$(x, y) \in \Gamma(\gamma, \zeta).$$

Thus, due to the boundedness of $\Gamma(\gamma, \zeta)$, we can conclude that $\Gamma(\gamma, \zeta)$ is weakly compact. $\square$

Theorem 2 shows that the solution set of Problem 1 is both non-empty and weakly compact. However, it raises the question of whether it is possible to prove the uniqueness of the solution under certain assumptions. Fortunately, the following theorems provide a positive answer to this issue.

Theorem 3. *Assume that* **(A), (B), (C), (D), (E)** *and* **(F)** *are satisfied. If, in addition, the following inequality holds,*

$$\begin{aligned}\langle \mathscr{A}(y_1, x_1) - \mathscr{A}(y_2, x_2), x_1 - x_2 \rangle_X + \langle \mathscr{B}(x_1, y_1) - \mathscr{B}(x_2, y_2), y_1 - y_2 \rangle_Y + \varrho_\varphi \|x_1 - x_2\|_X^2 \\ + \varrho_\phi \|y_1 - y_2\|_Y^2 > 0, \forall (x_1, y_1), (x_2, y_2) \in X \times Y \text{ with } (x_1, y_1) \neq (x_2, y_2).\end{aligned} \tag{27}$$

Then Problem 1 has a unique solution.

Proof. Theorem 2 assures that $\Gamma(\gamma, \zeta) \neq \emptyset$. We now prove the uniqueness of Problem 1. Let $(x_1, y_1), (x_2, y_2) \in \Gamma(\gamma, \zeta)$. Then, we have

$$\langle \mathscr{A}(y_i, x_i), v - x_i \rangle_X + \varphi(x_i, v) - \varphi(x_i, x_i) \geq \langle \gamma, v - x_i \rangle_X, \forall v \in \Omega, \tag{28}$$

and

$$\langle \mathscr{B}(x_i, y_i), w - y_i \rangle_Y + \phi(y_i, w) - \phi(y_i, y_i) \geq \langle \zeta, w - y_i \rangle_Y, \forall w \in \mho. \tag{29}$$

After setting $i = 1$ to correspond to $v = x_2$ and $i = 2$ to correspond to $v = x_1$ in equation (28), we add the two equations to obtain

$$\langle \mathscr{A}(y_1, x_1) - \mathscr{A}(y_2, x_2), x_1 - x_2 \rangle_X - \varphi(x_1, x_2) + \varphi(x_1, x_1) - \varphi(x_2, x_1) + \varphi(x_2, x_2) \leq 0. \tag{30}$$

Similarly, assigning $i = 1$ to correspond to $w = y_2$ and $i = 2$ to correspond to $w = y_1$ in equation (29), we add the two equations to get

$$\langle \mathscr{B}(x_1, y_1) - \mathscr{B}(x_2, y_2), y_1 - y_2 \rangle_Y - \phi(y_1, y_2) + \phi(y_1, y_1) - \phi(y_2, y_1) + \phi(y_2, y_2) \leq 0. \tag{31}$$

By using the (30), (31), and assertions **(C)** and **(E)**, we have

$$\langle \mathscr{A}(y_1, x_1) - \mathscr{A}(y_2, x_2), x_1 - x_2 \rangle_X + \langle \mathscr{B}(x_1, y_1) - \mathscr{B}(x_2, y_2), y_1 - y_2 \rangle_Y + \varrho_\varphi \|x_1 - x_2\|_X^2 + \varrho_\phi \|y_1 - y_2\|_Y^2 \leq 0.$$

This combined with the condition (27) implies that $x_1 = x_2$ and $y_1 = y_2$. Thus, Problem 1 has a unique solution. $\square$

By adding an additional condition to (27), the resulting theorem establishes a unique solution for Problem 1.

Theorem 4. *Assume that* **(A), (B), (C), (D), (E)** *and* **(F)** *are satisfied. If the following conditions also hold:*

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① *for each* $y \in Y$*, the function* $x \rightarrowtail \mathscr{A}(y,x)$ *is inversely relaxed monotone and Lipschitz continuous with constants* $\alpha_{\mathscr{A}} > 0$ *and* $\beta_{\mathscr{A}} > 0$*, respectively, and for each* $x \in X$ *the function* $y \rightarrowtail \mathscr{A}(y,x)$ *is Lipschitz continuous with constant* $\mathscr{L}_{\mathscr{A}} > 0$*,*

② *for each* $x \in X$*, the function* $y \rightarrowtail \mathscr{B}(x,y)$ *is inversely relaxed monotone and Lipschitz continuous with constants* $\alpha_{\mathscr{B}} > 0$ *and* $\beta_{\mathscr{B}} > 0$*, respectively, and for each* $y \in Y$ *the function* $x \rightarrowtail \mathscr{B}(x,y)$ *is Lipschitz continuous with constant* $\mathscr{L}_{\mathscr{B}} > 0$*,*

③ $\dfrac{\mathscr{L}_{\mathscr{A}}\mathscr{L}_{\mathscr{B}}}{(\alpha_{\mathscr{A}}\beta_{\mathscr{A}} + \varrho_{\varphi})(\alpha_{\mathscr{B}}\beta_{\mathscr{B}} + \varrho_{\phi})} < 1.$

Then Problem 1 has a unique solution.

Proof. Let (x_1, y_1) and (x_2, y_2) be two solutions to Problem 1. Then, it has

$$\langle \mathscr{A}(y_1, x_1) - \mathscr{A}(y_2, x_2), x_1 - x_2 \rangle_X - \varrho_{\varphi} \|x_1 - x_2\|_X^2 \leq 0, \tag{32}$$

and

$$\langle \mathscr{B}(x_1, y_1) - \mathscr{A}(x_2, y_2), y_1 - y_2 \rangle_Y - \varrho_{\phi} \|y_1 - y_2\|_Y^2 \leq 0. \tag{33}$$

Again, from the inverse relaxed monotonicity and Lipschitz continuity of $\mathscr{A}$, we have

$$\begin{aligned} -\alpha_{\mathscr{A}}\beta_{\mathscr{A}}\|x_1 - x_2\|_X^2 &\leq \langle \mathscr{A}(y_1, x_1 - \mathscr{A}(y_1, x_2), x_1 - x_2 \rangle_X \\ &\leq \langle \mathscr{A}(y_2, x_2) - \mathscr{A}(y_1, x_2), x_1 - x_2 \rangle_X \\ &\leq \mathscr{L}_{\mathscr{A}}\|y_1 - y_2\|_Y \|x_1 - x_2\|_X. \end{aligned} \tag{34}$$

Thus, from (32) and (34), we have

$$(-\alpha_{\mathscr{A}}\beta_{\mathscr{A}} - \varrho_{\varphi})\|x_1 - x_2\|_X^2 \leq \mathscr{L}_{\mathscr{A}}\|y_1 - y_2\|_Y \|x_1 - x_2\|_X. \tag{35}$$

Similarly, we obtain

$$\begin{aligned} -\alpha_{\mathscr{B}}\beta_{\mathscr{B}}\|y_1 - y_2\|_Y^2 &\leq \langle \mathscr{B}(x_1, y_1) - \mathscr{B}(x_1, y_2), y_1 - y_2 \rangle_Y \\ &\leq \langle \mathscr{B}(x_2, y_2) - \mathscr{B}(x_1, y_2), y_1 - y_2 \rangle_Y \\ &\leq \mathscr{L}_{\mathscr{B}}\|y_1 - y_2\|_Y \|x_1 - x_2\|_X. \end{aligned} \tag{36}$$

Again, from (33) and (36), we have

$$(-\alpha_{\mathscr{B}}\beta_{\mathscr{B}} - \varrho_{\phi})\|y_1 - y_2\|_Y^2 \leq \mathscr{L}_{\mathscr{B}}\|y_1 - y_2\|_Y \|x_1 - x_2\|_X. \tag{37}$$

Combining equations (35) and (37) yields

$$\|x_1 - x_2\|_X \leq \frac{\mathscr{L}_{\mathscr{A}}\mathscr{L}_{\mathscr{B}}}{(\alpha_{\mathscr{A}}\beta_{\mathscr{A}} + \varrho_{\varphi})(\alpha_{\mathscr{B}}\beta_{\mathscr{B}} + \varrho_{\phi})}\|x_1 - x_2\|_X. \tag{38}$$

However, the inequality $\frac{\mathscr{L}_{\mathscr{A}}\mathscr{L}_{\mathscr{B}}}{(\alpha_{\mathscr{A}}\beta_{\mathscr{A}} + \varrho_{\varphi})(\alpha_{\mathscr{B}}\beta_{\mathscr{B}} + \varrho_{\phi})} < 1$ implies that $x_1 = x_2$ and $y_1 = y_2$. Therefore, Problem 1 has a unique solution. □

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3. Stability Results

In this section, we delve into examining the stability of the system of nonlinear mixed variational inequality problems. Firstly, we present a set of regularized problems perturbed by duality mappings that correspond to Problem 1. Secondly, we produce a stability result that shows that any solution sequence of a regularized problem has at least one subsequence that leads to a solution of the original problem, which is Problem 1.

Recall that X and Y are two reflexive Banach spaces that can be renormed to become strictly convex. Without loss of generality, we may assume that X and Y are strictly convex. Let $J_X\colon X \longrightarrow X^*$ and $J_Y\colon Y \longrightarrow Y^*$ be the duality mappings of the spaces X and Y, *respectively*, namely:

$$J_X(x) = \{x^* \in X^* | \langle x^*, x\rangle_X = \|x\|_X^2 = \|x^*\|_{X^*}^2\},$$

$$J_Y(y) = \{y^* \in Y^* | \langle y^*, y\rangle_Y = \|y\|_Y^2 = \|y^*\|_{Y^*}^2\}.$$

Let real sequences $\{\varepsilon_n\}$ and $\{\delta_n\}$ be such that

$$\varepsilon_n > 0, \;\; \delta_n > 0, \;\;\; \varepsilon_n \longrightarrow 0 \text{ and } \delta_n \longrightarrow 0. \tag{39}$$

For each $n \in \mathbb{N}$, consider the following perturbated problem corresponding to Problem 1.

Problem 5. *Find* $(x_n, y_n) \in \Omega \times \mho$ *such that*

$$\langle \mathscr{A}(y_n, x_n) + \varepsilon_n J_X(x_n), v - x_n\rangle_X + \varphi(x_n, v) - \varphi(x_n, x_n) \geq \langle \gamma, v - x_n\rangle_X, \forall v \in \Omega, \tag{40}$$

and

$$\langle \mathscr{B}(x_n, y_n) + \delta_n J_Y(y_n), w - y_n\rangle_Y + \phi(y_n, w) - \phi(y_n, y_n) \geq \langle \zeta, w - y_n\rangle_Y, \forall w \in \mho. \tag{41}$$

We make the following assumptions.

(G): $x \rightarrowtail \mathscr{A}(y, x)$ and $y \rightarrowtail \mathscr{B}(x, y)$ are monotone, and satisfy

$$\limsup_{\lambda \longrightarrow 0} \langle \mathscr{A}(y, \lambda v + (1-\lambda)x), v - x\rangle_X \leq \langle \mathscr{A}(y, x), v - x\rangle_X,$$

$$\limsup_{\lambda \longrightarrow 0} \langle \mathscr{B}(x, \lambda w + (1-\lambda)y), w - y\rangle_Y \leq \langle \mathscr{B}(x, y), w - y\rangle_Y, \forall w, y \in Y \text{ and } v, x \in X.$$

(H): $x \rightarrowtail \mathscr{A}(y, x)$ is inverse relaxed monotone with constant $\alpha_{\mathscr{A}} > 0$ and Lipschitz continuous with constant $\beta_{\mathscr{A}} > 0$; similarly, $y \rightarrowtail \mathscr{B}(x, y)$ is inverse relaxed monotone with constant $\alpha_{\mathscr{B}} > 0$ and Lipschitz continuous with constant $\beta_{\mathscr{B}} > 0$, and and satisfy

$$\limsup_{\lambda \longrightarrow 0} \langle \mathscr{A}(y, \lambda v + (1-\lambda)x), v - x\rangle_X \leq \langle \mathscr{A}(y, x), v - x\rangle_X,$$

$$\limsup_{\lambda \longrightarrow 0} \langle \mathscr{B}(x, \lambda w + (1-\lambda)y), w - y\rangle_Y \leq \langle \mathscr{B}(x, y), w - y\rangle_Y, \forall w, y \in Y \text{ and } v, x \in X.$$

The theorem below guarantees the existence and convergence of solutions to Problem 5.

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Theorem 5. *Assume that* **(A), (B), (C), (D)***(ii)–(iv),* **(E)** *and* **(F)***(ii)–(iv) are satisfied. Then the following assertions hold,*

(i) if, in addition, **(G)** *holds, then for each $n \in \mathbb{N}$, Problem 5 has at least one solution $(x_n, y_n) \in \Omega \times \mho$;*

(ii) furthermore, if **(G)** *holds, then for any sequence of solutions $\{(x_n, y_n)\}$ of Problem 5, there exists a subsequence $\{(x_n, y_n)\}$, such that*

$$(x_n, y_n) \xrightarrow{w} (x, y) \in X \times Y \text{ as } n \longrightarrow \infty, \tag{42}$$

where $(x, y) \in \Omega \times \mho$ is a solution of Problem 5;

(iii) if **(H)** *holds, then for any sequence of solutions $\{(x_n, y_n)\}$ of Problem 5, there exists a subsequence $\{(x_n, y_n)\}$ such that*

$$(x_n, y_n) \longrightarrow (x, y) \in X \times Y \text{ as } n \longrightarrow \infty, \tag{43}$$

where $(x, y) \in \Omega \times \mho$ is a solution of Problem 1.

Proof. (i) Set

$$\mathscr{A}_n(y, x) = \mathscr{A}(y, x) + \varepsilon_n J_X(x)$$

and

$$\mathscr{B}_n(x, y) = \mathscr{B}(x, y) + \delta_n J_Y(y), \ \forall\ (x, y) \in X \times Y.$$

We will verify that $\mathscr{A}_n$ and $\mathscr{B}_n$ satisfy **(D)** and **(F)**, *respectively*. Note that J_X is demicontinuous and

$$0 \leq (\|x\|_X - \|v\|_X)^2 \leq \langle J_X(x) - J_X(v), x - v \rangle_X, \forall x, v \in X. \tag{44}$$

Using hypotheses **(G)**, we determine that **(D)**(i) is satisfied for each $y \in Y, x \rightarrowtail \mathscr{A}_n(y, x)$. By using the facts, $\|J_X(x)\|_X = \|x\|_X$ and

$$\langle J_X(x), x \rangle_X = \|x\|_X^2, \forall x \in X.$$

It is easy to show that $\mathscr{A}_n$ satisfies **(D)**(ii)–(iv). Similarly, $\mathscr{B}_n$ satisfies **(F)**. Consequently, by using Theorem 2, we can argue that Problem 5 has a solution.

(ii) Let $\{(x_n, y_n)\}$ be an arbitrary sequence of solutions of Problem 5. Next, a meticulous calculation yields

$$\begin{aligned} r(\|x_n\|_X, \|y_n\|_Y) &\leq r(\|x_n\|_X, \|y_n\|_Y) + \frac{\varepsilon_n \langle J_X(x_n), x_n \rangle_X}{\|x_n\|_X} \\ &\leq \frac{\varrho_{\mathscr{A}}(1 + \|x_n\|_X + \|y_n\|_Y)\|x_0\|_X}{\|x_n\|_X} + \frac{(\varepsilon_n \|J_X(x_n)\|_{X^*} + \|\gamma\|_{X^*})\|x_0\|_X + \varrho_\varphi(x)}{\|x_n\|_X} \\ &\quad + \varrho_\varphi(x) + \|\gamma\|_{X^*} \\ &= \frac{\varrho_{\mathscr{A}}(1 + \|x_n\|_X + \|y_n\|_Y)\|x_0\|_X}{\|x_n\|_X} + \frac{(\varepsilon_n \|x_n\|_X + \|\gamma\|_{X^*})\|x_0\|_X + \varrho_\varphi(x)}{\|x_n\|_X} \\ &\quad + \varrho_\varphi(x) + \|\gamma\|_{X^*}, \end{aligned} \tag{45}$$

and

$$\begin{aligned} \ell(\|y_n\|_Y, \|x_n\|_X) \leq &\frac{\varrho_{\mathscr{B}}(1 + \|x_n\|_X + \|y_n\|_Y)\|y_0\|_Y}{\|y_n\|_Y} + \frac{(\delta_n \|y_n\|_Y + \|\zeta\|_{Y^*})\|y_0\|_Y + \varrho_\phi(y)}{\|y_n\|_Y} \\ &+ \varrho_\phi(y) + \|\zeta\|_{Y^*}. \end{aligned} \tag{46}$$

The same arguments used in the proof of Lemma 3 that $\{(x_n, y_n)\}$ is bounded in $X \times Y$.
If necessitated we can proceed to a relabeled subsequence and presume that

$$(x_n, y_n) \xrightarrow{w} (x, y) \in X \times Y \text{ as } n \longrightarrow \infty, \text{ for some } (x, y) \in \Omega \times \mho. \tag{47}$$

By using the monotonicity of $x \rightarrowtail \mathscr{A}(y, x)$ and $y \rightarrowtail \mathscr{B}(x, y)$, we can make the following deduction

$$\langle \mathscr{A}(y_n, v) + \varepsilon_n J_X(x_n), v - x_n \rangle_X + \varphi(x_n, v) - \varphi(x_n, x_n) \geq \langle \gamma, v - x_n \rangle_X, \forall v \in \Omega, \tag{48}$$

and

$$\langle \mathscr{B}(x_n, w) + \delta_n J_Y(y_n), w - y_n \rangle_Y + \phi(y_n, w) - \phi(y_n, y_n) \geq \langle \zeta, w - y_n \rangle_Y, \forall w \in \mho. \tag{49}$$

Taking the upper limit as $n \longrightarrow \infty$ and using hypotheses **(D)**(ii) and **(F)**(ii), we infer that

$$\langle \mathscr{A}(y, v), v - x \rangle_X + \varphi(x, v) - \varphi(x, x) \geq \langle \gamma, v - x \rangle_X, \forall v \in \Omega,$$

and

$$\langle \mathscr{B}(x, w), w - y \rangle_Y + \phi(y, w) - \phi(y, y) \geq \langle \zeta, w - y \rangle_Y, \forall w \in \mho,$$

here we used the boundedness of $\{(x_n, y_n)\} \in X \times Y$. Using Minty approach, we find $(x, y) \in \Omega \times \mho$ solve Problem 1, *i.e.*,

$$(x, y) \in \Gamma(\gamma, \zeta).$$

(iii) It can be deduced from assertion (ii) that if we have a sequence of solutions denoted by $\{(x_n, y_n)\}$ for Problem 5, there will always exist a subsequence of $\{(x_n, y_n)\}$ which satisfies (42). We claim that the sequence $\{(x_n, y_n)\}$ converges strongly to (x, y). It is simple to demonstrate that:

$$\begin{aligned}
-\alpha_{\mathscr{A}} \beta_{\mathscr{A}} \|x_n - x\|_X^2 &\leq \langle \mathscr{A}(y_n, x_n) - \mathscr{A}(y_n, x), x_n - x \rangle_X \\
&\leq \langle \mathscr{A}(y, x) - \mathscr{A}(y_n, x), x_n - x \rangle_X + \varepsilon_n \langle J_X(x_n), x - x_n \rangle_X \\
&\Longrightarrow \\
\alpha_{\mathscr{A}} \beta_{\mathscr{A}} \|x_n - x\|_X^2 &\leq \langle \mathscr{A}(y, x) - \mathscr{A}(y_n, x), x - x_n \rangle_X + \varepsilon_n \langle J_X(x_n), x_n - x \rangle_X.
\end{aligned} \tag{50}$$

By using hypothesis **(D)**(ii) and taking the upper limit as n approaches infinity on the above inequality, we obtain

$$\begin{aligned}
0 &\leq \lim_{n \longrightarrow \infty} \inf \alpha_{\mathscr{A}} \beta_{\mathscr{A}} \|x_n - x\|_X^2 \\
&\leq \lim_{n \longrightarrow \infty} \sup \|x_n - x\|_X^2 \\
&\leq \langle \mathscr{A}(y, x) - \mathscr{A}(y_n, x), x - x_n \rangle_X + \lim_{n \longrightarrow \infty} \sup \varepsilon_n \|x_n\|_X \|x_n - x\|_X \\
&\leq 0.
\end{aligned}$$

This implies that

$$x_n \longrightarrow x \in X \text{ as } n \longrightarrow \infty.$$

On the other hand, it has

$$y_n \longrightarrow y \in Y \text{ as } n \longrightarrow \infty.$$

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□

4. Optimal Control

In this section, we explore optimal control for the system of nonlinear mixed variational inequality problems. Additionally, we consider an optimal control problem driven by the system of nonlinear mixed variational inequality problems and prove its solvability.

Consider two Banach spaces Z_1 and Z_2 with continuous embeddings from X to Z_1 and from Y to Z_2. Let $x_0 \in Z_1$ and $y_0 \in Z_2$ be two target profiles. We define subspaces $U \subset X^*$ and $V \subset Y^*$ such that the embeddings from U to X^* and V to Y^* are compact. Now, we investigate the following optimal control problem:

Problem 6. *Find $(\gamma^*, \zeta^*) \in U \times V$ such that*

$$\daleth(\gamma^*, \zeta^*) = \inf_{(\gamma,\zeta)\in U\times V} \daleth(\gamma, \zeta), \tag{51}$$

where the cost function $\daleth: U \times V \longrightarrow \mathbb{R}$ is defined by

$$\daleth(\gamma, \zeta) = \inf_{(x,y)\in\Gamma(\gamma,\zeta)} \left(\frac{\rho}{2}\|x - x_0\|^2_{Z_1} + \frac{\theta}{2}\|y - y_0\|^2_{Z_2} \right) + \Upsilon(\gamma, \zeta). \tag{52}$$

Here, $\Gamma(\gamma, \zeta)$ represents the solution set of Problem 1 for $(\gamma, \zeta) \in X^* \times Y^*$, with $\rho > 0$ and $\theta > 0$ as regularized parameters.

We assume that the function Υ satisfies the following conditions:

(K): $\Upsilon : U \times V \longrightarrow \mathbb{R}$ is such that

(i) Υ is bounded from below;
(ii) Υ is coercive on $U \times V$, namely it holds

$$\lim_{\substack{(\gamma,\zeta)\in U\times V \\ \|\gamma\|_U + \|\zeta\|_V \longrightarrow\infty}} \Upsilon(\gamma, \zeta) \longrightarrow +\infty;$$

(iii) Υ is weakly lower semicontinuous on $U \times V$, *i.e.*,

$$\lim_{n\longrightarrow\infty} \inf \Upsilon(\gamma_n, \zeta_n) \geq \Upsilon(\gamma, \zeta),$$

whenever $\{(\gamma_n, \zeta_n)\} \subset U \times V$ and $(\gamma, \zeta) \in U \times V$ are such that

$$(\gamma_n, \zeta_n) \xrightarrow{w} (\gamma, \zeta) \in U \times V \text{ as } n \longrightarrow \infty.$$

In this context, we are exploring the existence result for Problem 6.

Theorem 6. *Assume that **(A)**, **(B)**, **(C)**, **(D)**(ii)–(iv), **(E)** and **(F)**(ii)–(iv) hold. If, in addition, **(K)** and **(G)** are fulfilled. Then Problem 6 has an optimal control pair.*

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Proof. For each fix $(\gamma,\zeta)\in U\times V$, the closedness of $\Gamma(\gamma,\zeta)$ (*see* Theorem 2) guarantees that there exists $(\hat{x},\hat{y})\in\Gamma(\gamma,\zeta)$ such that

$$\frac{\rho}{2}\|\hat{x}-x_0\|_{Z_1}^2+\frac{\theta}{2}\|\hat{y}-y_0\|_{Z_2}^2=\inf_{(x,y)\in\Gamma(\gamma,\zeta)}\left(\frac{\rho}{2}\|x-x_0\|_{Z_1}^2+\frac{\theta}{2}\|y-y_0\|_{Z_2}^2\right), \tag{53}$$

is attainable.

From the definition of $\daleth$ and hypothesis **(K)**(i), there exists a minimizing sequence $\{(\gamma_n,\zeta_n)\}\subset U\times V$ such that

$$\lim_{n\longrightarrow\infty}\daleth(\gamma_n,\zeta_n)=\inf_{(\gamma,\zeta)\in U\times V}\daleth(\gamma,\zeta). \tag{54}$$

We assert that the sequence $\{(\gamma_n,\zeta_n)\}$ is bounded in $U\times V$. Arguing by contradiction, we suppose that

$$\|\gamma_n\|_U+\|\zeta_n\|_V\longrightarrow+\infty \ \text{ as } \ n\longrightarrow\infty.$$

The latter together with hypothesis **(K)**(ii), leads to the conclusion:

$$\begin{aligned}\inf_{(\gamma,\zeta)\in U\times V}\daleth(\gamma,\zeta)&=\lim_{n\longrightarrow\infty}\daleth(\gamma_n,\zeta_n)\\&\geq\lim_{n\longrightarrow\infty}\Upsilon(\gamma_n,\zeta_n)\\&=+\infty.\end{aligned} \tag{55}$$

This leads to a contradiction, so, $\{(\gamma_n,\zeta_n)\}$ is bounded in $U\longrightarrow V$. Passing to a relabeled subsequence if necessary, we may assume that

$$(\gamma_n,\zeta_n)\xrightarrow{w}(\gamma^*,\zeta^*)\in U\times V \ \text{ as } \ n\longrightarrow\infty, \ \text{ for some } \ (\gamma^*,\zeta^*)\in U\times V. \tag{56}$$

Let $\{(x_n,y_n)\}\subset\Omega\times\mho$ satisfy (53) by taking $\hat{x}=x_n$, $\hat{y}=y_n$, and $(\gamma,\zeta)=(\gamma_n,\zeta_n)$. We will now prove that $\{(x_n,y_n)\}\subset\Omega\times\mho$ is uniformly bounded in $X\times Y$. A direct computation shows that

$$\begin{aligned}r(\|x_n\|_X,\|y_n\|_Y)\leq&\frac{\varrho_{\mathscr{A}}(1+\|x_n\|_X+\|y_n\|_Y)\|x_0\|_X}{\|x_n\|_X}+\frac{(\varrho_\varphi(x)+\|\gamma_n\|_{X^*})\|x_0\|_X}{\|x_n\|_X}\\&+\varrho_\varphi(x)+\|\gamma_n\|_{X^*},\end{aligned} \tag{57}$$

and

$$\begin{aligned}\ell(\|y_n\|_Y,\|x_n\|_X)\leq&\frac{\varrho_{\mathscr{B}}(1+\|x_n\|_X+\|y_n\|_Y)\|y_0\|_Y}{\|y_n\|_Y}+\frac{(\varrho_\phi(y)+\|\zeta_n\|_{Y^*}\|y_0\|_Y}{\|y_n\|_Y}\\&+\varrho_\phi(y)+\|\zeta_n\|_{Y^*}.\end{aligned} \tag{58}$$

Since the embeddings from U to X^* and from V to Y^* are both continuous. We can apply the same arguments as in the proof of Lemma 3 to infer that $\{(x_n,y_n)\}\subset\Omega\times\mho$ is uniformly bounded in $X\times Y$. Without loss of generality, we assume that

$$(x_n,y_n)\xrightarrow{w}(x^*,y^*)\in X\times Y, \ \text{ and } \ Z_1\times Z_2 \ \text{ as } \ n\longrightarrow\infty, \ \text{ for some } \ (x^*,y^*)\in\Omega\times\mho. \tag{59}$$

Using the Minty approach yields

$$\langle \mathscr{A}(y_n, v), v - x_n \rangle_X + \varphi(x_n, v) - \varphi(x_n, x_n) \geq \langle \gamma_n, v - x_n \rangle_X, \forall v \in \Omega, \tag{60}$$

and

$$\langle \mathscr{B}(x_n, w), w - y_n \rangle_Y + \phi(y_n, w) - \phi(y_n, y_n) \geq \langle \zeta_n, w - y_n \rangle_Y, \forall w \in \mho. \tag{61}$$

The compactness of the embedding from (U, V) into (X^*, Y^*) and (56), implies that

$$(\gamma_n, \zeta_n) \longrightarrow (\gamma^*, \zeta^*) \in X^* \times Y^* \text{ as } n \longrightarrow \infty.$$

Taking the upper limit as $n \longrightarrow \infty$ for inequalities (60)–(61), we gain

$$\langle \mathscr{A}(y^*, v), v - x^* \rangle_X + \varphi(x^*, v) - \varphi(x^*, x^*) \geq \langle \gamma^*, v - x^* \rangle_X, \forall v \in \Omega,$$

and

$$\langle \mathscr{B}(x^*, w), w - y^* \rangle_Y + \phi(y^*, w) - \phi(y^*, y^*) \geq \langle \zeta^*, w - y^* \rangle_Y, \forall w \in \mho,$$

where we used the conditions **(F)**(ii) and **(D)**(ii). Once more using the Minty trick, we get

$$(x^*, y^*) \in \Gamma(\gamma^*, \zeta^*).$$

However, the weak lower semicontinuity of $\| \cdot \|_{Z_1}$ and $\| \cdot \|_{Z_2}$ suggests

$$\frac{\rho}{2}\|x^* - x_0\|_{Z_1}^2 + \frac{\theta}{2}\|y^* - y_0\|_{Z_2}^2 \leq \liminf_{n \longrightarrow \infty} \left[\frac{\rho}{2}\|x_n - x_0\|_{Z_1}^2 + \frac{\theta}{2}\|y_n - y_0\|_{Z_2}^2 \right]. \tag{62}$$

Note that Υ is weakly lower semicontinuous on $U \times V$, it implies

$$\Upsilon(\gamma^*, \zeta^*) \leq \liminf_{n \longrightarrow \infty} \Upsilon(\gamma_n, \zeta_n). \tag{63}$$

Referring to equations (62) and (63), we have

$$\begin{aligned}
\liminf_{n \longrightarrow \infty} \daleth(\gamma_n, \zeta_n) &\geq \liminf_{n \longrightarrow \infty} \inf_{(x,y) \in \Gamma(\gamma_n, \zeta_n)} \left(\frac{\rho}{2}\|x - x_0\|_{Z_1}^2 + \frac{\theta}{2}\|y - y_0\|_{Z_2}^2 \right) + \liminf_{n \longrightarrow \infty} \Upsilon(\gamma_n, \zeta_n) \\
&= \liminf_{n \longrightarrow \infty} \left(\frac{\rho}{2}\|x_n - x_0\|_X^2 + \frac{\theta}{2}\|y_n - y_0\|_Y^2 \right) + \liminf_{n \longrightarrow \infty} \Upsilon(\gamma_n, \zeta_n) \\
&\geq \frac{\rho}{2}\|x^* - x_0\|_{Z_1}^2 + \frac{\theta}{2}\|y^* - y_0\|_{Z_2}^2 + \Upsilon(\gamma^*, \zeta^*), \ \ (\text{where } (x^*, y^*) \in \Gamma(\gamma^*, \zeta^*)) \\
&\geq \inf_{(x,y) \in \Gamma(\gamma^*, \zeta^*)} \left(\frac{\rho}{2}\|x - x_0\|_{Z_1}^2 + \frac{\theta}{2}\|y - y_0\|_{Z_2}^2 \right) + \Upsilon(\gamma^*, \zeta^*) \\
&= \daleth(\gamma^*, \zeta^*).
\end{aligned} \tag{64}$$

We can use equation (64) along with (54) to arrive at the following conclusion:

$$\daleth(\gamma^*, \zeta^*) \leq \inf_{(\gamma,\zeta) \in U \times V} \daleth(\gamma, \zeta),$$

namely (γ^*, ζ^*) is an optimal control pair of Problem 6. □

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