Article

Not peer-reviewed version

Bayesian Identifying One or Two Close Sources by Gaussian Estimates of Planar Location under Double Emission

[Oleg I. Gerasimov](#) *

Posted Date: 26 April 2024

doi: 10.20944/preprints202404.1776.v1

Keywords: separation to the sources location; probability of estimates resolving; Statistical Resolution Limit; Bayesian inference; identification probability of one/two sources

Article

Bayesian Identifying One or Two Close Sources by Gaussian Estimates of Planar Location under Double Emission

Oleg I. Gerasimov

Independent researcher; helgi@inbox.ru

Abstract: When separation to the parameters of interest appears below resolution limit of the estimator, the ambiguity arises whether two parameter estimates relate to one source emitted twice or two close sources emitted once. In the paper, novel Bayes technique aimed to identify one/two sources below resolution limit by the pair of Gaussian estimates of a source(s) planar location as parameter is developed. Prior probabilities of the hypotheses on one/two sources are available from the analysis of physical characteristics of the emissions, assuming that they can be equally probable. The identifier recalculates a posteriori these probabilities subject to a distance between sources. It is applied to distinguish two location estimates obtained in the planar time difference of arrival mobile communication network. The work of the identifier is studied in the domain of closely spaced network users where covariance matrices of estimates are sufficiently approximated by the constant. The application gives an example of how the identifier revises the prior probabilities and can change thereby the initial preference of a hypothesis with a distance between users in some cases.

Keywords: separation to the sources location; probability of estimates resolving; Statistical Resolution Limit; Bayesian inference; identification probability of one/two sources

---

1. Introduction

Having two parameter estimates the problem to identify whether single source emits twice or two sources at some distance between parameters below resolution limit emit once takes place in application of the resolution methods. The resolution limit of the estimator is described by the Statistical Resolution Limit (SRL) defined as the minimal separation at which estimates are resolved correctly [1]. If the separation is less than SRL, then there will exist ambiguity about one or two sources. The concepts of SRL are primarily formulated in the detection theory [2–5] and the estimation accuracy approach [1], [6–8]. Herein, we rely on the estimation accuracy utilizing the Cramer-Rao Bound ($\mathbf{CRB}$) matrix-function of parameter, which under mild conditions represents a narrow lower bound on covariance matrix of any unbiased estimator [9]. The equality to $\mathbf{CRB}$ is achieved in the class of asymptotically efficient estimators to which we will address in the paper. While true parameter(s) is unknown, $\mathbf{CRB}$ can be well approximated by the same $\mathbf{CRB}$ matrix-function but taken at parameter estimate to gain its covariance matrix and thereby sufficient approximation of the SRL in the domain of small estimation errors.

We propose novel Byes formalism which can identify one or two sources below SRL by two planar location estimates and their covariance matrices. A handling with positional estimates on the plane is relevant in mobile communication, astrometry, microscopy and many other fields.

The sources have different spectrum, power, others features of the signal but they may be identical. The existing approaches in identifying closely spaced sources typically invoke the probability of number of sources – either from subjective expert experience or as posterior one obtained from Bayes inference, which operates with physical characteristics of the emissions. We come across the closest example of Bayes signal classification in radar [10], seismology [11] and so on, where, having the probabilities of each class for each emission, it is not difficult to deduce probabilities of all possible number of sources for a given number of emissions using probability

 doi:10.20944/preprints202404.1776.v1

theorems. In infrared optics, Bayesian calculation is applied to obtain probability of number of neighboring screen images to overcome the smearing effect [12]. A similar problem arises in localization of blinking objects in microscopy where Bayesian Information Criteria are to achieve probability of number of objects for the analysis below the diffraction limit [13]. We will refer to a solution that is able to support the required probabilities of one and two sources as the prior solution (PS). Prior probabilities in the Bayesian formalism will be the ones extracted from PS.

When resolving, the separation $|p_1 - p_2|$ to the parameters p_1 and p_2 is not to be less than some quantitative measure of random distance scattering between their estimates: $|p_1' - p_2'|$. For the design of identifier, resolution criteria are reformulated by the means of the notation of the resolving inequality: $|p_1' - p_2'| \le SRL$ (RI) in order to catch the probability that distance between parameter estimates is not bigger than SRL. Following this reformulation we propose new SRL concept for planar decoupled estimates when confidence circles of high (near to unity) probability around each of the two parameters are tangent. The concept provides approximately the same probability of the RI (PRI) as in scalar resolution criteria in the widest range of covariance matrix spectrum of difference between estimates.

Two mutually exclusive events constitute the Bayesian sample space: confidence circles around each of the sample estimates are either disjoint or intersectional if the distance between estimates is over or below SRL. Bayesian technique recalculates a posteriori given prior probabilities with regard to presumed distance between true locations. Doing that, it can change the probabilistic decision induced by PS. To the best of our knowledge, no solution aimed to discriminate single source emitted twice and two close sources emitted once for a given pair of location estimates, which would be parameterized by the distance between hypothesized sources is mentioned previously.

We illustrate the job of the Bayesian formalism by the example of distinguishing two positional estimates between one and two close each other users, obtained in the basic stations (BSs) network by the means of time difference of arrival (TDOA) technique. The signals are there classified in such a way that probabilities of one and two users can be derived. The solution must answer the following question: does one user emit twice or do two users at varying distance between them, carry out emissions consecutively. The identifier implementation is based on the radius of high confidence circle obtained in the paper for the confidence probability of 0.99. This radius will subsequently be denoted as $R99$.

The paper is organized as follows. In Section 2 the resolution criteria are surveyed. The problem is described in Section 3. New concept of SRL is founded in Section 4. Section 5 contains the design of Bayesian identifier. The algorithm for estimation of $R99$ is presented in Section 6. The application of the identifier in the BSs network is quantitatively studied in Section 7. Section 8 briefly draws the summary of the paper.

2. The Works Related to Resolution Criteria

One of the earliest was Lee resolution criterion [6] for scalar decoupled estimates, in our reformulation $|p_1' - p_2'| \le SRL_L$, $SRL_L = q\max\{\sigma_1, \sigma_2\}$, $\sigma_1 = \sqrt{CRB(p_1)}$, $\sigma_2 = \sqrt{CRB(p_2)}$, where σ_1, σ_2 are standard deviations of estimates p_1', p_2', which provide the resolving with "high probability" due to big factor $q = 8$: it is evident that the bigger q is chosen, the higher the probability is that a separation between estimates is not bigger than SRL_L (authors referencing to Lee consider $q = 2$). Delmas and Abeida [7] expressed $SRL_{D\&A}$ via standard deviation of the estimates difference: $SRL_{D\&A} = \sigma_{p_1' - p_2'} = \sqrt{\sigma_1^2 + \sigma_2^2}$, hence the probability of the inequality $|p_1' - p_2'| \le SRL_{D\&A}$ is smaller than with Lee, because inequality $SRL_{D\&A} < SRL_L$ being satisfied even at $q=2$. Regarding coupled estimates, Smith [1] offered

$SRL_S = \sigma_{p_1'-p_2'} = \sqrt{\sigma_1^2+\sigma_2^2-2\mathrm{cov}\left(p_1',p_2'\right)}$, $\mathrm{cov}\left(p_1',p_2'\right)=CRB\left(p_1,p_2\right)$, where $CRB\left(p_1,p_2\right)$ has been interpreted in [8] as an element of extended **CRB** matrix by using the change of variable formula. Both $SRL_{D\&A}$ and SRL_S are articulated in standard deviation of a difference between estimates. This fact offers an opportunity to achieve PRI in a scalar case for a known distribution of estimates.

In the work of Korso, Boyer, Renaux and Marcos [8] the general problem for multiple estimates was considered. Therein, $SRL = CRB\left(\delta\right)$, where separation δ between sets of parameters is the κ-norm distance (Minkowsky distance). Let us represent two scalar parameters $p^{(1)}$, $p^{(2)}$ in a vector form, $\boldsymbol{p}=\left[p^{(1)},p^{(2)}\right]^T$ and denote estimates of vector $\boldsymbol{p}$ obtained from the decoupled emissions *1* and *2* as $\boldsymbol{p}_1'$, $\boldsymbol{p}_2'$ respectively. Then, the RI is written for coupled estimates in metric of 1-norm ($\kappa=1$) of a difference vector $\Delta\boldsymbol{p} = \boldsymbol{p}_1' - \boldsymbol{p}_2'$ as $\delta\left(\Delta\boldsymbol{p}\right)=\left\|\Delta\boldsymbol{p}\right\|_1=\left|p_1'^{(1)}-p_2'^{(1)}\right|+\left|p_1'^{(2)}-p_2'^{(2)}\right|\le SRL_{K,B,R\&M}$,

$SRL^2_{K,B,R\&M}=\left(\sigma_1^{(1)}\right)^2+\left(\sigma_2^{(1)}\right)^2-2\mathrm{cov}\left(p_1'^{(1)},p_2'^{(1)}\right)+\left(\sigma_1^{(2)}\right)^2+\left(\sigma_2^{(2)}\right)^2-2\mathrm{cov}\left(p_1'^{(2)},p_2'^{(2)}\right)$, where variances and covariance of estimates are defined by the means of underlying CRB functions at true parameters $p^{(1)}$ and $p^{(2)}$, which are obtained by applying change of variable formula to the extended measurement model of the estimator. The result distinctively extends scalar SRL on vector case, however PRI , i.e. the probability of $\delta\left(\Delta\boldsymbol{p}\right)\le SRL_{K,B,R\&M}$, could hardly be determined here in comparison with scalar resolution criteria.

Clark [14], analyzing the resolution of estimator that produces vector decoupled Gaussian estimates, presented the metric of a distance between ellipsoidal confidence regions of probability 0.9 around each of the two parameters. The estimates are resolvable when ellipsoids are disjoint. SRL of the estimator is the metric from which they are tangent.

In the study, we transform Clark's idea about the ellipsoidal confidence region to develop the concept of SRL based on circular one of high confidence probability which is probabilistically consistent with conventional resolution criteria [1,7]. As with Clark, the estimates will be resolvable when the circles are disjoint, but their tangency will correspond to the SRL equal to the sum of their radii.

3. Statement of the Problem

During the two consecutive time intervals $t^I \in T^I$ and $t^{II}\in T^{II}$, $T^I \cap T^{II}=\varnothing$, L signals from emission *I* and L signals from emission *II* are received. The $k-$th signal depends upon vector parameter $\boldsymbol{q}$, $\boldsymbol{q}\in\left\{\boldsymbol{q}^I,\boldsymbol{q}^{II}\right\}$, which gathers physical characteristics of the emissions, and $m-$ dimensional measurement parameter $\boldsymbol{p}_k\left(\boldsymbol{\varphi}\right)$ which is known smooth function of two-dimensional unknown vector of the source location, $\boldsymbol{\varphi}\in\left\{\boldsymbol{\varphi}^I,\boldsymbol{\varphi}^{II}\right\}$:

$$\boldsymbol{V}\left(t^I\right)=\left[\boldsymbol{v}^T\left(t^I;\boldsymbol{q}^I,\boldsymbol{p}_1\left(\boldsymbol{\varphi}^I\right)\right),\ldots,\boldsymbol{v}^T\left(t^I;\boldsymbol{q}^I,\boldsymbol{p}_L\left(\boldsymbol{\varphi}^I\right)\right)\right]^T,$$

$$\boldsymbol{V}\left(t^{II}\right)=\left[\boldsymbol{v}^T\left(t^{II};\boldsymbol{q}^{II},\boldsymbol{p}_1\left(\boldsymbol{\varphi}^{II}\right)\right),\ldots,\boldsymbol{v}^T\left(t^{II};\boldsymbol{q}^{II},\boldsymbol{p}_L\left(\boldsymbol{\varphi}^{II}\right)\right)\right]^T, \quad (1)$$

$$\boldsymbol{v}\left(t;\boldsymbol{q},\boldsymbol{p}_k\left(\boldsymbol{\varphi}\right)\right)=\boldsymbol{g}\left(t;\boldsymbol{q},\boldsymbol{p}_k\left(\boldsymbol{\varphi}\right)\right)+\boldsymbol{n}_k\left(t\right),\ t\in\left\{T^I,T^{II}\right\},$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 April 2024 doi:10.20944/preprints202404.1776.v1

where $\boldsymbol{g}(t;\boldsymbol{q},\boldsymbol{p}_k)$ is a waveform vector-function of time and parameters $\boldsymbol{q}$, $\boldsymbol{p}_k$; $\boldsymbol{n}_k(t)$ is unbiased noise with $E\{\boldsymbol{n}_i(t^I)\boldsymbol{n}_j^T(t^{II})\}=\mathbf{0}$ for the all $i,j\le L$, $E\{\cdot\}$ denotes mathematical expectation operator.

Unbiased estimator $\Im_p$ establishes the inverse connection between signals (1) and related to emissions *I* and *II* measurement vectors $\boldsymbol{p}'$ and $\boldsymbol{p}''$, $\tilde{\boldsymbol{p}}=\Im_p\{V(t)\}$, $\tilde{\boldsymbol{p}}\in\{\boldsymbol{p}',\boldsymbol{p}''\}$:

$$\tilde{\boldsymbol{p}}=\boldsymbol{p}(\boldsymbol{\varphi})+\boldsymbol{\upsilon},\ \boldsymbol{p}(\boldsymbol{\varphi})=\left[\boldsymbol{p}_1^T(\boldsymbol{\varphi}),\ldots,\boldsymbol{p}_{L_p}^T(\boldsymbol{\varphi})\right]^T,\ L_p\le L,$$
$$\boldsymbol{\upsilon}\in\{\boldsymbol{\upsilon}',\boldsymbol{\upsilon}''\},\ \boldsymbol{\upsilon}'=\left[\boldsymbol{\upsilon}_1'^T,\ldots,\boldsymbol{\upsilon}_{L_p}'^T\right]^T,\ \boldsymbol{\upsilon}''=\left[\boldsymbol{\upsilon}_1''^T,\ldots,\boldsymbol{\upsilon}_{L_p}''^T\right]^T, \tag{2}$$
$$\boldsymbol{\upsilon}_k=\left[\upsilon_{k1},\ldots\upsilon_{km}\right]^T,\ k=\overline{1,L_p},$$

where $\boldsymbol{\upsilon}'$ and $\boldsymbol{\upsilon}''$ are the measurement errors corresponding to emissions *I* and *II* with $E\{\boldsymbol{\upsilon}'\boldsymbol{\upsilon}''^T\}=\mathbf{0}$, i.e. $\boldsymbol{p}'$ and $\boldsymbol{p}''$ are decoupled.

Unbiased estimator $\Im_\varphi$ uses measurements (2) to produce estimates $\boldsymbol{\varphi}'$ and $\boldsymbol{\varphi}''$ over emissions *I* and *II*: $\tilde{\boldsymbol{\varphi}}=\Im_\varphi(\tilde{\boldsymbol{p}})$, $\tilde{\boldsymbol{\varphi}}\in\{\boldsymbol{\varphi}',\boldsymbol{\varphi}''\}$, $\tilde{\boldsymbol{\varphi}}=\boldsymbol{\varphi}+\boldsymbol{\varsigma}$, $\boldsymbol{\varsigma}\in\{\Delta\boldsymbol{\varphi}',\Delta\boldsymbol{\varphi}''\}$, where $\Delta\boldsymbol{\varphi}'$ and $\Delta\boldsymbol{\varphi}''$ are the corresponding errors, which are also decoupled.

We consider (***a***) "regular enough" algorithms $\Im_p$ and $\Im_\varphi$ such that $\tilde{\boldsymbol{p}}$ and $\tilde{\boldsymbol{\varphi}}$ are both normally distributed [15]: $\tilde{\boldsymbol{p}}\sim\mathrm{N}(\boldsymbol{p},\boldsymbol{\Pi})$, $\tilde{\boldsymbol{\varphi}}\sim\mathrm{N}(\boldsymbol{\varphi},\boldsymbol{\Phi}(\boldsymbol{\varphi}))$ with a given covariance matrix $\boldsymbol{\Pi}$ of estimates $\tilde{\boldsymbol{p}}$ and an unknown one $\boldsymbol{\Phi}(\boldsymbol{\varphi})$ of estimates $\tilde{\boldsymbol{\varphi}}$. The estimators $\Im_p$ and $\Im_\varphi$ are assumed (***b***) to be efficient, hence $\boldsymbol{\Phi}$ is calculated as $\mathbf{CRB}$ by use of Fisher Information Matrix $\mathbf{FIM}(\boldsymbol{\varphi})$: $\boldsymbol{\Phi}(\boldsymbol{\varphi})=\mathbf{CRB}(\boldsymbol{\varphi})=\mathbf{FIM}^{-1}(\boldsymbol{\varphi})$. For Gaussian noise, $\mathbf{FIM}(\boldsymbol{\varphi})=\left[\dfrac{\partial\boldsymbol{p}(\boldsymbol{\varphi})}{\partial\boldsymbol{\varphi}}\right]^T\boldsymbol{\Pi}^{-1}\left[\dfrac{\partial\boldsymbol{p}(\boldsymbol{\varphi})}{\partial\boldsymbol{\varphi}}\right]$.

Hypotheses H_α and H_β serve to specify the sources of emissions: hypothesis H_α means that one source at $\boldsymbol{\varphi}_1=\boldsymbol{\varphi}^I=\boldsymbol{\varphi}^{II}$ with $\boldsymbol{q}_1=\boldsymbol{q}^I=\boldsymbol{q}^{II}$ emits twice, hypothesis H_β – that each of the two sources at $\boldsymbol{\varphi}_1=\boldsymbol{\varphi}^I$ and $\boldsymbol{\varphi}_2=\boldsymbol{\varphi}^{II}$ with $\boldsymbol{q}_1=\boldsymbol{q}^I$ and $\boldsymbol{q}_2=\boldsymbol{q}^{II}$ emit once. The probabilities $P\{\mathrm{H}_\alpha\}$ and $P\{\mathrm{H}_\beta\}$ of the hypotheses come from a preceding step of data processing where parameter $\boldsymbol{q}$ can be involved.

We define the circles $\mathbf{S}_1=\{\boldsymbol{\varphi}:\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_1\|\le R_1\}$ and $\mathbf{S}_2=\{\boldsymbol{\varphi}:\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_2\|\le R_2\}$, $\|\bullet\|$ denotes Euclidian norm, with radii R_1 and R_2 where estimates $\boldsymbol{\varphi}'$ and $\boldsymbol{\varphi}''$ fall with high probability P_S: $\tilde{\boldsymbol{\varphi}}\in\mathbf{S}_1$ for H_α and $\boldsymbol{\varphi}'\in\mathbf{S}_1$, $\boldsymbol{\varphi}''\in\mathbf{S}_2$ for H_β. Due to negligible probability $1-P_S^2$ of outliers beyond circles we will treat estimates $\boldsymbol{\varphi}'$ and $\boldsymbol{\varphi}''$ so that as if $\boldsymbol{\varphi}'$, $\boldsymbol{\varphi}''\in\mathbf{S}_1$ or $\boldsymbol{\varphi}'\in\mathbf{S}_1$, $\boldsymbol{\varphi}''\in\mathbf{S}_2$. Assumption (***c***) on small estimation errors is quantified inside the confidence circles as $\boldsymbol{\Phi}(\tilde{\boldsymbol{\varphi}})\approx\boldsymbol{\Phi}(\boldsymbol{\varphi}_1)$ at $\tilde{\boldsymbol{\varphi}}\in\mathbf{S}_1$ and $\boldsymbol{\Phi}(\boldsymbol{\varphi}'')\approx\boldsymbol{\Phi}(\boldsymbol{\varphi}_2)$ at $\boldsymbol{\varphi}''\in\mathbf{S}_2$.

The identification is performed in the area where positions $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ will be placed at the distance of not more than $R_S+\delta R_S$, where $R_S=R_1+R_2$, $\delta R_S<R_S$. We take the assumption (***d***) that $\boldsymbol{\Phi}(\boldsymbol{\varphi}_2)$ is sufficiently approximated by a constant for the all locations $\boldsymbol{\varphi}_2$ from the annulus $A=\{\boldsymbol{\varphi}_2:R_1<\|\boldsymbol{\varphi}_2-\boldsymbol{\varphi}_1\|\le R_S+\delta R_S\}$. Thus, it is permissible to use in that area sample estimates $\hat{\boldsymbol{\varphi}}'$ and

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 April 2024 doi:10.20944/preprints202404.1776.v1

$\hat{\boldsymbol{\varphi}}''$ of $\boldsymbol{\varphi}'$ and $\boldsymbol{\varphi}''$ for the calculation of sample approximations $\hat{\mathbf{\Phi}}' \equiv \mathbf{\Phi}\left(\hat{\boldsymbol{\varphi}}'\right)$, $\hat{\mathbf{\Phi}}'' \equiv \mathbf{\Phi}\left(\hat{\boldsymbol{\varphi}}''\right)$ of unknown true covariance matrix or matrices. As shown in the application, the stronger condition $\hat{\mathbf{\Phi}}' \approx \hat{\mathbf{\Phi}}''$ will be there fulfilled in the specific domain of closely spaced sources with a good accuracy.

Our purpose is to find the probabilities of that estimates $\boldsymbol{\varphi}'$ and $\boldsymbol{\varphi}''$ refer to one source in position $\boldsymbol{\varphi}_1$ and two sources in positions $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ for a given separation $r = \|\boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_2\|$ on conditions (***a***), (***b***), (***c***) and (***d***).

4.. Concept Based on High Confidence Circle

In the frame of this Section, random variables $\boldsymbol{\varphi}'$ and $\boldsymbol{\varphi}''$ are considered as just the Gaussian two-dimensional estimates, not necessarily of location coordinate, with $E\{\boldsymbol{\varphi}'\} = \boldsymbol{\varphi}_1$ and $E\{\boldsymbol{\varphi}''\} = \boldsymbol{\varphi}_2$. The difference between estimates $\boldsymbol{\psi} = \boldsymbol{\varphi}'' - \boldsymbol{\varphi}' = \bar{\boldsymbol{\psi}} + \Delta\boldsymbol{\psi}$, $\bar{\boldsymbol{\psi}} = \boldsymbol{\varphi}_2 - \boldsymbol{\varphi}_1$, $\Delta\boldsymbol{\psi} = \Delta\boldsymbol{\varphi}'' - \Delta\boldsymbol{\varphi}'$, is Gaussian with a mean $\bar{\boldsymbol{\psi}}$ and covariance matrix $\mathbf{W}_{\psi} = \mathbf{\Phi}(\boldsymbol{\varphi}_1) + \mathbf{\Phi}(\boldsymbol{\varphi}_2)$ with eigenvalues $l_1 > l_2 > 0$. We propose the concept of *SRL* based on a tangency of two confidence circles $\mathbf{S}_1$ and $\mathbf{S}_2$ of probability P_S when $\|\boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_2\| = R_S$, which comes from the following Theorem.

Theorem.

$$0.5P_S^2 - \frac{1}{\pi}\left[\frac{\pi}{2} + arctg\left(\sqrt{\frac{l_1}{l_2}}tg\frac{2\pi}{3}\right)\right] < P\left\{\left(\|\boldsymbol{\psi}\| \,\|\bar{\boldsymbol{\psi}}\| = R_S\right) \le R_S\right\} < 0.5P_S^2$$

Proof of Theorem.

1. The region $\boldsymbol{\Sigma}$ encompassing $\bar{\boldsymbol{\psi}}$ where $\boldsymbol{\psi}$ falls with probability P_S^2 in polar coordinates (r'', ϑ''), (r', ϑ') originated correspondently in $\boldsymbol{\varphi}_2$ and $\boldsymbol{\varphi}_1$ is $\boldsymbol{\Sigma}$ = $= \left\{\bar{\boldsymbol{\psi}} + \begin{bmatrix} r''\cos\vartheta'' - r'\cos\vartheta' \\ r''\sin\vartheta'' - r'\sin\vartheta' \end{bmatrix}\right\}$, $r'' \in (0, R_2]$, $r' \in (0, R_1]$ and ϑ'', $\vartheta' \in [0, 2\pi]$. The squared distance from $\bar{\boldsymbol{\psi}}$ to any point of $\boldsymbol{\Sigma}$ is $(r'')^2 + (r')^2 - 2r''r'\cos(\vartheta'' - \vartheta')$ maximum of which is achieved at $r'' = R_2$, $r' = R_1$ for $\vartheta'' - \vartheta' = \pi$ and is equal to $(R_1 + R_2)^2$. Consequently, $\boldsymbol{\Sigma}$ is a circle centered at $\bar{\boldsymbol{\psi}}$ with radius R_S and bounded by the circumference $\boldsymbol{\Sigma}_C$ including origin of coordinates.

2. The region $\boldsymbol{\Xi}$ where variable $\boldsymbol{\psi}$ falls with desired probability is formed by the intersection of the circles $\boldsymbol{\Sigma}$ and $\boldsymbol{\Lambda} = \{\boldsymbol{\psi} : \|\boldsymbol{\psi}\| \le R_S\}$, so long as $\boldsymbol{\psi} \in \boldsymbol{\Sigma}$, see Figure 1a. Straight line L connecting the intersection points of corresponding circumferences $\boldsymbol{\Sigma}_C$ and $\boldsymbol{\Lambda}_C$ is collinear in virtue of symmetry to the tangent T to $\boldsymbol{\Lambda}_C$ at the point $\bar{\boldsymbol{\psi}}$. It divides the line from origin to $\bar{\boldsymbol{\psi}}$ on two identical segments, hence the angle θ is equal to $\pi/6$, see Figure 1a again. We successively translate $\bar{\boldsymbol{\psi}}$ in origin and rotate coordinates to obtain probability integral for $\boldsymbol{\Xi}$ in terms of spectrum of $\mathbf{W}_{\psi}$. Changing to polar coordinates (s, ϕ) we get

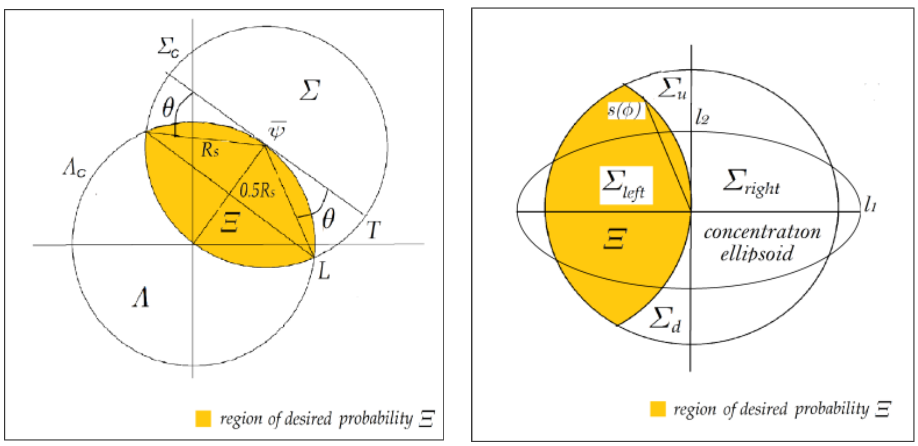
$$P\{\boldsymbol{\psi} \in \boldsymbol{\Xi}\} = \frac{1}{2\pi\sqrt{l_1 l_2}} \int\limits_{\pi/2+\pi/6}^{3\pi/2-\pi/6} d\phi \int\limits_0^{s(\phi)} s\exp\left[-\frac{s^2 g(\phi; l_1, l_2)}{2}\right] \mathrm{d}s\,, \tag{3}$$

where $g(\phi;l_1,l_2)\equiv\left(l_1^{-1}\cos^2\phi+l_2^{-1}\sin^2\phi\right)$. The ordinate divides $\boldsymbol{\Sigma}$ on two semicircles $\boldsymbol{\Sigma}_{left}$ and $\boldsymbol{\Sigma}_{right}$, but two identical regions $\boldsymbol{\Sigma}_u$ and $\boldsymbol{\Sigma}_d$ associated with $\pi/6$ complement $\boldsymbol{\Xi}$ to semicircle $\boldsymbol{\Sigma}_{left}$, Figure 1b. From the symmetry of probability density in integral (3) with respect to angle ϕ one has $P\{\boldsymbol{\psi}\in\boldsymbol{\Sigma}_{left}\}=P\{\boldsymbol{\psi}\in\boldsymbol{\Sigma}_{right}\}=0.5P_S^2$ and $P\{\boldsymbol{\psi}\in\boldsymbol{\Sigma}_u\}=P\{\boldsymbol{\psi}\in\boldsymbol{\Sigma}_d\}$.

3. We represent probability (3) as

$$P\{\boldsymbol{\psi}\in\boldsymbol{\Xi}\}=0.5P_S^2-\frac{1}{\pi\sqrt{l_1l_2}}\int_{\pi/2}^{2\pi/3}\mathrm{d}\phi\int_0^{s(\phi)}s\exp\left[-\frac{s^2g(\phi;l_1,l_2)}{2}\right]\mathrm{d}s, \tag{4}$$

where $s(\phi)=2R_S\sin(\phi-\pi/2)=-2R_S\cos\phi$, see Figure 1b again.



(a) (b)

Figure 1. The region $\boldsymbol{\Xi}$ of desired probability: (a) in initial coordinates, (b) in transformed coordinates.

Performing the integral in (4) over variable s leads to

$$P\{\boldsymbol{\psi}\in\boldsymbol{\Xi}\}=0.5P_S^2-\frac{1}{\pi\sqrt{l_1l_2}}\int_{\pi/2}^{2\pi/3}\frac{\left\{1-\exp\left[-2R_S^2g(\phi;l_1,l_2)\cos^2\phi\right]\right\}\mathrm{d}\phi}{g(\phi;l_1,l_2)}.$$

We maximize it over ϕ to evaluate $P\{\boldsymbol{\psi}\in\boldsymbol{\Xi}\}$ from below:

$$P\{\boldsymbol{\psi}\in\boldsymbol{\Xi}\}>0.5P_S^2-\max_{\phi}\left\{1-\exp\left[-2R_S^2g(\phi;l_1,l_2)\cos^2\phi\right]\right\}\frac{1}{\pi\sqrt{l_1l_2}}\int_{\pi/2}^{2\pi/3}\frac{\mathrm{d}\phi}{g(\phi;l_1,l_2)}. \tag{5}$$

It can be easily verified that the derivative of the exponential function from (5) is non-positive, thus required maximum is achieved at $\phi=2\pi/3$ and is equal to $\left(1-\mathrm{e}^{-\frac{R_S^2}{2}\left(\frac{1}{4l_1}+\frac{3}{4l_2}\right)}\right)$. We substitute l_1 with l_2 in exponent and perform integration in (5) [16] to achieve the inequality

$$P\{\boldsymbol{\psi}\in\boldsymbol{\Xi}\}>0.5P_S^2-\left(1-\mathrm{e}^{-\frac{R_S^2}{2l_2}}\right)\frac{1}{\pi}\left[\frac{\pi}{2}+arctg\left(\overline{e}tg\frac{2\pi}{3}\right)\right],$$

where $\overline{e}\equiv\sqrt{l_1/l_2}$. The expression in parentheses with exponent is the probability that realizations of $\boldsymbol{\psi}$ will be in the concentration ellipsoid centered at origin with principal semi-minor axis R_S and

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 April 2024 doi:10.20944/preprints202404.1776.v1

semi-major axis $\overline{e}R_S$. Region $\boldsymbol{\Sigma}$ which contains $\boldsymbol{\psi}$ with probability P_S^2 is the circle inscribed in the ellipsoid, so its confidence probability will be closer to unity than P_S^2. And if so, we substitute it with unity and end the Proof □.

Proceeding from the Theorem, we establish $SRL = R_S$ in RI $\|\boldsymbol{\psi}\| \le SRL$ ensuring $PRI < 0.5P_S^2$, low bound of which depends on spectral characteristic $\overline{e}$. Function $1/2+$ $+\pi^{-1}arctg\left[\overline{e}tg(2\pi/3)\right]$ is monotone decreasing by $\overline{e}$: it approaches $1/6$ at $\overline{e}\to 1$, hence $PRI > 0.4905 - 0.1667 = 0.3238$ when $P_S = 0.99$. In other hand, it approaches zero at $\overline{e}\to\infty$, hence $PRI \to 0.4905$. Since, say $\overline{e} > 10$ we get $PRI > 0.4721$ and can declare that PRI occurs here in a proximity of $1/2$. When $\overline{e}\to 1$ concentration ellipsoid is degenerating into the circle with radius $l = l_1 = l_2$ yet $g(\phi; l_1, l_2) = l^{-1}$ therefore the contribution of $2P\{\boldsymbol{\psi}\in\boldsymbol{\Sigma}_u\}$ in probability $P\{\boldsymbol{\psi}\in\boldsymbol{\Sigma}_{left}\}$ is in fact $2(\pi/6)\times\times 0.5P_S^2/\pi = P_S^2/6$. On the contrary, at $\overline{e}\to\infty$ it elongates along the semi-axis l_1, $l_1 \gg l_2$, and $2P\{\boldsymbol{\psi}\in\boldsymbol{\Sigma}_u\}$ hereby approaches zero.

The consistency of the proposed SRL with the SRL for scalar estimates in the sense of PRI is obvious for stretched concentration ellipsoid: SRL from [1,7] for scalar ψ is achieved at $|\overline{\psi}| = \sigma_{\Delta\psi}$ with $PRI = P\{-\sigma_{\Delta\psi} \le \psi \le \sigma_{\Delta\psi}\} = 0.477$ which also is not far from $1/2$. In general, PRI lies in the interval (0.3238, 0.4905) at $P_S = 0.99$.

5. Bayesian Identification Formalism

Sample analogues $\hat{\boldsymbol{\Phi}}'$ and $\hat{\boldsymbol{\Phi}}''$ of one or two true covariance matrices are employed to achieve the approximation $\overline{R}_S$ of R_S as $\overline{R}_S = R_S' + R_S''$ where estimates R_S', R_S'' approximate R_1 twice or R_1, R_2 once. Bayes sample space C is defined to consist of two events, when the confidence circles encompassing sample estimates $\hat{\boldsymbol{\varphi}}'$ and $\hat{\boldsymbol{\varphi}}''$ intersect: $\left\|\hat{\boldsymbol{\varphi}}'' - \hat{\boldsymbol{\varphi}}'\right\| \le \overline{R}_S$ at $\mathrm{C} = \mathrm{C}_1$ or do not intersect: $\left\|\hat{\boldsymbol{\varphi}}'' - \hat{\boldsymbol{\varphi}}'\right\| > \overline{R}_S$ at $\mathrm{C} = \mathrm{C}_2$. Probabilities of the events C_1 and C_2 are: $P\{\mathrm{C}_1|\mathrm{H}_\alpha\} = P\left\{\left(\|\boldsymbol{\psi}\|^2 \middle| \overline{\boldsymbol{\psi}} = \mathbf{0}\right) \le \overline{R}_S^2\right\}$ and $P\{\mathrm{C}_1|\mathrm{H}_\beta\} = P\left\{\left(\|\boldsymbol{\psi}\|^2 \middle| \overline{\boldsymbol{\psi}} \ne \mathbf{0}\right) \le \overline{R}_S^2\right\}$, $P\{\mathrm{C}_2|\mathrm{H}_\alpha\} = P\left\{\left(\|\boldsymbol{\psi}\|^2 \middle| \overline{\boldsymbol{\psi}} = \mathbf{0}\right) > \overline{R}_S^2\right\}$ and $P\{\mathrm{C}_2|\mathrm{H}_\beta\} == P\left\{\left(\|\boldsymbol{\psi}\|^2 \middle| \overline{\boldsymbol{\psi}} \ne \mathbf{0}\right) > \overline{R}_S^2\right\}$. Probability $P\{\mathrm{C}_1|\mathrm{H}_\alpha\}$ is very near to unity thus $P\{\mathrm{C}_2|\mathrm{H}_\alpha\}$ is very near to zero. Probability $P\{\mathrm{C}_1|\mathrm{H}_\beta\}$ decreases asymptotically with the rise of $\|\overline{\boldsymbol{\psi}}\|$ while $P\{\mathrm{C}_2|\mathrm{H}_\beta\}$ asymptotically increases. We define all possible $\overline{\boldsymbol{\psi}}$ on the circumference $\mathbb{C}_r = \{\overline{\boldsymbol{\psi}} : \|\overline{\boldsymbol{\psi}}\| = r\}$ in order to pass from unknown $P\{\mathrm{C}|\mathrm{H}_\beta\}(r) \equiv P\{\mathrm{C}|\mathrm{H}_\beta\}\big|_{\overline{\boldsymbol{\psi}}\in\mathbb{C}_r}$ to its minimum $P_{\mathrm{C}}^-(r) = \min\limits_{\overline{\boldsymbol{\psi}}\in\mathbb{C}_r} P\{\mathrm{C}|\mathrm{H}_\beta\}(r)$ and maximum $P_{\mathrm{C}}^+(r) = \max\limits_{\overline{\boldsymbol{\psi}}\in\mathbb{C}_r} P\{\mathrm{C}|\mathrm{H}_\beta\}(r)$ on $\mathbb{C}_r$. To develop Bayesian scheme we need the following Lemma.

Lemma. $P_{\mathrm{C}_1}^-(r) > P_{\mathrm{C}_1}^-(r')$, $P_{\mathrm{C}_1}^+(r) > P_{\mathrm{C}_1}^+(r')$ at $r < r' \le \overline{R}_S$, and $P_{\mathrm{C}_2}^-(r') > P_{\mathrm{C}_2}^-(r)$, $P_{\mathrm{C}_2}^+(r') > P_{\mathrm{C}_2}^+(r)$ at $\overline{R}_S < r < r'$.

Proof of Lemma. For the pair of some vectors $\overline{\boldsymbol{\psi}}_r = r\overline{\boldsymbol{\psi}}_1$ and $\overline{\boldsymbol{\psi}}_{r'} = r'\overline{\boldsymbol{\psi}}_1$, where $\overline{\boldsymbol{\psi}}_1 \in \mathbb{C}_1$, one has $\overline{\boldsymbol{\psi}}_r \in \mathbb{C}_r$ and $\overline{\boldsymbol{\psi}}_{r'} \in \mathbb{C}_{r'}$, and $\|\boldsymbol{\psi}_r\|^2 = r^2 + r2\overline{\boldsymbol{\psi}}_1^T\Delta\boldsymbol{\psi} + \|\Delta\boldsymbol{\psi}\|^2$, $\|\boldsymbol{\psi}_{r'}\|^2 = r'^2 + r'2\overline{\boldsymbol{\psi}}_1^T\Delta\boldsymbol{\psi} + +\|\Delta\boldsymbol{\psi}\|^2$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 April 2024 doi:10.20944/preprints202404.1776.v1

Variances of $r'2\bar{\psi}_1^T\Delta\psi$ and $r2\bar{\psi}_1^T\Delta\psi$ are increased by factor r'^2 and r^2 correspondently as compared to the variance of $2\bar{\psi}_1^T\Delta\psi$, hence due to the properties of Gaussian distribution $P\left\{\left(r^2+r2\bar{\psi}_1^T\Delta\psi\right)\le\bar{R}_S^2\right\}>P\left\{\left(r'^2+r'2\bar{\psi}_1^T\Delta\psi\right)\le\bar{R}_S^2\right\}$ and $P\left\{\left(r'^2+r'2\bar{\psi}_1^T\Delta\psi\right)>\bar{R}_S^2\right\}>P\left\{\left(r^2+r2\bar{\psi}_1^T\Delta\psi\right)>\bar{R}_S^2\right\}$. Thus, $P\left\{\left\|\psi_r\right\|^2\le\bar{R}_S^2\right\}>$ $>P\left\{\left\|\psi_{r'}\right\|^2\le\bar{R}_S^2\right\}$ and $P\left\{\left\|\psi_{r'}\right\|^2>\bar{R}_S^2\right\}>P\left\{\left\|\psi_r\right\|^2>\bar{R}_S^2\right\}$ that completes the Proof □.

As comes from Lemma, $P\{\mathrm{C}_1|\mathrm{H}_\beta\}(r)$ is smaller than $P\{\mathrm{C}_1|\mathrm{H}_\alpha\}$ and approaches it from below at $r\to 0$; both $P_{\mathrm{C}_1}^-(r)$ and $P_{\mathrm{C}_1}^+(r)$ decrease monotonically with the rise of r while $P_{\mathrm{C}_2}^-(r)$ and $P_{\mathrm{C}_2}^+(r)$ monotonically increase.

We apply binary Bayesian theorem to construct maximum $P_{\max}^\beta(\mathrm{C};r)\equiv\ \equiv\max_{\bar{\psi}\in\mathbb{C}_r}P\{\mathrm{H}_\beta|\mathrm{C}\}$ and minimum $P_{\min}^\beta(\mathrm{C};r)\equiv\min_{\bar{\psi}\in\mathbb{C}_r}P\{\mathrm{H}_\beta|\mathrm{C}\}$ of the posterior probability of two sources,

$$P_{\max}^\beta(\mathrm{C};r)=\frac{P\{\mathrm{H}_\beta\}P_\mathrm{C}^+(r)}{P\{\mathrm{H}_\alpha\}P\{\mathrm{C}|\mathrm{H}_\alpha\}+P\{\mathrm{H}_\beta\}P_\mathrm{C}^+(r)},\tag{6}$$

$$P_{\min}^\beta(\mathrm{C};r)=\frac{P\{\mathrm{H}_\beta\}P_\mathrm{C}^-(r)}{P\{\mathrm{H}_\alpha\}P\{\mathrm{C}|\mathrm{H}_\alpha\}+P\{\mathrm{H}_\beta\}P_\mathrm{C}^-(r)}\tag{7}$$

and minimum $P_{\min}^\alpha(\mathrm{C};r)$ and maximum $P_{\max}^\alpha(\mathrm{C};r)$ of the posterior probability of one source,

$$P_{\min}^\alpha(\mathrm{C};r)=\frac{P\{\mathrm{H}_\alpha\}P\{\mathrm{C}|\mathrm{H}_\alpha\}}{P\{\mathrm{H}_\alpha\}P\{\mathrm{C}|\mathrm{H}_\alpha\}+P\{\mathrm{H}_\beta\}P_\mathrm{C}^+(r)},\tag{8}$$

$$P_{\max}^\alpha(\mathrm{C};r)=\frac{P\{\mathrm{H}_\alpha\}P\{\mathrm{C}|\mathrm{H}_\alpha\}}{P\{\mathrm{H}_\alpha\}P\{\mathrm{C}|\mathrm{H}_\alpha\}+P\{\mathrm{H}_\beta\}P_\mathrm{C}^-(r)}.\tag{9}$$

The preference of a hypothesis at small r depends on which probability, $P\{\mathrm{H}_\alpha\}$ or $P\{\mathrm{H}_\beta\}$ is bigger: if *a)* $P\{\mathrm{H}_\beta\}>P\{\mathrm{H}_\alpha\}$ then $P_{\min}^\beta(\mathrm{C}_1;r)\ge P_{\max}^\alpha(\mathrm{C}_1;r)$, otherwise *b)* $P_{\max}^\beta(\mathrm{C}_1;r)<P_{\min}^\alpha(\mathrm{C}_1;r)$. Both $P_{\max}^\beta(\mathrm{C}_1;r)$ (6) and $P_{\min}^\beta(\mathrm{C}_1;r)$ (7) decrease as r goes up while probabilities $P_{\min}^\alpha(\mathrm{C}_1;r)$ (8) and $P_{\max}^\alpha(\mathrm{C}_1;r)$ (9) grow. Hence, hypothesis H_α becomes more and more probable with r from zero up to $\bar{R}_S$, while hypothesis H_β is less and less probable. Starting with a certain r, we get for the case *a)* that $P_{\min}^\beta(\mathrm{C}_1;r)<P_{\max}^\alpha(\mathrm{C}_1;r)$ but still $P_{\max}^\beta(\mathrm{C}_1;r)>P_{\min}^\alpha(\mathrm{C}_1;r)$, at which preference is ambiguous, however with a further increase in r hypothesis H_α begins to prevail, $P_{\max}^\beta(\mathrm{C}_1;r)<P_{\min}^\alpha(\mathrm{C}_1;r)$. For the case *b)* hypothesis H_α prevails at all $r\le R_S$. As $P_{\mathrm{C}_2}^-(r)\gg P\{\mathrm{C}_2|\mathrm{H}_\alpha\}$, probability $P_{\min}^\beta(\mathrm{C}_2;r)$ is much bigger than $P_{\max}^\alpha(\mathrm{C}_2;r)$ even though $P\{\mathrm{H}_\alpha\}>P\{\mathrm{H}_\beta\}$ for sensible prior probabilities.

We define for $\mathrm{C}=\mathrm{C}_1$ identification probabilities (IPs) $P_{\mathrm{C}_1}^\alpha(r)\equiv P_{\min}^\alpha(\mathrm{C}_1;r)$ of one and $P_{\mathrm{C}_1}^\beta(r)\equiv P_{\max}^\beta(\mathrm{C}_1;r)$ of two sources if $P\{\mathrm{H}_\beta\}P_{\mathrm{C}_1}^+(r)<P\{\mathrm{H}_\alpha\}P\{\mathrm{C}_1|\mathrm{H}_\alpha\}$, and $P_{\mathrm{C}_1}^\alpha(r)\equiv\ \equiv P_{\max}^\alpha(\mathrm{C}_1;r)$, $P_{\mathrm{C}_1}^\beta(r)\equiv P_{\min}^\beta(\mathrm{C}_1;r)$ if $P\{\mathrm{H}_\beta\}P_{\mathrm{C}_1}^-(r)>P\{\mathrm{H}_\alpha\}P\{\mathrm{C}_1|\mathrm{H}_\alpha\}$. IPs are not defined if

$P^{-}_{C_1}(r) \le P\{C_1|H_\alpha\}P\{H_\alpha\}/P\{H_\beta\} \le P^{+}_{C_1}(r)$ when a hypothesis could not be preferred. As for $C = C_2$, $P^{\alpha}_{C_2}(r) \equiv P^{\alpha}_{\max}(C_2;r)$ and $P^{\beta}_{C_2}(r) \equiv P^{\beta}_{\min}(C_2;r)$. To achieve preference the function $\delta_H(r;C)$ of relative difference between IPs is compiled:

$$\delta_H(r;C_1) = \begin{cases} \dfrac{P\{H_\alpha\}P\{C_1|H_\alpha\}}{P\{H_\beta\}P^{-}_{C_1}(r)} - 1, P\{H_\beta\}P^{-}_{C_1}(r) > P\{H_\alpha\}P\{C_1|H_\alpha\} \\ 1 - \dfrac{P\{H_\beta\}P^{+}_{C_1}(r)}{P\{H_\alpha\}P\{C_1|H_\alpha\}}, P\{H_\beta\}P^{+}_{C_1}(r) < P\{H_\alpha\}P\{C_1|H_\alpha\} \end{cases},$$

$$\delta_H(r;C_2) = 1 - P\{H_\alpha\}P\{C_2|H_\alpha\}/\left(P\{H_\beta\}P^{-}_{C_2}(r)\right), \qquad (10)$$

which is compared with a threshold $t_H > 0$. At $|\delta_H(r;C)| < t_H$ preference is ambiguous. If $\delta_H(r;C_1) \ge t_H$ or $\delta_H(r;C_2) \ge t_H$ then hypotheses H_α or H_β respectively prevail. When $\delta_H(r;C_1) \le -t_H$ we select H_β.

Probability distribution of squared distance between two Gaussian vectors (in our case it will be $\|\boldsymbol{\psi}\|^2$) is referred to the distribution of a quadratic form $\boldsymbol{\psi}^T\mathbf{A}\boldsymbol{\psi}$, where $\boldsymbol{\psi}$ is non-central Gaussian vector with a mean $\bar{\boldsymbol{\psi}}$ and covariance matrix $\mathbf{W}$ (for us $\mathbf{W} = \hat{\mathbf{\Phi}}' + \hat{\mathbf{\Phi}}''$), $\mathbf{A}$ is symmetric and nonnegative definite matrix. We obtain the desired distribution by equating $\mathbf{A}$ to the identity matrix. Although its mathematical structure is complicated, modern computer resources allow implement supporting functions precisely [17].

The separation r is to be chosen everywhere in the range of $0 < r \le \bar{R}_S$ for the event C_1 or of $r > \bar{R}_S$ for the event C_2 in the use of identifier (10). Nevertheless, when it approaches $\bar{R}_S$ probability of both events may grow nearer to $1/2$, where they tend to be equally probable. To improve identification reliability we invent the probability threshold $t_C > 0.5$ if necessary for determining such distances $r_1(t_C) < \bar{R}_S$ or $r_2(t_C) > \bar{R}_S$ that would afford a more valid event: $P^{+}(r_1)| = t_C$ for C_1 or $P^{-}(r_2)| = t_C$ for C_2. Beginning from $\bar{R}_S$ we decrease the distance r up to r_1 or increase it up to r_2. Probability t_C is assigned to be the probabilistic measure of event validity, which prescribes that events C_1 at $r \le r_1$ or C_2 at $r \ge r_2$, are both valid with probability no smaller than t_C. The region of $r \in (r_1, r_2)$ is defined to be of invalid event.

To summarize the results of Section 5 we present the core pseudo-code of the identification procedure:

if $C = C_1$ **then**

Step 1. Identify one/two sources at $r \in (0, r_1]$

if $P^{-}_{C_1}(r) \le P\{C_1|H_\alpha\}P\{H_\alpha\}/P\{H_\beta\} \le P^{+}_{C_1}(r)$

then No preference of a hypothesis **else**

if $\delta_H(r;C_1) \ge t_H$ **then** *Prefer* H_α

else if $\delta_H(r;C_1) \le -t_H$ **then** *Prefer* H_β

else No preference of a hypothesis

end if end if end if

else $/C = C_2/$

Step 2. Identify one/two sources at $r \ge r_2$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 April 2024 doi:10.20944/preprints202404.1776.v1

if $\delta_H\left(r;\mathrm{C}_2\right)<t_H$ **then** *No preference of a hypothesis*

else Prefer H_β

end if end if

6. Estimation of $R99$

The topic of a confidence circle has been addressed in a number of publications regarding circular error probability (CEP) integral [18–22], occurring in navigation and surveillance systems. CEP is the radius of a confidence circle which contains a random variable with probability 0.5.

The probability density for zero mean bivariate Gaussian variable $\mathbf{z}=\left[x \quad y\right]^T$ with covariance matrix $\mathbf{K}$ is defined as

$$f_{\mathbf{z}}\left(\mathbf{z},\mathbf{K}\right)=\frac{1}{2\pi\sqrt{\det\mathbf{K}}}\exp\left(-\frac{1}{2}\mathbf{z}^T\mathbf{K}^{-1}\mathbf{z}\right),\quad \mathbf{K}=\begin{bmatrix}\sigma_x^2 & \rho_{\mathrm{cv}}\\ \rho_{\mathrm{cv}} & \sigma_y^2\end{bmatrix}, \tag{11}$$

where σ_x, σ_y and ρ_{cv} are standard deviations and covariance of x and y. Hence, function (11) is

$$f_{xy}\left(x,y\right)=\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2\left(1-\rho^2\right)}\left(\frac{x^2}{\sigma_x^2}-\frac{2\rho xy}{\sigma_x\sigma_y}+\frac{y^2}{\sigma_y^2}\right)\right], \tag{12}$$

where $\rho=\rho_{cv}/\left(\sigma_x\sigma_y\right)$ is a correlation coefficient.

Among others, Krempasky CEP estimator [22] is the most attractive both owing to its numerical simplicity (it is closed-form) and high accuracy (the deviation of CEP estimate from the exact value is less than 2 percent). Krempasky first rotates the x and y coordinates by the angle $\varGamma$: $\mathrm{tg}\varGamma=\left(\sigma_y^2-\sigma_x^2\right)^{-1}\left[-2\rho_{cv}+\sqrt{4\rho_{cv}^2+\left(\sigma_y^2-\sigma_x^2\right)^2}\right]$ such that $\sigma_x'^2=\sigma_y'^2=\left(\sigma'\right)^2=\sigma_x^2\cos^2\varGamma+\sigma_y^2\sin^2\varGamma+\rho_{cv}\sin 2\varGamma$ in transformed coordinates x' and y'. Correlation coefficient is $\rho'=\left(\sigma'\right)^{-2}\left[0.5\left(\sigma_y^2-\sigma_x^2\right)\sin 2\varGamma+\rho_{cv}\cos 2\varGamma\right]$ in the new coordinates. With polar coordinates the originating in (12) integral for CEP, expressed through variables x' and y', is written with angular coordinate ϑ after the integral over radial coordinate is performed:

$$0.5=\int_o^{2\pi}\mathrm{d}\vartheta\left\{\frac{-1}{2\pi\varpi\sqrt{1-\rho'}}\left[\exp\left(-\frac{\varpi CEP^2}{2\left(\sigma'\right)^2}\right)-1\right]\right\},\quad \varpi=\frac{1-\rho'\sin\vartheta}{1-\rho'^2}. \tag{13}$$

The integral (13) is then expanded to the fourth order in the correlation coefficient ρ', and after much manipulation the final CEP estimate is found to be

$$\begin{gathered}CEP=CEP_{00}\left[1-0.5C_2\rho'^2-0.5\left(C_4+0.25C_2^2\right)\rho'^4\right],\quad CEP_{00}=\sqrt{2\ln 2}\sigma',\\ C_4=C_2\left(\ln 2-0.25\ln^2 2-0.5\right)-0.5C_2^2\ln 2-0.5625\ln 2+\\ +0.1875\ln^2 2-0.015625\ln^3 2+0.375.\end{gathered} \tag{14}$$

We substitute 0.5 with 0.99 in (13) to rework Krempasky CEP estimator to estimate $R99$. Repeating the same manipulation as in [22], one can make sure that $R99$ estimate is deduced from (14) by substitution $\ln 2$ with $2\ln 10$. Finally, the expression for $R99_K$ is derived:

$$R99_K=R_0\left[1-0.5B_2\rho'^2-0.5\left(B_4+0.25B_2^2\right)\rho'^4\right],\quad R_0=2\sqrt{\ln 10}\sigma',$$

$$B_2 = 0.5\left(1-\ln 10\right),\ B_4 = B_2\left(2\ln 10 - \ln^2 10 - 0.5\right) - B_2^2 \ln 10 - 1.125\ln 10 +$$

$$+0.75\ln^2 10 - 0.125\ln^3 10 + 0.375 .$$

To assess the performance of $R99_K$ we propose the alternate estimate $R99_A$ based on the two specific behaviors of the integrand (11) in Gaussian probability integral $0.99 = \int\limits_{\|\mathbf{z}\|\le R99} f_{\mathbf{z}}\left(\mathbf{z},\mathbf{K}\right)\mathrm{d}\mathbf{z}$ in the vicinity of zero and beyond. Initially, it is reformulated as an integral along one variable [23]:

$$0.99 = \int\limits_0^{R99} F\left(s\right)\mathrm{d}s\,,\ F\left(s\right) = \frac{s}{\sqrt{\lambda_1\lambda_2}}\exp\left\{-\left(\frac{1}{4\lambda_1}+\frac{1}{4\lambda_2}\right)s^2\right\} I_0\left\{\left(\frac{1}{4\lambda_2}-\frac{1}{4\lambda_1}\right)s^2\right\}, \quad (15)$$

where $\lambda_1 > \lambda_2 > 0$ are eigenvalues of $\mathbf{K}$, $\lambda_1,\lambda_2 = 0.5\times\left[\sigma_x^2+\sigma_y^2 \pm\sqrt{\left(\sigma_x^2-\sigma_y^2\right)^2+4\rho_{\mathrm{cv}}^2}\right]$; $I_0(\bullet)$ is a modified Bessel function of the first kind and zero order. It is approximated with the use of the asymptotic expansions [24]:

$$I_0\left(\Omega s^2\right) \approx \begin{cases} 1+\dfrac{\Omega^2 s^4}{1\times 4}+\dfrac{\Omega^4 s^8}{1\times 2\times 32}+\dfrac{\Omega^6 s^{12}}{1\times 2\times 3\times 128}+\ldots, s\le R_\lambda \\ \wp\left(s\right)\left(1+\dfrac{1^2}{1!8^1\Omega s^2}+\dfrac{1\times 3^2}{2!8^2\Omega^2 s^4}+\dfrac{1\times 3^2\times 5^2}{3!8^3\Omega^3 s^6}+\ldots\right), s> R_\lambda \end{cases}, \quad (16)$$

$$\Omega \equiv \frac{\lambda_1-\lambda_2}{4\lambda_1\lambda_2},\ \wp\left(s\right)\equiv\frac{\exp\left(\Omega s^2\right)}{s\sqrt{2\pi\Omega}},\ R_\lambda = 2\sqrt{\lambda_1\lambda_2/\left(\lambda_1-\lambda_2\right)}.$$

Equation (15) is now rewritten as $0.99 = J_0^{R99}$, $J_0^{R99} = \int\limits_0^{R99} F\left(s\right)\mathrm{d}s$ at $R99 \le R_\lambda$ ***(i)*** or as $0.99 = J_0^{R_\lambda}+J_{R_\lambda}^{R99}$, $J_0^{R_\lambda} = \int\limits_0^{R_\lambda} F\left(s\right)\mathrm{d}s$, $J_{R_\lambda}^{R99} = \int\limits_{R_\lambda}^{R99} F\left(s\right)\mathrm{d}s$ at $R99 > R_\lambda$ ***(ii)***. The integral $J_0^{R_\lambda}$ for the case ***(i)*** is the probability which is bigger than or equal to 0.99, hence $\mu = 0.99 - J_0^{R_\lambda} \le 0$, otherwise for the case ***(ii)*** – $\mu > 0$. We use representation from (16) at $s \le R_\lambda$ instead of $I_0\left(\Omega s^2\right)$ in integral $J_0^{R_\lambda}$ to get its approximation $\widehat{J}_0^{R_\lambda}$ and $\widehat{\mu} = 0.99 - \widehat{J}_0^{R_\lambda}$ afterwards. If $\widehat{\mu}\le 0$ then case ***(i)*** is recognized, or otherwise – case ***(ii)*** for which representation from (16) at $s > R_\lambda$ is used instead of $I_0\left(\Omega s^2\right)$ in integral $J_{R_\lambda}^{R99}$ to achieve approximation $\widehat{J}_{R_\lambda}^{R99_A}$. By doing so, we come to the approximation of the right part of (15) and obtain:

$$0.99 = \begin{cases} \widehat{J}_0^{R99_A} & (i): \widehat{\mu}\le 0 \\ \widehat{J}_0^{R_\lambda}+\widehat{J}_{R_\lambda}^{R99_A} & (ii): \widehat{\mu} > 0 \end{cases},$$

where $\widehat{J}_0^{R99_A}$ is the same approximation as $\widehat{J}_0^{R_\lambda}$ with $R99_A$ instead of R_λ. The integration techniques detailed in Appendixes lead to the following analytical expressions:

$$\widehat{J}_0^{B} = \frac{2\sqrt{\lambda_1\lambda_2}}{\lambda_1+\lambda_2}\times$$

$$\times\left\{1+\sum_1^{M_0}\frac{\Omega^{2k}}{k!2^{3k-1}}4k\left(4k-2\right)\ldots 4\times 2\gamma^{2k} - e^{-\frac{\rho}{2\gamma}}\left(\sum_1^{M_0}\frac{\Omega^{2k}}{k!2^{3k-1}}\left\{\rho^{2k}+4k\gamma\rho^{2k-1}+\ldots+4k\left(4k-2\right)\ldots 4\times 2\gamma^{2k}\right\}+1\right)\right\},$$

 doi:10.20944/preprints202404.1776.v1

$$\gamma = 2\lambda_1\lambda_2/(\lambda_1+\lambda_2),\ \rho \equiv B^2,\ B \in \{R_\lambda, R99_A\},$$

see Appendix A, and

$$\hat{J}^{R99}_{R_\lambda} = \sqrt{\frac{\lambda_1}{\lambda_1-\lambda_2}}\left\{\left[erf\left(\frac{R99_A}{\sqrt{2\lambda_1}}\right)-erf\left(\frac{R_\lambda}{\sqrt{2\lambda_1}}\right)\right]\left[1+\sum_{k=1}^{M_\infty}\frac{(-2)^k\prod_{s=1}^{k}(2s-1)^2}{(16\lambda_1\Omega)^k\,k!\prod_{s=1}^{k}[2(k-s)+1]}\right]+\right.$$

$$\left.+\frac{2}{\sqrt{\pi}}\sum_{k=1}^{M_\infty}\frac{\prod_{s=1}^{k}(2s-1)^2}{(16\lambda_1\Omega)^k\,k!}\sum_{\ell=1}^{k}\frac{(-2)^{\ell-1}}{\prod_{s=1}^{\ell}[2(k-s)+1]}\left[\frac{\exp\left(-\frac{R_\lambda^2}{2\lambda_1}\right)}{\left(\frac{R_\lambda}{\sqrt{2\lambda_1}}\right)^{2(k-\ell)+1}}-\frac{\exp\left(-\frac{R99_A^2}{2\lambda_1}\right)}{\left(\frac{R99_A}{\sqrt{2\lambda_1}}\right)^{2(k-\ell)+1}}\right]\right\},$$

see Appendix B, where $erf(R)=\frac{2}{\sqrt{\pi}}\int_0^R e^{-s^2}\mathrm{d}s$; M_0 and M_∞ are the orders of approximation. As $R99_A > R_\lambda$ the member $\frac{\exp\left(-R99_A^2/(2\lambda_1)\right)}{\left(R99_A/\sqrt{2\lambda_1}\right)^{2(k-\ell)+1}}$ in $\hat{J}^{R99_A}_{R_\lambda}$ is small as compared to $\frac{\exp\left(-R_\lambda^2/(2\lambda_1)\right)}{\left(R_\lambda/\sqrt{2\lambda_1}\right)^{2(k-\ell)+1}}$ and further ignored.

We assign $M_0=2$ in $\hat{J}^B_0$ after that case (*i*) is reduced to the following equation for $R99_A$:

$$e^{-\frac{\rho}{2\gamma}}\left(\sum_1^2\frac{\Omega^{2k}}{k!2^{3k-1}}\left\{\rho^{2k}+4k\gamma\rho^{2k-1}+\ldots+4k(4k-2)\ldots4\times2\gamma^{2k}\right\}+1\right)=$$

$$=1+\sum_1^2\frac{\Omega^{2k}}{k!2^{3k-1}}4k(4k-2)\ldots4\times2\gamma^{2k}-0.99\frac{\lambda_1+\lambda_2}{2\sqrt{\lambda_1\lambda_2}},\quad R99_A=\rho^{1/2}.$$

As it is observable, the derivative along ρ of the function from the left side of this equation is negative, supplying the solution uniqueness.

For the case (***ii***) we set $M_\infty=2$ in $\hat{J}^{R99_A}_{R_\lambda}$ and obtain the closed-form solution:

$$erf\left(\frac{R99_A}{\sqrt{2\lambda_1}}\right)=erf\left(\frac{R_\lambda}{\sqrt{2\lambda_1}}\right)+$$

$$+\frac{\hat{\mu}\sqrt{\frac{\lambda_1-\lambda_2}{\lambda_1}}-\frac{2\,e^{-\frac{R_\lambda^2}{2\lambda_1}}}{\sqrt{\pi}}\sum_{k=1}^{2}\frac{\prod_{s=1}^{k}(2s-1)^2}{(16\lambda_1\Omega)^k\,k!}\sum_{\ell=1}^{k}\frac{(-2)^{\ell-1}}{\left(\frac{R_\lambda}{\sqrt{2\lambda_1}}\right)^{2(k-\ell)+1}\prod_{s=1}^{\ell}[2(k-s)+1]}}{1+\sum_{k=1}^{2}\frac{(-2)^k\prod_{s=1}^{k}(2s-1)^2}{(16\lambda_1\Omega)^k\,k!\prod_{s=1}^{k}[2(k-s)+1]}},$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 April 2024 doi:10.20944/preprints202404.1776.v1

The performance of estimate $R99_K$ versus $R99_A$ is studied by simulation in the full range of $\hat{\mu}$ from -0.0098 to 0.9888, which correspond to varying of the correlation coefficient and sigma ratio (ratio of smaller to larger standard deviation) in the intervals (-1,1) and (0,1) respectively. The performance index for comparison is the probability integral (15) performed numerically with estimates $R99_K$ and $R99_A$. Table 1 contains the extraction from the simulation results.

Table 1. Simulation results on performance of $R99_K$ versus $R99_A$.

$\hat{\mu}$	$\int_0^R F(s)\,\mathrm{d}s, R \in \{R99_K, R99_A\}$		$\left(\frac{R99_K}{R99_A} - 1\right) \times 100$
	Krempasky	**Alternate**	
-0.0098	0.9900	0.9900	0.00
-0.0097	0.9899	0.9899	0.00
-0.0064	0.9899	0.9897	0.12
-0.0032	0.9899	0.9880	1.90
0.0014	0.9900	0.9921	-2.50
0.0086	0.9898	0,9874	2.70
0.1083	0.9900	0.9875	2.70
0.1446	0.9899	0.9880	2.40
0.1985	0.9902	0.9887	1.90
0.2502	0.9903	0.9893	1.30
0.3045	0.9906	0.9899	0.85
0.4009	0.9911	0.9909	0.19
0.4346	0.9912	0.9912	0.00
0.5112	0.9916	0.9916	0.00
0.5429	0.9916	0.9916	0.00
0.5862	0.9917	0.9915	0.22
0.6115	0.9917	0.9915	0.33
0.6465	0.9919	0.9914	0.60
0.7025	0.9921	0.9914	1.00
0.7949	0.9925	0.9911	1.80
0.8512	0.9926	0.9909	2.70
0.8840	0.9926	0.9905	3.10
0.9208	0.9927	0.9905	3.50
0.9794	0.9932	0.9900	4.40
0.9867	0.9958	0.9900	5.40
0.9883	0.9967	0.9900	6.50

As we can see from the Table 1, at $\hat{\mu} < 0$ radii $R99_K$ and $R99_A$ are both of excellent accuracy and close to each other. As $\hat{\mu}$ approaches zero from below, estimate $R99_K$ keeps excellent accuracy while estimate $R99_A$ degrades. The bigger $\hat{\mu}$ after zero is, the better $R99_A$ becomes, while $R99_K$ degrades gradually. It is safe to conclude, that estimate $R99_A$ begins to outperform $R99_K$ visibly from the $\hat{\mu} \approx 0.8$ onwards. As a result, radius $R99$ is estimated as follows:

$$R99 = \begin{cases} R99_K, \hat{\mu} \le 0.8 \\ R99_A, \hat{\mu} > 0.8 \end{cases}.$$

7. Distinguishing Two Estimates between One or Two close users in the TDOA BSs Network

Bayesian formalism is applied to distinguish two positional estimates of the user(s) which are obtained in the TDOA planar positioning network. The received signals are classified by parameter $\boldsymbol{q}$ with the probabilities $\left\{P_j^I\right\}_{j=1}^{j=Q}$ and $\left\{P_j^{II}\right\}_{j=1}^{j=Q}$ over emissions *I* and *II*, where Q is the number of different users. Proceeding from these probabilities, one can easily get $P\left\{\mathrm{H}_\alpha\right\} = \sum_{j=1}^{Q} P_j^I P_j^{II}$ and $P\left\{\mathrm{H}_\beta\right\} = 1 - P\left\{\mathrm{H}_\alpha\right\}$.

In TDOA technique, output signal at k -th BS is with parameter τ_k $(m=1)$ of time difference of emission arrival on that BS and reference one, placed in (0, 0), from a user located at $\boldsymbol{\varphi}$: $p_k(\boldsymbol{\varphi}) = c\tau_k(\boldsymbol{\varphi}) = \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_k\| - \|\boldsymbol{\varphi}\|$, $k = \overline{1, L_p}$, $L_p = L - 1$, where c is the speed of light. The model (1) is here reduced to the following one:

$$\nu\left(t^I; \boldsymbol{q}^I, p_k\left(\boldsymbol{\varphi}^I\right)\right) = g\left(t^I - c^{-1} p_k\left(\boldsymbol{\varphi}^I\right); \boldsymbol{q}^I\right) + n_k\left(t^I\right),$$

$$\nu\left(t^{II}; \boldsymbol{q}^{II}, p_k\left(\boldsymbol{\varphi}^{II}\right)\right) = g\left(t^{II} - c^{-1} p_k\left(\boldsymbol{\varphi}^{II}\right); \boldsymbol{q}^{II}\right) + n_k\left(t^{II}\right),$$

$$\nu\left(t^I; \boldsymbol{q}^I\right) = g\left(t^I; \boldsymbol{q}^I\right) + n_L\left(t^I\right), \nu\left(t^{II}; \boldsymbol{q}^{II}\right) = g\left(t^{II}; \boldsymbol{q}^{II}\right) + n_L\left(t^{II}\right)$$

with scalar signals, waveforms and noises instead of vector ones as in (1) at $k = \overline{1, L_p}$. Scalar processes $\nu(t; \boldsymbol{q})$, $n_L(t)$ and function $g(t; \boldsymbol{q})$ with $p_L = 0$ are given at reference BS.

If signal at each of L_p BSs is uncorrelated with the signal at reference BS: $E\left\{n_k(t) n_L(t)\right\} = 0$, $\Im_p$ as maximum likelihood estimator is efficient asymptotically [25]. As far as $\Im_\varphi$ is concerned, the constrained weighted least square estimator [26], which combine core pillars of the quadratic and linear correction techniques [27,28], is to be efficient for uncorrelated measurement errors $\Delta\tau_k$ of parameters τ_k, $\boldsymbol{\upsilon} = \left[\Delta\tau_1, \ldots, \Delta\tau_{L_p}\right]^T$. Despite the fact that they nevertheless correlate through signal at reference BS, it is presumed that matrix $E\left\{\boldsymbol{\upsilon}\boldsymbol{\upsilon}^T\right\}$ is to be near to a diagonal one. Reasoning from the aforementioned, the covariance matrix in TDOA BSs network is calculated as

$$\boldsymbol{\Phi}(\boldsymbol{\varphi}) = \left(\left[\frac{\partial c\boldsymbol{\tau}(\boldsymbol{\varphi})}{\partial \boldsymbol{\varphi}}\right]^T diag\left\{\frac{1}{\sigma_k^2}\right\}_1^{L_p} \left[\frac{\partial c\boldsymbol{\tau}(\boldsymbol{\varphi})}{\partial \boldsymbol{\varphi}}\right]\right)^{-1},$$

$$\boldsymbol{\tau}(\boldsymbol{\varphi}) = \left[\tau_1(\boldsymbol{\varphi}), \ldots, \tau_{L_p}(\boldsymbol{\varphi})\right]^T, \ \sigma_k^2 = E\left\{\left(c\Delta\tau_k\right)^2\right\}.$$

There are five BSs, including the reference BS. Four BSs in conjunction with the reference one, produce the range differences of arrival $c\tau_k$, corrupted by the noise with standard deviations $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.04$m (m hereinafter refers to meters). They are situated at (1000, -700)m, (-1000, 2000)m, (4000, 3000)m and (-4000, 1000)m. The identifier is investigated in the close users domain (CUD) to be associated with the position $\boldsymbol{\varphi}_1$ at (14000, 12600)m.

Since the exact solution for the distribution of $\|\psi\|^2$ [17] is very expansive, the simpler closed-form Lui-Tang-Zhang approximation [29] is utilized in the study.

7.1. Characterization of the CUD

We intend to construct the CUD around φ_1 in such a manner to characterize it by the constant characteristics SRL and $\mathbf{W}$ which would be sufficient approximations of $SRL(\varphi_2) = R99_1 + R99(\varphi_2)$, where $R99_1$ and $R99(\varphi_2)$ are the radii of the confidence circles $\mathbf{S}_1$ and $\mathbf{S}_2$ centered at φ_1 and φ_2 in charge of probability 0.99, and $\mathbf{W}_\psi(\varphi_2) = \mathbf{\Phi}(\varphi_1) + \mathbf{\Phi}(\varphi_2)$ for the all φ_2 from CUD. The estimates φ'' cover subdomain $\mathbf{S}_2^{C_1} = \left\{\varphi'' : \|\varphi'' - \varphi_1\| \le R99_1 + \min R99(\varphi_2)\big|_{\|\varphi_2-\varphi_1\|=R99_1}\right\}$ when locations φ_2 fill $\mathbf{S}_1$. Event C_1 would predictably be observed in $\mathbf{S}_2^{C_1}$. To get event C_2 we have to withdraw φ_2 from the subdomain $\mathbf{S}_2^{C_1}$ into the subdomain $\mathbf{S}_2^{C_2}$ where $r = \|\varphi_2 - \varphi_1\|$ is bigger than $R99_1 + \max R99(\varphi_2)\big|_{\|\varphi_2-\varphi_1\|=r}$.

In the simulation, $R99(\varphi_2)$ is computed at N points $\{\varphi_2^k\}$ evenly spaced along circumference of radius r centered at φ_1: $R99_2^k(r) = R99(\varphi_2^k)\big|_{\|\varphi_2^k-\varphi_1\|=r}$, $k=\overline{1,N}$. Thereafter, mean values $\overline{R99}_2(r) = N^{-1}\sum_{k=1}^{N} R99_2^k(r)$, $\overline{SRL}(r) = \; = R99_1 + \overline{R99}_2(r)$ and the square root from the mean of squared deviations of $R99_2^k(r)$ from $\overline{R99}_2(r)$, $\bar{\varepsilon}(r) = \sqrt{N^{-1}\sum_{k=1}^{N}\left(R99_2^k(r) - \overline{R99}_2(r)\right)^2}$ are determined.

We start with $r^{(1)} = R99_1$, as a result of which $\overline{SRL}(r^{(1)})$, $\bar{\varepsilon}(r^{(1)})$ are obtained. The next distance $r^{(2)} = \overline{SRL}(r^{(1)})$ is to find $\overline{SRL}(r^{(2)})$ with $\bar{\varepsilon}(r^{(2)})$ and compare it to $\overline{SRL}(r^{(1)})$: if $\overline{SRL}(r^{(2)})$ is nearly identical to $\overline{SRL}(r^{(1)})$ then the circumference of radius $r^{(2)}$ will be the boundary of subdomain $\mathbf{S}_2^{C_1}$ characterized by the $\overline{SRL}(r^{(2)})$ with the accuracy $\bar{\varepsilon}(r^{(2)})$. With regard to the subdomain $\mathbf{S}_2^{C_2}$, $\overline{SRL}(r)$, $\bar{\varepsilon}(r)$ are computed at some $r^{(3)} > r^{(2)}$ and at $r^{(4)} > r^{(3)}$. If $\overline{SRL}(r^{(3)})$ and $\overline{SRL}(r^{(4)})$ are nearly equal to $\overline{SRL}(r^{(1)})$, then we acquire at $r^{(3)} \le r \le r^{(4)}$ the subdomain $\mathbf{S}_2^{C_2}$ and at $0 < r \le r^{(4)}$ entire CUD characterized by the $SRL = \overline{SRL}(r^{(4)}) \pm \bar{\varepsilon}(r^{(4)})$, wherein $\bar{\varepsilon}(r^{(4)})$ is the worst approximation accuracy of $SRL(\varphi_2)$ in CUD by the constant.

The simulation results for N =1000 with $R99_1$ equal to 14.5133m are given in the Table 2.

Table 2. CUD characterization on SRL.

Subdomain	**Distance** r	$\overline{R99}_2(r)$	$\overline{SRL}(r)$	$\bar{\varepsilon}(r)$
↑	14.5133m	14.5133m	29.0266m	0.0253m
$\mathbf{S}_2^{C_1}$	29.0266m	14.5132m	29.0265m	0.0507m

 doi:10.20944/preprints202404.1776.v1

TransitionalPlea subdomain from the event C_1 *to the event* C_2

$\mathbf{S}_2^{\mathrm{C}_2}$	33m	14.5132m	29.0265m	0.0576m
↓	38m	14.5132m	29.0265m	0.0664m

Taking into consideration the column $\bar{\varepsilon}(r)$ from the Table 2, we can infer that $SRL(\boldsymbol{\varphi}_2)\in$ [28.9601, 29.0929]m. Having interval centered at $\overline{SRL}(r)$ where true $SRL(\boldsymbol{\varphi}_2)$ is varied, we accept it to be constant, namely 29.02m with the accuracy $\pm$ 0.07m. Thus, $SRL(\boldsymbol{\varphi}_2)$ is approximated in CUD by the constant with a good accuracy. Therefore, we have reason to approximate the true covariance matrices $\mathbf{W}_\psi(\boldsymbol{\varphi}_2)$ by the constant everywhere in CUD as well. To this end, the mean of $\boldsymbol{\Phi}(\boldsymbol{\varphi}_2)$ over 1000 points evenly spaced along the circumference of radius $r^{(4)}$, $\overline{\boldsymbol{\Phi}}_2=10^{-3}\sum_{k=1}^{1000}\boldsymbol{\Phi}\left(\boldsymbol{\varphi}_2^k\right)\Big|_{\|\boldsymbol{\varphi}_2^k-\boldsymbol{\varphi}_1\|=r^{(4)}}$, and the mean $\overline{\mathbf{W}}=\boldsymbol{\Phi}(\boldsymbol{\varphi}_1)+\overline{\boldsymbol{\Phi}}_2$ are computed. To reveal the scattering of $\mathbf{W}_\psi(\boldsymbol{\varphi}_2)$ towards $\overline{\mathbf{W}}$ in CUD we additionally determine the matrices $\mathbf{W}_{\min}=\boldsymbol{\Phi}(\boldsymbol{\varphi}_1)+\boldsymbol{\Phi}(\boldsymbol{\varphi}_{2\min})$, $\mathbf{W}_{\max}==\boldsymbol{\Phi}(\boldsymbol{\varphi}_1)+\boldsymbol{\Phi}(\boldsymbol{\varphi}_{2\max})$, where $\boldsymbol{\varphi}_{2\min}$, $\boldsymbol{\varphi}_{2\max}$ correspond to the minimum and maximum of $R99(\boldsymbol{\varphi}_2)$ on the same circumference: $\boldsymbol{\varphi}_{2\min}=\arg\min\left\{R99_2^k\left(r^{(4)}\right)\right\}_1^{1000}$, $\boldsymbol{\varphi}_{2\max}==\arg\max\left\{R99_2^k\left(r^{(4)}\right)\right\}_1^{1000}$. Simulation data in square meters are: $\boldsymbol{\Phi}(\boldsymbol{\varphi}_1)==\begin{bmatrix}19.2589 & 15.2837\\15.2837 & 12.1876\end{bmatrix}$, $\overline{\boldsymbol{\Phi}}_2=\begin{bmatrix}19.2593 & 15.2837\\15.2837 & 12.1874\end{bmatrix}$, hence $\overline{\mathbf{W}}=\begin{bmatrix}38.5182 & 30.5673\\30.5673 & 24.3749\end{bmatrix}$; $\mathbf{W}_{\min}=\begin{bmatrix}38.2219 & 30.3798\\30.3798 & 24.2637\end{bmatrix}$ and $\mathbf{W}_{\max}=\begin{bmatrix}38.8155 & 30.7555\\30.7555 & 24.4864\end{bmatrix}$. Following the same logic as in characterization on SRL, we approximate $\mathbf{W}_\psi(\boldsymbol{\varphi}_2)$ in CUD by the $\begin{bmatrix}38.51 & 30.56\\30.56 & 24.37\end{bmatrix}$ with the accuracy $\pm\begin{bmatrix}0.30 & 0.19\\0.19 & 0.11\end{bmatrix}$. Spectral characteristic $\bar{e}$ of the $\mathbf{W}$ is equal to 29.3636, hence $0.4842<PRI<0.4905$.

7.2. Identification in CUD: Simulation and Discussion of Results

There is no need to employ the estimator [26] in the simulation because the CUD is already structured into the subdomains $\mathbf{S}_2^{\mathrm{C}_1}$ and $\mathbf{S}_2^{\mathrm{C}_2}$ where all the estimates fall. We use SRL and $\mathbf{W}$ established previously in Subsection 7.1 of this Section to find IPs and build the identifier function $\delta_H(r;\mathrm{C})$ under prior probabilities $P\{\mathrm{H}_\alpha\}$=0.2, 0.48, 0.5, 0.52, 0.8 related to the probabilities $P\{\mathrm{H}_\beta\}$ =0.8, 0.52, 0.5, 0.48, 0.2 and compare it with the threshold t_H=0.05. The threshold for valid events is t_C=0.55 under which region of invalid event $r\in$(28.90, 29.76)m is determined. The results of simulation are shown in Table 3, where α, β and δ denote the rows for IPs $P_{\mathrm{C}}^{\alpha}(r)$, $P_{\mathrm{C}}^{\beta}(r)$ and identifier function $\delta_H(r;\mathrm{C})$ respectively; abbreviation "*undef*" denotes undefined IPs and $\delta_H(r;\mathrm{C})$.

Table 3. Identification in the CUD.

$P\{\mathrm{H}_\alpha\}$		$\mathrm{C}=\mathrm{C}_1$								$\mathrm{C}=\mathrm{C}_2$
		Distance between users r [m]								
		10	14	17	20	24	26	26.5	28.9	30.0
0.2	α	0.201	0.204	0.210	0.223	0.256	0.283	0.291	0.434	0.0001
	β	0.799	0.796	0.790	0.777	0.744	0.717	0.709	0.566	0.9999
	δ	-0.749	-0.743	-0.734	-0.713	-0.656	-0.606	-0.590	-0.234	0.9999
0.48	α	0.481	0.486	0.496	*undef*	*undef*	0.505	0.514	0.646	0.0004
	β	0.519	0.514	0.504	*undef*	*undef*	0.495	0.486	0.354	0.9996
	δ	-0.071	-0.054	-0.017	*undef*	*undef*	0.021	0.053	0.452	0.9996
0.5	α	0.500	0.500	0.500	0.502	0.509	0.525	0.534	0.664	0.0004
	β	0.499	0.499	0.499	0.498	0.491	0.475	0.466	0.336	0.9996
	δ	0.000	0.001	0.002	0.007	0.036	0.096	0.126	0.494	0.9996
0.52	α	0.520	0.520	0.521	0.522	0.529	0.545	0.553	0.682	0.0005
	β	0.480	0.480	0.479	0.478	0.471	0.455	0.447	0.318	0.9995
	δ	0.077	0.078	0.079	0.084	0.110	0.166	0.193	0.533	0.9995
0.8	α	0.800	0.800	0.800	0.801	0.806	0.816	0.821	0.888	0.0018
	β	0.199	0.199	0.199	0.199	0.194	0.184	0.179	0.112	0.9982
	δ	0.750	0.750	0.751	0.752	0.759	0.774	0.782	0.874	0.9982

In the Table above the fragments of rows δ that are colored blue conform to the one user identification, those colored brown conform to the two users identification, and the uncolored ones do not conform to any identification. Green column is to highlight the region of invalid event. Figure 2 visualizes identification probability $P_{\mathrm{C}}^{\alpha}(r)$ from Table 3.

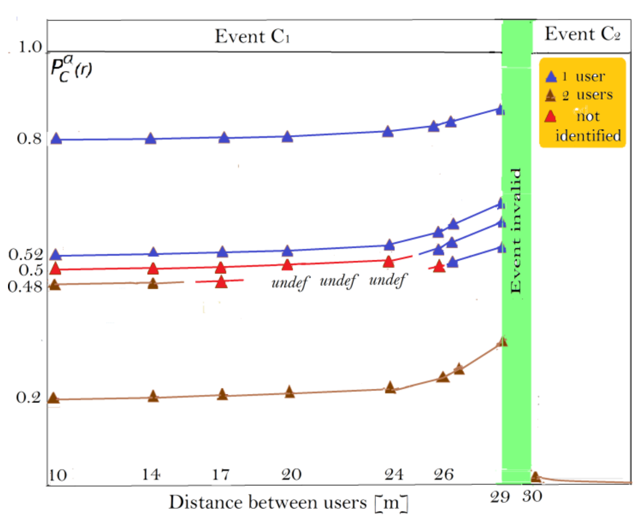


Figure 2. Identification probability $P_{\mathrm{C}}^{\alpha}(r)$ in the CUD.

 doi:10.20944/preprints202404.1776.v1

To facilitate the analysis of the simulation results from Table 3 we consider two different users, Q=2.

When $P\{\mathrm{H}_\alpha\}$=0.2 one has high probability $P\{\mathrm{H}_\beta\}$=0.8 that two users emit due to high probabilities P_2^I and P_1^{II}, as in the example P_1^I=0.125, P_2^I=0.875, P_1^{II}=0.9, P_2^{II}=0.1. Although $P_{\mathrm{C}_1}^{\beta}(r)$ decreases with separation r from 0.8 to 0.566, it still remains big enough in the subdomain $\mathbf{S}_2^{\mathrm{C}_1}$ till 28.9m to identify two users. On the contrary, for the probabilities P_1^I=0.875, P_2^I=0.125, P_1^{II}=0.9 and P_2^{II}=0.1 one user emits twice with probability $P\{\mathrm{H}_\alpha\}$=0.8, most probable user 1 ($P_1^I P_1^{II} - P_2^I P_2^{II}$=0.775). $P_{\mathrm{C}_1}^{\alpha}(r)$ rises with separation of two hypothesized users in $\mathbf{S}_2^{\mathrm{C}_1}$ from 0.8 to 0.888. Probability $P\{\mathrm{H}_\alpha\}$=0.48, which may be provided by the collection P_1^I=0.4, P_2^I=0.6, P_1^{II}=0.6 and P_2^{II}=0.4, is slightly less than probability $P\{\mathrm{H}_\beta\}=0.52$. However, $P_{\mathrm{C}_1}^{\beta}(r)$ decreases with separation from 0.52 to 0.514 at 14m where two users are identified. As $|\delta_H(17\mathrm{m};\mathrm{C}_1)|<t_H$, the preference at 17m becomes ambiguous. $P_{\mathrm{C}_1}^{\alpha}(r)$ grows up to 0.505 at 26m, but as $|\delta_H(26\mathrm{m};\mathrm{C}_1)|<t_H$ and in the region of $r\in$[20, 24]m IPs are undefined, the identifier is not there able to select a hypothesis. Since 26.5m up to the 28.9m one user 1 or user 2 ($P_1^I P_1^{II} - P_2^I P_2^{II}$=0) is identified with probability from 0.514 to 0.646. The identifier changes here the prior preference on two users in favor of one user with growth of separation. Probability $P\{\mathrm{H}_\alpha\}$=0.52, say for P_1^I=0.6, P_2^I=0.4, P_1^{II}=0.6, P_2^{II}=0.4, is slightly bigger than $P\{\mathrm{H}_\beta\}$=0.48 but $P_{\mathrm{C}_1}^{\alpha}(r)$ grows with r providing the identification of one user everywhere in $\mathbf{S}_2^{\mathrm{C}_1}$ with probability from 0.52 to 0.682, little bit more probable of user 1 ($P_1^I P_1^{II} - P_2^I P_2^{II}$ =0.2). The case of equally probable hypotheses, $P\{\mathrm{H}_\alpha\}=P\{\mathrm{H}_\beta\}$ =0.5 at $P_1^I = P_2^I = P_1^{II} = P_2^{II}$=0.5, takes place rather when it is hard to classify the emissions. For the separation up to 24m the growth of $P_{\mathrm{C}_1}^{\alpha}(r)$ is negligible. However, for bigger separation from 26m up to the bound of $\mathbf{S}_2^{\mathrm{C}_1}$ one user is identified with probability from 0.525 to 0.664, user 1 or user 2 ($P_1^I P_1^{II} - P_2^I P_2^{II}$=0). Even though prior preference is uncertain, the identifier comes to hypothesis H_α with the rise of r.

In the subdomain $\mathbf{S}_2^{\mathrm{C}_2}$ where the separation is not less than 29.76m, two users are identified with near to unity probability regardless of the prior probabilities. We could interpret this as follows: at $P\{\mathrm{H}_\beta\}$=0.8, 0.52 – two different users, at $P\{\mathrm{H}_\alpha\}$=0.52, 0.8 – user 1, at $P\{\mathrm{H}_\alpha\}$=0.5 – one and the same or different users for our examples of the classification probabilities. Transitional subdomain between Bayes events is the one of the invalid event where the identifier is incapable to work.

8. Concluding Remarks

Novel Bayesian formalism aimed to identify under double emission one or two closely spaced sources parameterized by their separation, through the use the two location estimates and their covariance matrices, is designed. Prior probabilities of the hypotheses on one and two sources are extracted from the prior solution (abbreviated as PS in the paper), assuming that they can be equally probable. The job of the formalism below and over resolution limit of the estimator is supported by the new resolution criterion emphasizing the probability of the resolving of planar parameter decoupled estimates.

As illustrated in the application, identification technique revises the probabilities induced by PS in dependence upon the distance between sources: if prior probability of a hypothesis is large enough, the identifier only recalculates the initial probability below resolution limit leaving PS-decision unchanged; if prior probability of one source is equal to or slightly less than one of two sources the identifier can change with distance the PS-decision in favor of one source; two sources are identified over resolution limit irrespective of PS-decision with near to unity probability where the identifier works as a routine resolver.

In the future we plan to extend proposed technique on a more general problem of identifying the number of sources, each of which can emit one or several times by a given set of location estimates.

Funding: This research received no external funding

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. On Integration of $\widehat{J}_0^B$

Integral $\widehat{J}_0^B$ is equal to

$$\widehat{J}_0^B = \frac{1}{\sqrt{\lambda_1\lambda_2}}\int_0^B s\exp\left[-\left(\frac{1}{4\lambda_1}+\frac{1}{4\lambda_2}\right)s^2\right]\left[1+\sum_{k=1}^{M_0}\frac{\Omega^{2k}s^{4k}}{2^{3k-1}k!}\right]\mathrm{d}s =$$

$$= \frac{2\sqrt{\lambda_1\lambda_2}}{\lambda_1+\lambda_2}\left(1-\mathrm{e}^{-\frac{\rho}{2\gamma}}\right)+\sum_{k=1}^{M_0}\frac{\Omega^{2k}}{2^{3k-1}k!}J_k,$$

$$J_k = \frac{1}{\sqrt{\lambda_1\lambda_2}}\int_0^B s^{4k+1}\exp\left[-\left(\frac{1}{4\lambda_1}+\frac{1}{4\lambda_2}\right)s^2\right]\mathrm{d}s.$$

Applying to integral J_k the rule of by parts integration we conclude that

$$\upsilon^{-1}J_k = -\int_0^B s^{4k}\,\mathrm{d}\exp\left[-\left(\frac{1}{4\lambda_1}+\frac{1}{4\lambda_2}\right)s^2\right] =$$

$$= -s^{4k}\exp\left[-\left(\frac{1}{4\lambda_1}+\frac{1}{4\lambda_2}\right)s^2\right]\Bigg|_0^B + 4k\int_0^B s^{4k-1}\exp\left[-\left(\frac{1}{4\lambda_1}+\frac{1}{4\lambda_2}\right)s^2\right]\mathrm{d}s =$$

$$= 4k(4k-2)\ldots4\times2\gamma^{2k} - e^{-\frac{\rho}{2\gamma}}\left\{\rho^{2k}+4k\gamma\rho^{2k-1}+\ldots+4k(4k-2)\ldots4\times2\gamma^{2k}\right\},$$

$$\upsilon = 2\sqrt{\lambda_1\lambda_2}\big/\left(\lambda_1+\lambda_2\right).$$

Appendix B. On Integration of $\widehat{J}_{R_\lambda}^{R99_A}$

Change of variable $s = u\big/\sqrt{2\lambda_1}$ in integral $\widehat{J}_{R_\lambda}^{R99_A}$ leads to

$$\widehat{J}_{R_\lambda}^{R99_A} = \frac{2}{\sqrt{\pi}}\sqrt{\frac{\lambda_1}{\lambda_1-\lambda_2}}\left\{\int_{\frac{R_\lambda}{\sqrt{2\lambda_1}}}^{\frac{R99_A}{\sqrt{2\lambda_1}}}\mathrm{e}^{-u^2}\mathrm{d}u+\sum_1^{M_\infty}\frac{\prod_{s=1}^{k}(2s-1)^2}{(16\lambda_1\Omega)^k\,k!}J_k^\infty\right\}.$$

The integral $J_k^{\infty} = \int\limits_{\frac{R_\lambda}{\sqrt{2\lambda_1}}}^{\frac{R99_A}{\sqrt{2\lambda_1}}} \frac{\mathrm{e}^{-u^2}}{u^{2k}} \mathrm{d}u$ is performed by parts:

$$J_k^{\infty} = \frac{-1}{(2k-1)} \int\limits_{\frac{R_\lambda}{\sqrt{2\lambda_1}}}^{\frac{R99_A}{\sqrt{2\lambda_1}}} \mathrm{e}^{-u^2}\, \mathrm{d}\frac{1}{u^{2k-1}} = \frac{-1}{(2k-1)} \left(\frac{\mathrm{e}^{-u^2}}{u^{2k-1}} \Bigg|_{\frac{R_\lambda}{\sqrt{2\lambda_1}}}^{\frac{R99_A}{\sqrt{2\lambda_1}}} + 2 \int\limits_{\frac{R_\lambda}{\sqrt{2\lambda_1}}}^{\frac{R99_A}{\sqrt{2\lambda_1}}} \frac{\mathrm{e}^{-u^2}}{u^{2k-2}} \mathrm{d}u \right) =$$

$$= \frac{-1}{(2k-1)} \left(\frac{\mathrm{e}^{-u^2}}{u^{2k-1}} \Bigg|_{\frac{R_\lambda}{\sqrt{2\lambda_1}}}^{\frac{R99_A}{\sqrt{2\lambda_1}}} + \frac{-2}{(2k-3)} \left(\frac{\mathrm{e}^{-u^2}}{u^{2k-3}} \Bigg|_{\frac{R_\lambda}{\sqrt{2\lambda_1}}}^{\frac{R99_A}{\sqrt{2\lambda_1}}} + 2 \int\limits_{\frac{R_\lambda}{\sqrt{2\lambda_1}}}^{\frac{R99_A}{\sqrt{2\lambda_1}}} \frac{\mathrm{e}^{-u^2}}{u^{2k-4}} \mathrm{d}u \right) \right) =$$

$$= \frac{1}{(2k-1)} \left[\frac{e^{-\frac{R_\lambda^2}{2\lambda_1}}}{\left(\frac{R_\lambda}{\sqrt{2\lambda_1}}\right)^{2k-1}} - \frac{e^{-\frac{R99_A^2}{2\lambda_1}}}{\left(\frac{R99_A}{\sqrt{2\lambda_1}}\right)^{2k-1}} \right] - \frac{2}{(2k-1)(2k-3)} \left[\frac{e^{-\frac{R_\lambda^2}{2\lambda_1}}}{\left(\frac{R_\lambda}{\sqrt{2\lambda_1}}\right)^{2k-3}} - \frac{e^{-\frac{R99_A^2}{2\lambda_1}}}{\left(\frac{R99_A}{\sqrt{2\lambda_1}}\right)^{2k-3}} \right] +$$

$$+ \frac{4}{(2k-1)(2k-3)(2k-5)} \left[\frac{e^{-\frac{R_\lambda^2}{2\lambda_1}}}{\left(\frac{R_\lambda}{\sqrt{2\lambda_1}}\right)^{2k-5}} - \frac{e^{-\frac{R99_A^2}{2\lambda_1}}}{\left(\frac{R99_A}{\sqrt{2\lambda_1}}\right)^{2k-5}} \right] - \ldots + \frac{(-2)^{\kappa-1}}{(2k-1)\ldots\times 5\times 3\times 1} \left[\frac{e^{-\frac{R_\lambda^2}{2\lambda_1}}}{\left(\frac{R_\lambda}{\sqrt{2\lambda_1}}\right)} - \frac{e^{-\frac{R99_A^2}{2\lambda_1}}}{\left(\frac{R99_A}{\sqrt{2\lambda_1}}\right)} \right] +$$

$$+ \frac{(-2)^{\kappa}}{(2k-1)\ldots\times 5\times 3\times 1} \int\limits_{\frac{R_\lambda}{\sqrt{2\lambda_1}}}^{\frac{R99_A}{\sqrt{2\lambda_1}}} \mathrm{e}^{-u^2}\, \mathrm{d}u = \frac{(-2)^{k}}{\prod\limits_{s=1}^{k} \left[2(k-s)+1 \right]} \int\limits_{\frac{R_\lambda}{\sqrt{2\lambda_1}}}^{\frac{R99_A}{\sqrt{2\lambda_1}}} \mathrm{e}^{-u^2}\, \mathrm{d}u +$$

$$+ \sum_{\ell=1}^{k} \frac{(-2)^{\ell-1}}{\prod\limits_{s=1}^{\ell} \left[2(k-s)+1 \right]} \left[\frac{e^{-\frac{R_\lambda^2}{2\lambda_1}}}{\left(\frac{R_\lambda}{\sqrt{2\lambda_1}}\right)^{2(k-\ell)+1}} - \frac{e^{-\frac{R99_A^2}{2\lambda_1}}}{\left(\frac{R99_A}{\sqrt{2\lambda_1}}\right)^{2(k-\ell)+1}} \right].$$

References

1. Smith, S.T. Statistical resolution limits and the complexified Cramer-Rao bound. *IEEE Trans. Signal Process.* **2005**, *53*, 5, 1597-1609.
2. Abramovich, Y.I.; Johnson, B.A.; Spencer, N.K. Statistical nonidentifiability of close emitters: Maximum-likelihood estimation breakdown. In Proceedings of the 17th European Signal Processing Conference, EUSIPCO 2009, Glasgow, Scotland, UK, Aug. 24-28, 2009.
3. Amar, A.; Weiss, A.J. Fundamental resolution limits of closely spaced random signals. *IET Radar, Sonar Nav.* **2008**, 2, 3, 170–179.

 doi:10.20944/preprints202404.1776.v1

4. Abramovich, Y.I.; Johnson, B.A. Detection-estimation of very close emitters: performance breakdown, ambiguity, and general statistical analysis of maximum-likelihood estimation. *IEEE Trans. Signal Process.* **2010**, 58, 7, 3647-3660.
5. Sun, M.; Jiang, D; Song, H.; Liu Y. Statistical resolution limit analysis of two closely spaced signal sources using Rao test. *IEEE Access* **2017**, 5, 22013-22022.
6. Lee, H.B. The Cramer-Rao bound on frequency of signals closely spaced in frequency. *IEEE Trans. Signal Process.* **1992**, 40, 6, 1508-1517.
7. Delmas, J.P.; Abeida, H. Statistical resolution limits of DOA for discrete sources. In Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing, Toulouse, France, May 14-19, 2006.
8. Korso, M.N.El.; Boyer, R.; Renaux, A.; Marcos, S. Statistical resolution limit for multiple parameters of interest for multiple signals. In Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing, Dallas, Texas, USA, March 14-19, 2010.
9. Cramer, H. *Mathematical methods of statistics*; Princeton University Press: New York, NY, USA, 1946.
10. Xiao Z.; Yan Z. Radar emitter identification based on naive Bayesian algorithm. In Proceedings of the 2020 IEEE 5th Information Technology and Mechatronics Engineering Conference (ITOEC), Chongqing, China, June 12-14, 2020.
11. Tempola, F.; Muhamad M.; Khairan A. Naive Bayes classifier for prediction of volcanic status in Indonesia. In Proceedings of the 2018 5th International Conference on Information Technology, Computer, and Electrical Engineering (ICITACEE), Semarang, Indonesia, Sep. 27-28, 2018.
12. Bretthorst, G.L.; Smith, C.R. Bayesian analysis of signals from closely-spaced objects. *Infrared Systems and Components III: 16-17 Jan 1989, Los Angeles, California – Proceedings of SPIE Series*, **1989**, 1050, 93–104.
13. Fazel, M.; Wester, M.J. Analysis of super-resolution single molecule localization microscopy data: A tutorial. *AIP Advances* **2022**, 12, 1.
14. Clark, M.P. On the resolvability of normally distributed vector parameter estimates. *IEEE Trans. Signal Process.* **1995**, 43, 12, 2975-2981.
15. Delmas, J.P. Asymptotic performance of second-order algorithms. *IEEE Trans. Signal Process.* **2002**, 50, 1, 49-57.
16. Dwight, H.B. *Tables of integrals and other mathematical data*, 3rd ed.; The Macmillan Company: New York, NY, USA, 1957; p. 93.
17. Duchesne, P.; De Micheaux, P.L. Computing the distribution of quadratic forms: further comparisons between the Lui-Tang-Zhang approximation and exact methods. *Comput. Stat. Data Anal.* **2010**, 54, 858-862.
18. Gillis, J.T. Computation of circular error probability integral. *IEEE Trans. Aero. Electron. Syst.* **1991**, 27, 6, 906-910.
19. Pyati, V.P. Computation of the circular error probability (CEP) integral, *IEEE Trans. Aero. Electron. Syst.* **1993**, 29, 3, 1023-1024.
20. Shnildman, D.A. Efficient computation of the circular error probability (CEP) integral. *IEEE Trans. Autom. Control* **1995**, 40, 8, 1472-1474.
21. Ignagni, M. Determination of circular and spherical position-error bounds in system performance analysis. *J. Guid., Control Dynam.* **2010**, 33, 4, 1301-1305.
22. Krempasky, J.J. CEP equation exact to the fourth order. *Navigation-US* **2003**, 50, 3, 143-149.
23. Torrieri, D.J. Statistical theory of passive location systems. *IEEE Trans. Aero. Electron. Syst.* **1984**, AES-20, 2, 183-198.
24. Abramowitz, M.; Stegun, I.A. *Handbook of mathematical function: with formulas, graphs, and mathematical tables*; Nat. Bur. Standards Appl. Math. Series, U.S. Gov't Printing Office: Washington, D.C., USA, 1972, pp. 375, 377.
25. Huang, Y.; Benesty, J.; Chen, J. Time delay estimation and acoustic source localization. In *Acoustic MIMO Signal Processing (Signals and Communication Technology)*; Springer: Berlin, Heidelberg, FRG, 2006, pp. 215-259.
26. Cheung, K.W.; So, H.C.; Ma, W.K.; Chan, Y.T. A constrained least squares approach to mobile positioning: Algorithms and optimality. *EURASIP J. Appl. Signal Process.* **2006**, 2006, 1-23.
27. Chan, Y.T.; Ho, K.C. A simple and efficient estimator for hyperbolic location. *IEEE Trans. Signal Process.* **1994**., 42, 8, 1905-1915.
28. Huang, Y.; Benesty, J.; Elko, G.W.; Mersereati, R.M. Real-time passive source localization: a practical linear-correction least-square approach. *IEEE Trans. Speech, Audio Process.* **2001**, 9, 8, 943-956.
29. Lui, H.; Tang, Y.; Zhang, H. H. A new chi-square approximation to the distribution of non-negative definite quadratic forms in non-central normal variables. *Compute. Stat. Data Anal.* **2009**, 53, 853-856.