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Posted Date: 28 April 2024

doi: 10.20944/preprints202404.1843.v1

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Randomly Stopped Sums, Minima and Maxima for Heavy-tailed and Light-Tailed Distributions

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Abstract: This paper investigates the randomly stopped sums, minima and maxima of heavy- and light-tailed random variables. The conditions on the primary random variables, which are independent but generally not identically distributed, and counting random variable are given in order that the randomly stopped sum, random minimum and maximum is heavy/light tailed. The results generalize some existing ones in the literature. The examples illustrating the results are provided.

Keywords: heavy tail; light tail; randomly stopped sums; randomly stopped minima; randomly stopped maxima

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1. Introduction

This paper is devoted to the randomly stopped sums, minima and maxima of heavy- and light-tailed random variables (r.v.s). Such objects appear when the number of the random variables under consideration is unknown and is described by some random integer. In particular, randomly stopped sums appear in such fields as insurance and financial mathematics, survival analysis, risk theory, computer and communication networks, etc. The area of randomly stopped sums for heavy-tailed r.v.s is well-developed for more than 50 years and covers mainly the case of independent identically distributed (i.i.d.) r.v.s. In this paper we consider the case where the underlying r.v.s are not necessary identically distributed although are independent.

Specifically, suppose that $X_1, X_2, \dots$ are r.v.s defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define a sequence of partial sums $\{S_n, n \geq 0\}$ by

$$S_0 := 0, \;\; S_n := X_1 + \cdots + X_n, \; n \geq 1. \tag{1}$$

The main subject of the paper lies in the study of *randomly stopped sums*

$$S_\nu := X_1 + \cdots + X_\nu,$$

where n in (1) is replaced by a random variable ν taking values in $\mathbb{N}_0 := \{0, 1, 2, \dots\}$. Throughout the paper, we assume that ν is not degenerate at zero, i.e. $\mathbb{P}(\nu > 0) > 0$. We will call such ν a *counting random variable*.

Further, we will assume that r.v.s $X_1, X_2, \dots$ are independent and counting r.v. ν is independent of the sequence $\{X_1, X_2, \dots\}$. In general, r.v.s $X_1, X_2, \dots$ can be not identically distributed, each having a distribution function (d.f.) $F_{X_k}(x) = \mathbb{P}(X_k \leq x)$, respectively. Consider the d.f.

$$F_{S_\nu}(x) = \mathbb{P}(S_\nu \leq x) \;=\; \sum_{n=0}^{\infty} \mathbb{P}(S_n \leq x)\mathbb{P}(\nu = n).$$

The main task considered in the paper is to give conditions guaranteeing that F_{S_ν} is heavy/light-tailed provided that some of the d.f.s F_{X_k} or F_ν are heavy/light-tailed.

 doi:10.20944/preprints202404.1843.v1

Other objects of the paper are the randomly stopped minima and maxima. By the *randomly stopped minimum of sums* we call the minimum of partial sums:

$$S_{(\nu)} = \begin{cases} \min\{S_1, \ldots, S_\nu\}, & \nu \geq 1, \\ 0, & \nu = 0, \end{cases}$$

and by the *randomly stopped maximum of sums* we call the maximum of partial sums:

$$S^{(\nu)} = \max\{0, S_1, \ldots, S_\nu\}.$$

Also, we provide some results for *randomly stopped minimum*

$$X_{(\nu)} = \begin{cases} \min\{X_1, \ldots, X_\nu\}, & \nu \geq 1, \\ 0, & \nu = 0, \end{cases}$$

and *randomly stopped maximum*

$$X^{(\nu)} = \max\{0, X_1, \ldots, X_\nu\}.$$

Similarly, we are interested when $F_{X_{(\nu)}}$, $F_{X^{(\nu)}}$, $F_{S_{(\nu)}}$ and $F_{S^{(\nu)}}$ are heavy-tailed or light tailed. For various distribution classes, similar questions were studied in [1–10], [11–20], [21–30]. We mention also the paper [31], where two independent heavy-tailed r.v.s, such that their minimum is not heavy tailed, were constructed.

The structure of the paper is as follows. In Section 2 we introduce heavy- and light-tailed distributions and formulate two auxiliary lemmas. Main results are formulated in Section 3. Some examples of non-standard heavy-tailed and light-tailed distributions are presented in Section 4. The heaviness of the distribution tails presented in Section 4 is determined on the basis of the statements formulated in Section 3. The proofs of main results are presented in Section 5. The last Section 6 is devoted to the discussion of the obtained results in the broadest context together with the highlight of the future research directions.

2. Heavy-tailed and light-tailed distributions

For any distribution F, define its Laplace-Stieltjes transform as

$$\widehat{F}(\lambda) := \int_{-\infty}^{\infty} \mathrm{e}^{\lambda x} \mathrm{d}F(x), \;\; \lambda \in \mathbb{R}.$$

A distribution F is said to be *heavy-tailed*, denoted $F \in \mathcal{H}$, if

$$\widehat{F}(\lambda) = \infty \;\text{ for any }\; \lambda > 0.$$

Otherwise, F is said to be *light-tailed*. Common examples of heavy-tailed distributions are Pareto, Log-normal, Weibull with shape parameter $\tau \in (0, 1)$, Burr, Student's t distributions. For the detailed exposition of the heavy-tailed distributions and their properties we refer to the monographs [32–34], [35–38].

We formulate two lemmas that will be used in the proofs of several main propositions. Although the results of the lemmas are well-known and can be found, e.g., in [35], [37], [38], we provide the proofs for the sake of convenience. First lemma gives equivalent conditions for the distribution F to be heavy/light tailed.

Lemma 1. *Suppose that F is a d.f. of a real-valued r.v. The following statements are equivalent:*

(i) *F is heavy-tailed,*

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(ii) $\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\overline{F}(x) = \infty$ *for any* $\lambda > 0$,

(iii) $\limsup_{x\to\infty} x^{-1}\log\overline{F}(x) = 0$.

Similarly, the equivalent are the following statements:

(i') *F is light-tailed*,

(ii') $\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\,\overline{F}(x) < \infty$ *for some* $\lambda > 0$,

(iii') $\limsup_{x\to\infty} x^{-1}\log\overline{F}(x) < 0$.

Proof. We prove only the first part of the lemma.
(i) $\Rightarrow$ (iii). Suppose that $\widehat{F}(\lambda) = \infty$ for any $\lambda > 0$. Let, on the contrary,

$$\limsup_{x\to\infty} \frac{\log\overline{F}(x)}{x} < 0.$$

Then there exist constants $c > 0$ and $x_c > 0$ such that $x^{-1}\log\overline{F}(x) \le -c$ for $x \ge x_c$ or, equivalently,

$$\overline{F}(x) \le \mathrm{e}^{-cx}, \quad x \ge x_c. \tag{2}$$

For any $\delta \in (0, c)$, using (2) and the alternative expectation formula (see [39], for instance), we obtain

$$\begin{aligned}
\int_{[0,\infty)} \mathrm{e}^{\delta u}\mathrm{d}F(u) &= 1 + \delta\int_0^\infty \mathrm{e}^{\delta u}\overline{F}(u)\mathrm{d}u \\
&= 1 + \left(\int_1^{\mathrm{e}^{\delta x_c}} + \int_{\mathrm{e}^{\delta x_c}}^\infty\right)\overline{F}(\delta^{-1}\log u)\mathrm{d}u \\
&\le \mathrm{e}^{\delta x_c} + \int_{\mathrm{e}^{\delta x_c}}^\infty \mathrm{e}^{-c\delta^{-1}\log u}\mathrm{d}u \\
&= \mathrm{e}^{\delta x_c} + \int_{\mathrm{e}^{\delta x_c}}^\infty u^{-c\delta^{-1}}\mathrm{d}u.
\end{aligned}$$

Since $c\delta^{-1} > 1$, the last integral is finite, hence

$$\widehat{F}(\delta) \le F(0) + \int_{[0,\infty)} \mathrm{e}^{\delta u}\mathrm{d}F(u) \ < \ \infty,$$

leading to a contradiction.
(iii) $\Rightarrow$ (ii). From the condition

$$\limsup_{x\to\infty} x^{-1}\log\overline{F}(x) = 0$$

we get that there exists an infinitely increasing sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} x_n^{-1}\log\overline{F}(x_n) = 0.$$

For any given $\lambda > 0$, this implies that there exists $n_\lambda \ge 1$ such that

$$x_n^{-1}\log\overline{F}(x_n) \ge -\lambda/2$$

for all $n \ge n_\lambda$. Equivalently,

$$\mathrm{e}^{\lambda x_n}\overline{F}(x_n) \ge \mathrm{e}^{\lambda x_n/2}, \quad n \ge n_\lambda.$$

Hence, $e^{\lambda x_n}\overline{F}(x_n)$ tends to infinity as $n \to \infty$, and thus

$$\limsup_{x\to\infty} e^{\lambda x}\overline{F}(x) \geq \lim_{n\to\infty} e^{\lambda x_n}\overline{F}(x_n) = \infty.$$

Since this holds for any $\lambda > 0$, we have (ii).
(ii) $\Rightarrow$ (i). Let

$$\limsup_{x\to\infty} e^{\lambda x}\overline{F}(x) = \infty$$

for any $\lambda > 0$. For any $x \in \mathbb{R}$ write

$$\int_{-\infty}^{\infty} e^{\lambda u} dF(u) \geq \int_{(x,\infty)} e^{\lambda u} dF(u) \geq e^{\lambda x}\overline{F}(x).$$

Thus,

$$\widehat{F}(\lambda) \geq \limsup_{x\to\infty} e^{\lambda x}\overline{F}(x) = \infty \text{ for any } \lambda > 0.$$

Lemma 1 is proved. □

The next lemma implies that $\mathcal{H}$ and $\mathcal{H}^c$ are closed with respect to weak tail equivalence.

Lemma 2. *Let F and G be two distributions of real-valued r.v.s.*
(i) *If $F \in \mathcal{H}$ and*

$$\liminf_{x\to\infty} \frac{\overline{G}(x)}{\overline{F}(x)} > 0, \tag{3}$$

then $G \in \mathcal{H}$.
(ii) *If $F \in \mathcal{H}^c$, and $\overline{G}(x) \leq \tilde{c}\,\overline{F}(x)$ for some $\tilde{c} > 0$ and large x ($x > x_{\tilde{c}}$), then $G \in \mathcal{H}^c$.*

Proof. Consider part (i). By condition (3) we get that

$$\overline{G}(x) \geq \hat{c}\,\overline{F}(x)$$

for some $\hat{c}$ and sufficiently large x ($x > x_{\hat{c}}$). Therefore,

$$\limsup_{x\to\infty} e^{\lambda x}\,\overline{G}(x) \geq \hat{c}\,\limsup_{x\to\infty} e^{\lambda x}\,\overline{F}(x) = \infty$$

for any positive λ implying $G \in \mathcal{H}$ by Lemma 1 (ii).
Proof of part (ii) can be constructed in a similar way by using Lemma 1 (ii') showing that

$$\limsup_{x\to\infty} e^{\lambda x}\,\overline{G}(x) < \infty$$

for some $\lambda > 0$. Lemma 2 is proved. □

3. Main results

In this section we formulate the main results of the paper. We start with the randomly stopped sums. We notice that the d.f. F_{S_ν} can become heavy-tailed because of heavy tail of some element in $\{F_{X_1}, F_{X_2}, \ldots\}$ or because of heavy tail of the counting random variable ν.

Proposition 1. *Let $X_1, X_2, \ldots$ be independent real-valued r.v.s and let ν be a counting r.v. independent of the sequence $\{X_1, X_2, \ldots\}$. Distribution F_{S_ν} is heavy-tailed if at least one of the following conditions is satisfied:*
(i) $\inf_{k\geq 1} \mathbb{E}\, e^{\lambda X_k} > 1$ *for any $\lambda > 0$, and $F_\nu \in \mathcal{H}$;*

 doi:10.20944/preprints202404.1843.v1

(ii) $\inf_{k\geq 1} \mathbb{P}(X_k \geq a) = 1$ *for some* $a > 0$, *and* $F_\nu \in \mathcal{H}$;

(iii) $F_{X_\varkappa} \in \mathcal{H}$ *for some* $\varkappa \geq 1$, *and* $\overline{F_\nu}(x) > 0$ *for all* $x \in \mathbb{R}$;

(iv) $F_{X_\varkappa} \in \mathcal{H}$ *for some* $1 \leq \varkappa \leq \max\{\operatorname{supp}(\nu)\}$ *and* $\operatorname{supp}(\nu) < \infty$.

Distribution F_{S_ν} *is light-tailed if at least one of the following conditions is satisfied:*

(v) $F_{X_1} \in \mathcal{H}^c$, $F_\nu \in \mathcal{H}^c$, $\overline{F_{X_1}}(x) > 0$ *for all* $x \in \mathbb{R}$ *and*

$$\limsup_{x\to\infty} \sup_{k\geq 1} \frac{\overline{F_{X_k}}(x)}{\overline{F_{X_1}}(x)} < \infty; \tag{4}$$

(vi) $\sup_{k\geq 1} \mathbb{E}\, \mathrm{e}^{\lambda X_k} < \infty$ *for some* $\lambda > 0$, *and* $F_\nu \in \mathcal{H}^c$.

Our next statement is about the randomly stopped maximum of r.v.s. We observe that some conditions under which the distribution of the randomly stopped maximum $F_{X^{(\nu)}}$ becomes heavy-tailed are the same as in Proposition 1. Unfortunately, we did not find how to make a heavy-tailed distribution $F_{X^{(\nu)}}$ from the light-tailed primary r.v.s $\{X_1, X_2, \ldots\}$.

Proposition 2. *Let* $X_1, X_2, \ldots$ *be independent real-valued r.v.s and let* ν *be a counting r.v. independent of the sequence* $\{X_1, X_2, \ldots\}$.

(i) *If* $F_{X_\varkappa} \in \mathcal{H}$ *for some* $\varkappa \geq 1$ *and* $\overline{F_\nu}(x) > 0$ *for all* $x \in \mathbb{R}$, *then* $F_{X^{(\nu)}} \in \mathcal{H}$;

(ii) *If* $F_{X_\varkappa} \in \mathcal{H}$ *for some* $\varkappa \leq \max\{\operatorname{supp}(\nu)\} < \infty$, *then* $F_{X^{(\nu)}} \in \mathcal{H}$;

(iii) *Distribution* $F_{X^{(\nu)}}$ *belongs to the class* $\mathcal{H}^c$ *if* $F_{X_1} \in \mathcal{H}^c$, $\overline{F_{X_1}}(x) > 0$ *for all* $x \in \mathbb{R}$, $\mathbb{E}\nu < \infty$ *and*

$$\limsup_{x\to\infty} \sup_{n\geq 1} \frac{1}{n} \sum_{k=1}^{n} \frac{\overline{F_{X_k}}(x)}{\overline{F_{X_1}}(x)} < \infty. \tag{5}$$

The statement below is on the distribution of the randomly stopped minimum of r.v.s. From the formulation below, we observe that the tail of the d.f. $F_{X_{(\nu)}}$ has much less chance of becoming heavy compared to the d.f.s F_{S_ν} and $F_{X^{(\nu)}}$.

Proposition 3. *Let* $X_1, X_2, \ldots$ *be independent real-valued r.v.s and let* ν *be a counting r.v. independent of the sequence* $\{X_1, X_2, \ldots\}$.

(i) *If* $F_{X_1} \in \mathcal{H}$ *and*

$$\liminf_{x\to\infty} \min_{1\leq k\leq \varkappa} \frac{\overline{F_{X_k}}(x)}{\overline{F_{X_1}}(x)} > 0$$

for $\varkappa = \min\{\operatorname{supp}(\nu) \setminus \{0\}\}$, *then* $F_{X_{(\nu)}} \in \mathcal{H}$ *and*

$$\overline{F_{X_{(\nu)}}}(x) \underset{x\to\infty}{\sim} \overline{F_{X_{(\varkappa)}}}(x);$$

(ii) *If* $F_{X_k} \in \mathcal{H}^c$ *for* $1 \leq k \leq \varkappa = \min\{\operatorname{supp}(\nu) \setminus \{0\}\}$, *then* $F_{X_{(\nu)}} \in \mathcal{H}^c$.

The next two statements are on the heaviness of randomly stopped minimum of sums and randomly stopped maximum of sums. It can be seen from the presented formulations that some of the conditions were already present in the previous statements. However, for the sake of clarity, we present the full statements on the heaviness of $F_{S_{(\nu)}}$ and $F_{S^{(\nu)}}$.

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Proposition 4. *Let $X_1, X_2, \ldots$ be independent real-valued r.v.s and let ν be a counting r.v. independent of the sequence $\{X_1, X_2, \ldots\}$.*

(i) *If $F_{X_1} \in \mathcal{H}$ and $\min\limits_{1\le k\le \varkappa} \mathbb{P}(X_k \ge 0) > 0$ for $\varkappa = \min\{\mathrm{supp}(\nu) \setminus \{0\}\}$, then $F_{S_{(\nu)}} \in \mathcal{H}$ and*

$$\overline{F_{S_{(\nu)}}}(x) \underset{x\to\infty}{\asymp} \overline{F_{X_1}}(x). \tag{6}$$

(ii) *If $F_{X_1} \in \mathcal{H}^c$, then $F_{S_{(\nu)}} \in \mathcal{H}^c$ for any r.v. ν.*

Proposition 5. *Let $\{X_1, X_2, \ldots\}$ and ν be r.v.s. such as in Propositions 1-4. Then $F_{S^{(\nu)}} \in \mathcal{H}$ if at least one of the following conditions is satisfied:*

(i) $\inf\limits_{k\ge1} \mathbb{E}\,\mathrm{e}^{\lambda X_k} > 1$ *for all* $\lambda > 0$ *and* $F_\nu \in \mathcal{H}$;

(ii) $\inf\limits_{k\geqslant 1} \mathbb{P}(X_k \geqslant a) = 1$ *for some* $a > 0$ *and* $F_\nu \in \mathcal{H}$;

(iii) $F_{X_1} \in \mathcal{H}$;

(iv) $F_{X_\varkappa} \in \mathcal{H}$ *for some* $\varkappa \ge 1$ *in the case of infinite* $\mathrm{supp}(\nu)$ *or for some* $1 \le \varkappa \le \max\{\mathrm{supp}(\nu)\}$ *in the case of finite* $\mathrm{supp}(\nu)$.

Distribution $F_{S^{(\nu)}}$ is light-tailed if at least one of the following two conditions is satisfied:

(v) $\sup\limits_{k\ge1} \mathbb{E}\,\mathrm{e}^{\lambda X_k} < \infty$ *for some* $\lambda > 0$ *and* $F_\nu \in \mathcal{H}^c$.

In the i.i.d. case, Proposition 1 immediately implies the following corollaries. Note that the first two corollaries can be found in monograph [35] as Problems 2.12 and 2.13.

Corollary 1. *Let $X_1, X_2, \ldots$ be i.i.d. real-valued r.v.s with common distribution F_{X_1}, and let ν be a counting r.v. independent of $\{X_1, X_2, \ldots\}$. If $F_{X_1} \in \mathcal{H}^c$ and $F_\nu \in \mathcal{H}^c$ then $F_{S_\nu} \in \mathcal{H}^c$.*

Corollary 2. *Let $X_1, X_2, \ldots$ be i.i.d. nonnegative not degenerate at zero r.v.s, and let ν be a counting r.v. independent of $\{X_1, X_2, \ldots\}$. If $F_\nu \in \mathcal{H}$ then $F_{S_\nu} \in \mathcal{H}$.*

Corollary 3. *Let $X_1, X_2, \ldots$ be i.i.d. real-valued r.v.s with common distribution F_{X_1}, and let ν be a counting r.v. independent of $\{X_1, X_2, \ldots\}$. If $F_{X_1} \in \mathcal{H}$ then $F_{S_\nu} \in \mathcal{H}$.*

Analogous corollaries can be formulated for randomly stopped minima and maxima.

4. Examples

In this section, we present two examples showing how one concretely can construct heavy-tailed distributions by using the above randomly stopped structures.

Example 1. *Let $\{X_1, X_2, \ldots\}$ be a sequence of independent r.v.s such that the first member X_1 has the Pareto distribution*

$$F_{X_1}(x) = \left(1 - \frac{1}{(1+x)^3}\right)\mathbb{1}_{[0,\infty)}(x),$$

and other elements of the sequence are identically exponentially distributed:

$$F_{X_k}(x) = \left(1 - \mathrm{e}^{-x}\right)\mathbb{1}_{[0,\infty)}(x),\;\; k \in \{2,3,\ldots\}$$

According to Proposition 1 (parts (iii) and (iv)) and Proposition 5 (iii) distributions F_{S_ν} and $F_{S^{(\nu)}}$ are heavy-tailed for any counting r.v. independent of the sequence $\{X_1, X_2, \ldots\}$. For instance, in the

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case of the discrete uniform counting r.v. with parameter $N \geq 2$, we have that distributions with the tail

$$
\begin{aligned}
\overline{F_{S_\nu}}(x) &= \overline{F_{S^{(\nu)}}}(x) \\
&= \mathbb{1}_{(-\infty,0)}(x) + \Big(\frac{1}{(1+x)^3} + \frac{1}{N}\sum_{k=1}^{N-1}\frac{1}{(k-1)!}\int_0^x \frac{y^{k-1}\mathrm{e}^{-y}}{(1+(x-y))^3}\,\mathrm{d}y\Big)\mathbb{1}_{[0,\infty)}(x)
\end{aligned}
$$

belong to the class $\mathcal{H}$. Proposition 2 (ii) implies that distribution $F_{X^{(\nu)}}$ belongs to the class $\mathcal{H}$ for any counting r.v. ν independent of $\{X_1, X_2, \ldots\}$. Meanwhile Proposition 3 (i) and Proposition 4 (i) imply that $F_{X_{(\nu)}}$ and $F_{S_{(\nu)}}$ are heavy-tailed for counting r.v. under condition $1 \in \mathrm{supp}(\nu)$. In the case of the discrete uniform counting r.v. ν with parameter $N = 3$, we have that $F_{S_{(\nu)}} = F_{X_1}$ and distributions with the following tails are heavy-tailed:

$$
\begin{aligned}
\overline{F_{X^{(\nu)}}}(x) &= \mathbb{1}_{(-\infty,0)}(x) + \bigg(\frac{1}{(1+x)^3} + \Big(\mathrm{e}^{-x} - \frac{\mathrm{e}^{-2x}}{3}\Big)\Big(1 - \frac{1}{(1+x)^3}\Big)\bigg)\mathbb{1}_{[0,\infty)}(x), \\
\overline{F_{X_{(\nu)}}}(x) &= \mathbb{1}_{(-\infty,0)}(x) + \frac{1}{3(1+x)^3}\big(1 + \mathrm{e}^{-x} + \mathrm{e}^{-2x}\big)\mathbb{1}_{[0,\infty)}(x).
\end{aligned}
$$

Example 2. *Let $\{X_1, X_2, \ldots\}$ be a sequence of independent r.v.s uniformly distributed on the interval $[0,1]$, i.e.*

$$F_{X_k}(x) = x\mathbb{1}_{[0,1)}(x) + \mathbb{1}_{[1,\infty)}(x)$$

for each $k \in \mathbb{N}$.

Obviously,

$$\mathbb{E}\mathrm{e}^{\lambda X_k} = \frac{\mathrm{e}^{\lambda} - 1}{\lambda} > 1$$

for any $\lambda > 0$ and all $k \in \mathbb{N}$. Therefore, by Proposition 1 (i) and Proposition 5 (i) we get that distributions F_{S_ν} and $F_{S^{(\nu)}}$ are heavy-tailed for an arbitrary heavy tailed counting r.v. ν independent of $\{X_1, X_2, \ldots\}$. Suppose that counting r.v. ν is distributed according to the zeta distribution with parameter 2:

$$\mathbb{P}(\nu = n) = \frac{1}{n^2}\frac{1}{\zeta(2)}, \ \ n \in \mathbb{N},$$

where

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, \ s \in \mathbb{C},$$

denotes the Riemann zeta function. Such ν is heavy-tailed. Propositions 1 (i) and 5 (i) imply that distribution

$$F_{S_\nu}(x) = F_{S^{(\nu)}}(x) = \frac{1}{\zeta(2)}\sum_{n=1}^{\infty}\frac{1}{n^2}F_{X_1}^{*n}(x)\mathbb{1}_{[0,n]}(x)$$

belongs to the class $\mathcal{H}$, where

$$F_{X_1}^{*n}(x) = \frac{1}{n!}\sum_{k=0}^{\lfloor x\rfloor}(-1)^k\binom{n}{k}(x-k)^n$$

is the well-known Irwin-Hall distribution with parameter n, see [40,41] or Section 26.9 in [42]. Meanwhile propositions 3 (ii) and 4 (ii) imply that distributions with tails

$$\begin{aligned}\overline{F_{S^{(\nu)}}}(x) &= \overline{F_{X_1}}(x),\\ \overline{F_{X_{(\nu)}}}(x) &= \mathbb{1}_{(-\infty,0)}(x) + \frac{1}{\zeta(2)}\sum_{n=1}^{\infty}\frac{1}{n^2}(1-x)^n\,\mathbb{1}_{[0,1)}(x)\end{aligned}$$

are light-tailed despite the fact that counting r.v. ν distributed according to the zeta distribution is heavy-tailed.

5. Proofs of the main results

In this section, we present the proofs of all main propositions. We assign a separate subsection to the proof of each proposition.

5.1. Proof of Proposition 1

Proof of part (i). For any $\lambda > 0$ and an arbitrary $K \geq 1$, we have

$$\begin{aligned}\mathbb{E}\,\mathrm{e}^{\lambda S_\nu} &= \mathbb{E}\left(\mathrm{e}^{\lambda S_\nu}\sum_{n=0}^{\infty}\mathbb{1}_{\{\nu=n\}}\right) = \mathbb{E}\left(\sum_{n=0}^{\infty}\mathrm{e}^{\lambda S_n}\mathbb{1}_{\{\nu=n\}}\right)\\ &\geq \mathbb{E}\left(\sum_{n=0}^{K}\mathrm{e}^{\lambda S_n}\mathbb{1}_{\{\nu=n\}}\right)\\ &= \sum_{n=0}^{K}\mathbb{E}\,\mathrm{e}^{\lambda S_n}\mathbb{P}(\nu=n).\end{aligned} \tag{7}$$

From the condition

$$\inf_{k\geq 1}\mathbb{E}\,\mathrm{e}^{\lambda X_k} > 1$$

we derive that the estimate

$$\min_{1\leq k\leq K}\mathbb{E}\,\mathrm{e}^{\lambda X_k} \geq \Delta$$

holds for some $\Delta = \Delta(\lambda) > 1$. Therefore, for all $n \in \{1,\ldots,K\}$ we obtain

$$\mathbb{E}\,\mathrm{e}^{\lambda S_n} = \prod_{k=1}^{n}\mathbb{E}\,\mathrm{e}^{\lambda X_k} \geq \Delta^n. \tag{8}$$

This together with (7) imply that

$$\mathbb{E}\,\mathrm{e}^{\lambda S_\nu} \geq \sum_{n=0}^{K}\Delta^n\mathbb{P}(\nu=n).$$

Since $F_\nu \in \mathcal{H}$, we have

$$\sum_{n=0}^{K}\Delta^n\mathbb{P}(\nu=n) = \mathbb{E}\,\mathrm{e}^{\nu\log\Delta}\mathbb{I}_{\{\nu\leq K\}} \underset{K\to\infty}{\to} \infty.$$

Hence, $\mathbb{E}\,\mathrm{e}^{\lambda S_\nu} = \infty$ implying $F_{S_\nu} \in \mathcal{H}$ by definition. Part (i) of the proposition is proved. □

Proof of part (ii). Let us fix an arbitrary $\lambda > 0$. Due to the conditions of part (ii), for such λ we have

$$\begin{aligned}\inf_{k\geq 1} \mathbb{E}\, e^{\lambda X_k} &= \inf_{k\geq 1} \left(\mathbb{E}\, e^{\lambda X_k} \mathbb{1}_{\{X_k \geq a\}} + \mathbb{E}\, e^{\lambda X_k} \mathbb{1}_{\{X_k < a\}}\right)\\ &\geq \inf_{k\geq 1} \mathbb{E}\, e^{\lambda X_k} \mathbb{1}_{\{X_k \geq a\}}\\ &\geq \inf_{k\geq 1} e^{\lambda a}\, \mathbb{P}(X_k \geq a)\\ &= e^{\lambda a} > 1.\end{aligned}$$

Hence the assertion of part (ii) follows from part (i) of the proposition. $\square$

Proof of part (iii). The requirement $\overline{F_\nu}(x) > 0$ for all $x \in \mathbb{R}$ implies that counting r.v. ν has an unbounded support. Thus we can find $K \geq \varkappa$ such that $\mathbb{P}(\nu = K) > 0$. Let λ be any positive number and $M \geq 1$. Then

$$\begin{aligned}\mathbb{E}\, e^{\lambda S_K} &\geq \mathbb{E} \exp\left\{\lambda \sum_{k=1}^{K} X_k \mathbb{I}_{\{X_k \leq M\}}\right\}\\ &= e^{\lambda X_\varkappa \mathbb{I}_{\{X_\varkappa \leq M\}}} \prod_{\substack{k=1\\ k\neq\varkappa}}^{K} \mathbb{E}\, e^{\lambda X_k \mathbb{I}_{\{X_k \leq M\}}} \underset{M\to\infty}{\longrightarrow} \infty\end{aligned}$$

because $F_\varkappa \in \mathcal{H}$ and $\mathbb{E}e^{\lambda X_k} > 0$ for each $k \in \{1, \ldots, K\}$. Therefore, $F_{S_K} \in \mathcal{H}$. By representation (7) we get that

$$\mathbb{E}\, e^{\lambda S_\nu} \geq \mathbb{P}(\nu = K)\mathbb{E}\, e^{\lambda S_K}$$

implying $F_{S_\nu} \in \mathcal{H}$. This completes the proof of part (iii) of the proposition. $\square$

Proof of part (iv). Let K be such that $\mathbb{P}(\nu = K) > 0$ and $\varkappa \leq K$. Clearly, conditions of part (iv) imply the existence of such K. To finish the proof of this part, it is sufficient to repeat the arguments of part (iii). $\square$

Proof of part (v). Suppose that $0 < \delta \leq \lambda$, and $\lambda > 0$ is such that $\mathbb{E}\, e^{\lambda X_1^+} < \infty$ with $X_1^+ := X_1 \mathbb{I}_{\{X_1 \geq 0\}}$. By the standard representation (7) we have

$$\begin{aligned}\mathbb{E}\, e^{\delta S_\nu} &= \sum_{n=0}^{\infty} \mathbb{E}\, e^{\delta S_n} \mathbb{P}(\nu = n)\\ &\leq \sum_{n=0}^{\infty} \mathbb{E}\, e^{\delta S_n^+} \mathbb{P}(\nu = n),\end{aligned} \tag{9}$$

where $S_0^+ = 0$ and

$$S_n^+ = \sum_{k=1}^{n} X_k^+ = \sum_{k=1}^{n} X_k \mathbb{I}_{\{X_k \geq 0\}},\ n \in \{1, 2, \ldots\}.$$

Condition (4) implies

$$\overline{F_{X_k}}(x) \leq c_1 \overline{F_{X_1}}(x) \tag{10}$$

 doi:10.20944/preprints202404.1843.v1

for some $c_1 > 0$, all $k \geq 1$ and all $x \in \mathbb{R}$. Therefore, by the alternative expectation formula (see, for instance, [39]), we derive from (10) that

$$\begin{aligned}\mathbb{E}\,\mathrm{e}^{\delta X_k^+} &= 1 + \delta \int_0^\infty \mathrm{e}^{\delta u}\, \overline{F_{X_k^+}}(u)\,\mathrm{d}u \\ &\leq 1 + \delta c_1 \int_0^\infty \mathrm{e}^{\lambda u}\, \overline{F_{X_1}}(u)\,\mathrm{d}u \\ &= 1 + \frac{\delta}{\lambda} c_1 \Big(\mathbb{E}\,\mathrm{e}^{\lambda X_1^+} - 1 \Big) := c_2(\delta)\end{aligned}$$

for any $k \geq 1$, where $1 < c_2(\delta) < \infty$ for $0 < \delta \leq \lambda$, and

$$\lim_{\delta \downarrow 0} c_2(\delta) = 1.$$

Since $X_1^+, X_2^+, \ldots$ are independent r.v.s, we obtain

$$\mathbb{E}\,\mathrm{e}^{\delta S_n^+} = \prod_{k=1}^{n} \mathbb{E}\,\mathrm{e}^{\delta X_k^+} \leq \big(c_2(\delta)\big)^n.$$

Hence, by inequality (9) and condition $F_\nu \in \mathcal{H}^c$ we derive that

$$\mathbb{E}\,\mathrm{e}^{\delta S_\nu} \leq \sum_{n=0}^{\infty} \big(c_2(\delta)\big)^n\, \mathbb{P}(\nu = n) = \mathbb{E}\,\mathrm{e}^{\nu \log c_2(\delta)} < \infty$$

if $\delta \in (0, \lambda]$ is chosen sufficiently small. This implies that $F_{S_\nu} \in \mathcal{H}^c$. $\square$

Proof of part (vi). The statement of this part can be proved analogously to the statement of part (v). Namely, conditions of part (vi) imply that

$$\sup_{k \geq 1} \mathbb{E}\,\mathrm{e}^{\lambda X_k^+} = c_\lambda$$

for some constants $\lambda > 0$ and $c_\lambda \geq 1$. Therefore, using the alternative expectation formula, we derive

$$\begin{aligned}\mathbb{E}\,\mathrm{e}^{\delta X_k^+} &= 1 + \delta \int_{[0,\infty)} \mathrm{e}^{\delta u} \overline{F_{X_k}}(u)\,\mathrm{d}u \\ &\leq 1 + \frac{\delta}{\lambda} \Big(\lambda \int_{[0,\infty)} \mathrm{e}^{\lambda u} \overline{F_{X_k}}(u)\,\mathrm{d}u \Big) \\ &= 1 + \frac{\delta}{\lambda} (c_\lambda - 1)\end{aligned}$$

for all $\delta \in (0, \lambda)$ and $k \geq 1$. The last estimation and inequality (9) imply that

$$\mathbb{E}\,\mathrm{e}^{\delta S_\nu} \leq \sum_{n=0}^{\infty} \prod_{k=1}^{n} \mathbb{E}\,\mathrm{e}^{\delta X_k^+}\, \mathbb{P}(\nu = n) \leq \mathbb{E}\,\mathrm{e}^{\nu \log \left(1 + \frac{\delta}{\lambda}(c_\lambda - 1)\right)}.$$

If $\delta \in (0, \lambda]$ is sufficiently small, then the last expectation is finite because of $F_\nu \in \mathcal{H}^c$. Hence $F_{S_\nu} \in \mathcal{H}^c$ as well. Part (vi) of the proposition is proved. $\square$

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5.2. Proof of Proposition 2

Proof of part (i). By the standard representation we have

$$\begin{aligned}\overline{F_{X^{(\nu)}}}(x) &= \sum_{n=1}^{\infty} \mathbb{P}(X^{(n)} > x)\mathbb{P}(\nu = n) \\ &\geq \mathbb{P}(X^{(K)} > x)\mathbb{P}(\nu = K) \end{aligned} \tag{11}$$

for $x > 0$ and any K such that $\mathbb{P}(\nu = K) > 0$, $K \geq \varkappa$. Due to the conditions of part (ii) there exists a sequence of numbers K with the above property. Obviously,

$$\begin{aligned}\mathbb{P}(X^{(K)} > x) &= \mathbb{P}(\max\{0, X_1, \ldots, X_K\} > x) \\ &\geq \mathbb{P}(X_\varkappa > x). \end{aligned} \tag{12}$$

Consequently, for an arbitrary $\lambda > 0$, we get from (11) and (12)

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\, \overline{F_{X^{(\nu)}}}(x) \geq \mathbb{P}(\nu = K) \limsup_{x\to\infty} \mathrm{e}^{\lambda x}\, \overline{F_{X_\varkappa}}(x).$$

The assertion of part (i) follows now by Lemma 1. □

Proof of part (ii). The proof of this part is similar to the proof of part (i), because the conditions of part (ii) imply that there exists at least one K such that $K \geq \varkappa$ and $\mathbb{P}(\nu = K) > 0$. □

Proof of part (iii). The standard representation implies that

$$\overline{F_{X^{(\nu)}}}(x) = \sum_{n=1}^{\infty} \mathbb{P}(X^{(n)} > x)\mathbb{P}(\nu = n) \tag{13}$$

$$\begin{aligned} &= \sum_{n=1}^{\infty} \mathbb{P}\left(\bigcup_{k=1}^{n} \{X_k > x\}\right)\mathbb{P}(\nu = n) \\ &\leq \sum_{n=1}^{\infty} \mathbb{P}(\nu = n) \sum_{k=1}^{n} \overline{F_{X_k}}(x) \end{aligned} \tag{14}$$

for positive x.

Due to Lemma 1, there is $\lambda > 0$ such that

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\, \overline{F_{X_1}}(x) < \infty. \tag{15}$$

It follows from the estimate (13) that

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\, \overline{F_{X^{(\nu)}}}(x) \leq \limsup_{x\to\infty} \mathrm{e}^{\lambda x} \sum_{n=1}^{\infty} \mathbb{P}(\nu = n) \sum_{k=1}^{n} \overline{F_{X_k}}(x).$$

Condition (5) of part (iii) implies that

$$\sum_{k=1}^{n} \overline{F_{X_k}}(x) \leq c_4 n \overline{F_{X_1}}(x) \tag{16}$$

for all $n \geq 1$, for some $c_4 > 0$ and for sufficiently large x ($x \geq x_1$). Therefore, by (15) and (16) we get that

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\, \overline{F_{X^{(\nu)}}}(x) \leq c_4 \mathbb{E}\nu \limsup_{x\to\infty} \mathrm{e}^{\lambda x}\, \overline{F_{X_1}}(x) < \infty.$$

The assertion of part (iii) follows now by Lemma 1. □

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5.3. *Proof of Proposition 3*

Proof of part (i). By the standard representation we have

$$
\begin{aligned}
\overline{F_{X_{(\nu)}}}(x) &= \sum_{n=1}^{\infty} \mathbb{P}\big(\min\{X_1, \ldots, X_n\} > x\big)\mathbb{P}(\nu = n) \\
&= \sum_{n=1}^{\infty} \mathbb{P}(\nu = n) \prod_{k=1}^{n} \overline{F_{X_k}}(x) \\
&= \overline{F_{X_{(\varkappa)}}}(x)\mathbb{P}(\nu = \varkappa) + \sum_{n=\varkappa+1}^{\infty} \mathbb{P}(\nu = n)\overline{F_{X_{(\varkappa)}}}(x) \prod_{k=\varkappa+1}^{n} \overline{F_{X_k}}(x) \\
&\leq \overline{F_{X_{(\varkappa)}}}(x)\mathbb{P}(\nu = \varkappa)\left(1 + \overline{F_{X_{\varkappa+1}}}(x)\frac{\mathbb{P}(\nu > \varkappa)}{\mathbb{P}(\nu = \varkappa)}\right),
\end{aligned} \tag{17}
$$

and

$$\overline{F_{X_{(\nu)}}}(x) \geq \overline{F_{X_{(\varkappa)}}}(x)\mathbb{P}(\nu = \varkappa)$$

for each positive x. In addition, conditions of part (i) give that $\overline{F_{X_{(\varkappa)}}}(x) > 0$ for all positive x. Therefore

$$\overline{F_{X_{(\nu)}}}(x) \underset{x\to\infty}{\sim} \overline{F_{X_{(\varkappa)}}}(x).$$

We get from this, by using Lemma 2, that $F_{X_{(\nu)}} \in \mathcal{H}$ if $F_{X_{(\varkappa)}} \in \mathcal{H}$. Hence, to prove the assertion of part (i) it is enough to prove that $F_{X_{(\varkappa)}} \in \mathcal{H}$ for $1 \leq \varkappa \leq \min\{\mathrm{supp}(\nu) \setminus \{0\}\}$.

Due to the condition $F_{X_1} \in \mathcal{H}$ and Lemma 1 we have

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\, \overline{F_{X_1}}(x) = \infty \tag{18}$$

for an arbitrary $\lambda > 0$. The requirement

$$\liminf_{x\to\infty} \min_{1\leqslant k\leqslant \varkappa} \frac{\overline{F_{X_k}}(x)}{\overline{F_{X_1}}(x)} > 0$$

implies that

$$\overline{F_{X_k}}(x) \geq c_5\overline{F_{X_1}}(x)$$

for some positive c_5, sufficiently large x ($x \geq x_2$) and for all $1 \leq k \leq \varkappa$. Therefore, for any positive λ and large x ($x \geq x_2$) we obtain

$$
\begin{aligned}
\mathrm{e}^{\lambda x}\, \overline{F_{X_{(\varkappa)}}}(x) &= \mathrm{e}^{\lambda x} \prod_{k=1}^{\varkappa} \overline{F_{X_k}}(x) \\
&\geq c_5^{\varkappa}\, \mathrm{e}^{\lambda x}\, \big(\overline{F_{X_1}}(x)\big)^{\varkappa} \\
&= \big(c_5\, \mathrm{e}^{\lambda x/\varkappa}\, \overline{F_{X_1}}(x)\big)^{\varkappa}.
\end{aligned}
$$

By relation (18) we derive that

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\, \overline{F_{X_{(\varkappa)}}}(x) = \infty$$

implying that $F_{X_{(\varkappa)}} \in \mathcal{H}$. Part (i) of the proposition is proved. □

 doi:10.20944/preprints202404.1843.v1

Proof of part (ii). According to the inequality (17) and Lemma 2, $F_{X_{(\nu)}} \in \mathcal{H}^c$ if $F_{X_{(\varkappa)}} \in \mathcal{H}^c$. Since $\varkappa$ is finite, conditions $F_{X_k} \in \mathcal{H}^c, k \in \{1, 2, \ldots, \varkappa\}$ and Lemma 1 imply that

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\, \overline{F_{X_k}}(x) < \infty \tag{19}$$

for some $\lambda > 0$ and each $k \in \{1, 2, \ldots, \varkappa\}$. For this λ and an arbitrary positive x, we have

$$\mathrm{e}^{\lambda x}\, \overline{F_{X_{(\varkappa)}}}(x) = \prod_{k=1}^{\varkappa} \left(\mathrm{e}^{\lambda x/\varkappa} \overline{F_{X_k}}(x)\right).$$

Since $\lambda/\varkappa \leq \lambda$, due to (19),

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x/\varkappa}\, \overline{F_{X_k}}(x) < \infty$$

for each $k \in \{1, 2, \ldots, \varkappa\}$. Therefore,

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\, \overline{F_{X_{(\varkappa)}}}(x) < \infty$$

implying that $F_{X_{(\varkappa)}} \in \mathcal{H}^c$ by Lemma 1. Hence $F_{X_{(\nu)}} \in \mathcal{H}^c$ as well, and part (ii) of the proposition is proved. $\square$

5.4. Proof of Proposition 4

Proof of part (i). If $\varkappa = 1$, then for $x > 0$ we have

$$\begin{aligned}
\overline{F_{S_{(\nu)}}}(x) &= \sum_{n \in \mathrm{supp}(\nu)\setminus\{0\}} \overline{F_{S_{(n)}}}(x)\, \mathbb{P}(\nu = n) \\
&\geq \overline{F_{S_{(1)}}}(x)\, \mathbb{P}(\nu = 1) \\
&= \overline{F_{X_1}}(x)\, \mathbb{P}(\nu = 1),
\end{aligned}$$

and

$$\begin{aligned}
\overline{F_{S_{(\nu)}}}(x) &= \sum_{n=1}^{\infty} \overline{F_{S_{(n)}}}(x)\, \mathbb{P}(\nu = n) \\
&= \sum_{n=1}^{\infty} \mathbb{P}(\min\{S_1, \ldots, S_n\} > x)\, \mathbb{P}(\nu = n) \\
&= \sum_{n=1}^{\infty} \mathbb{P}\left(\bigcap_{k=1}^{n} \{S_k > x\}\right) \mathbb{P}(\nu = n) \\
&\leq \sum_{n=1}^{\infty} \mathbb{P}(S_1 > x)\, \mathbb{P}(\nu = n) \\
&= \overline{F_{X_1}}(x)\, \mathbb{P}(\nu \geq 1).
\end{aligned}$$

The derived estimates imply the asymptotic relation (6) in the case $\varkappa = 1$.

Let us now suppose that $\varkappa > 1$. Due to the conditions of part (i)

$$\mathbb{P}(X_k \geqslant 0) \geqslant c_6$$

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for some $c_6 > 0$ and all $1 \le k \le \varkappa$. Hence by the standard decomposition we get that for positive x

$$\begin{aligned}
\overline{F_{S_{(\nu)}}}(x) &= \sum_{n=1}^{\infty} \overline{F}_{S_{(n)}}(x)\, \mathbb{P}(\nu = n) \\
&\ge \overline{F_{S_{(\varkappa)}}}(x) \mathbb{P}(\nu = \varkappa) \\
&= \mathbb{P}(\min\{S_1, \ldots, S_\varkappa\} > x)\, \mathbb{P}(\nu = \varkappa) \\
&= \mathbb{P}\Big(\bigcap_{k=1}^{\varkappa} \{X_1 + \ldots + X_k > x\} \Big) \mathbb{P}(\nu = \varkappa) \\
&\ge \mathbb{P}(X_1 > x, X_2 \ge 0, \ldots, X_\varkappa \ge 0)\, \mathbb{P}(\nu = \varkappa) \\
&= \mathbb{P}(X_1 > x) \prod_{k=2}^{\varkappa} \mathbb{P}(X_k \ge 0) \mathbb{P}(\nu = \varkappa) \\
&\ge c_6^{\varkappa - 1}\, \mathbb{P}(\nu = \varkappa)\, \overline{F_{X_1}}(x). \qquad (20)
\end{aligned}$$

On the other hand, similarly as in the case $\varkappa = 1$, we have

$$\begin{aligned}
\overline{F_{S_{(\nu)}}}(x) &= \sum_{n \in \mathrm{supp}(\nu) \setminus \{0\}} \mathbb{P}\Big(\bigcap_{k=1}^{n} \{S_k > x\} \Big) \mathbb{P}(\nu = n) \\
&\le \sum_{n \in \mathrm{supp}(\nu) \setminus \{0\}} \mathbb{P}(S_1 > x) \mathbb{P}(\nu = n) \\
&= \overline{F_{X_1}}(x) \mathbb{P}(\nu \ge \varkappa). \qquad (21)
\end{aligned}$$

Estimates (20) and (21) imply that the asymptotic relation (6) holds for any possible $\varkappa$. In addition, we observe that, by Lemma 2, distribution $F_{S_{(\nu)}}$ belongs to $\mathcal{H}$ together with F_{X_1}. Part (i) of the proposition is proved. □

Proof of part (ii). The statement of this part follows immediately from the estimate (21) and Lemma 1 because

$$\limsup_{x \to \infty} \mathrm{e}^{\lambda x}\, \overline{F_{S_{(\nu)}}}(x) \le \mathbb{P}(\nu \ge 1) \limsup_{x \to \infty} \mathrm{e}^{\lambda x}\, \overline{F_{X_1}}(x)$$

for any $\lambda > 0$. □

5.5. Proof of Proposition 5

Proof of part (i). Proof of this part is similar to the proof of part (i) of Proposition 1. Namely, for $\lambda > 0$ and $K \ge 2$ by using (8), we get that

$$\begin{aligned}
\mathbb{E}\, \mathrm{e}^{\lambda S^{(\nu)}} &\ge \mathbb{E}\Big(\mathrm{e}^{\lambda S^{(\nu)}} \mathbb{1}_{\{\nu \le K\}} \Big) \\
&= \sum_{n=0}^{K} \mathbb{E}\, \mathrm{e}^{\lambda S^{(n)}} \mathbb{P}(\nu = n) \\
&\ge \sum_{n=0}^{K} \mathbb{E}\, \mathrm{e}^{\lambda S_n} \mathbb{P}(\nu = n) \\
&\ge \sum_{n=0}^{K} \Delta^n \mathbb{P}(\nu = n) \\
&= \mathbb{E}\left(\mathrm{e}^{\nu \log \Delta} \mathbb{1}_{\{\nu \le K\}} \right)
\end{aligned}$$

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with $\Delta = \Delta(\lambda) = \inf_{k\geq 1} \mathbb{E}e^{\lambda X_k} > 1$. The condition $F_\nu \in \mathcal{H}$ implies that

$$\lim_{K\to\infty} \mathbb{E}\left(e^{\nu \log \Delta} \mathbb{1}_{\{\nu \leq K\}}\right) = \infty.$$

Therefore, $\mathbb{E}e^{\lambda S^{(\nu)}} = \infty$ for an arbitrary $\lambda > 0$, i.e. $F_{S^{(\nu)}} \in \mathcal{H}$. Part (i) of the proposition is proved. $\square$

Proof of part (ii). The assertion of this part is obvious because condition $\inf_{k\geq 1} \mathbb{P}(X_k \geq a) = 1$ with $a > 0$ implies that $\inf_{k\geq 1} \mathbb{E}e^{\lambda X_k} > 1$ for any $\lambda > 0$. The details of this implication are presented in the proof of Proposition 1(ii). $\square$

Proof of part (iii). For positive x we have

$$\begin{aligned}
\overline{F_{S^{(\nu)}}}(x) &= \sum_{n=1}^{\infty} \overline{F_{S^{(n)}}}(x)\mathbb{P}(\nu = n) \\
&= \sum_{n=1}^{\infty} \mathbb{P}\Big(\bigcup_{k=1}^{n} \{S_k > x\}\Big)\mathbb{P}(\nu = n) \\
&\geq \sum_{n=1}^{\infty} \mathbb{P}(S_1 > x)\,\mathbb{P}(\nu = n) \\
&= \overline{F_{X_1}}(x)\,\mathbb{P}(\nu \geq 1). \qquad (22)
\end{aligned}$$

The assertion of part (iii) follows now from Lemma 1 because by (22)

$$\limsup_{x\to\infty} e^{\lambda x}\, \overline{F_{S^{(\nu)}}}(x) \geq \mathbb{P}(\nu \geq 1) \limsup_{x\to\infty} e^{\lambda x}\, \overline{F_{X_1}}(x)$$

for an arbitrary positive λ. $\square$

Proof of part (iv). Conditions of this part and and Proposition 1 (parts (iii) and (iv)) imply that $F_{S_\nu} \in \mathcal{H}$. In addition, for positive x

$$\begin{aligned}
\overline{F_{S^{(\nu)}}}(x) &= \sum_{n=1}^{\infty} \mathbb{P}\big(\max\{S_1, S_2, \ldots, S_n\} > x\big)\mathbb{P}(\nu = n) \\
&\geq \sum_{n=1}^{\infty} \mathbb{P}(S_n > x)\mathbb{P}(\nu = n) \\
&= \overline{F_{S_\nu}}(x).
\end{aligned}$$

Hence $F_{S^{(\nu)}} \in \mathcal{H}$ according to the Lemma 2. Part (iv) of the proposition is proved. $\square$

Proof of part (v). Let $\lambda > 0$ be a positive number from the condition of part (v), i.e.

$$\sup_{k\geq 1} \mathbb{E}\, e^{\lambda X_k} = \hat{c}_\lambda$$

with some positive constant $\hat{c}_\lambda$. For this λ we have

$$\begin{aligned}
\sup_{k\geq 1} \mathbb{E}\, e^{\lambda X_k^+} &= \sup_{k\geq 1} \mathbb{E}\Big(e^{\lambda X_k^+} \mathbb{1}_{\{X_k \geq 0\}} + e^{\lambda X_k^+} \mathbb{1}_{\{X_k < 0\}}\Big) \\
&= \sup_{k\geq 1} \mathbb{E}\Big(e^{\lambda X_k} \mathbb{1}_{\{X_k \geq 0\}} + \mathbb{1}_{\{X_k < 0\}}\Big) \\
&\leq \hat{c}_\lambda + 1,
\end{aligned}$$

where $X_k^+ = X_k \mathbb{I}_{\{X_k \geq 0\}}$ for $k \in \{1, 2, \ldots, \}$. Due to Proposition 1(vi) d.f. $F_{S_\nu^+}$ belongs to the class $\mathcal{H}^c$ with r.v. $S_\nu^+ = X_1^+ + \ldots + X_\nu^+$.

According to the standard representation, for positive x, we have

$$\begin{aligned}
\overline{F_{S(\nu)}}(x) &= \sum_{n=1}^{\infty} \mathbb{P}\big(\max\{S_1, S_2, \ldots, S_n\} > x\big)\mathbb{P}(\nu = n) \\
&\leq \sum_{n=1}^{\infty} \mathbb{P}\big(\max\{S_1^+, S_2^+, \ldots, S_n^+\} > x\big)\mathbb{P}(\nu = n) \\
&= \sum_{n=1}^{\infty} \mathbb{P}(S_n^+ > x)\mathbb{P}(\nu = n) \\
&= \overline{F_{S_\nu^+}}(x).
\end{aligned}$$

By applying Lemma 2 we get that d.f. $F_{S(\nu)}$ is light-tailed due to the light tail of d.f. $F_{S_\nu^+}$. Part (v) of the proposition is proved. $\square$

6. Concluding remarks

In this paper, we show that both heavy-tailed and light-tailed classes of distributions have quite a number of interesting properties related to the randomly stopped structures. Based on our results, various heavy-tailed or light-tailed distributions can be constructed. On the other hand, according to the propositions we proved, in most cases it is easier to determine whether the considered distribution is light-tailed or heavy-tailed. The main novelty of our work consists in the fact that we study randomly stopped structures in a set of independent but possibly *differently distributed* primary random variables. In Section 1 it was mentioned that randomly stopped structures together with heavy-tailed distributions appear in such fields as insurance and financial activity, survival analysis, risk management, computer and communication networks, etc. Recently, many articles have been written on the heavy-tailed distributions, both in scientific and popular science journals. Let us mention a few of such works. Heavy-tailed distributions applied to financial losses and stochastic returns are described and discussed in the articles [43–45]. The influence of heavy-tailed distributions on actuarial statistics is examined in works [46,47]. The performance of heavy-tailed distributions in social and medical research is discussed in the papers [48,49]. The application of heavy-tailed distributions of a special form to study computer systems and telecommunication networks is presented in [50–52]. From the content of the mentioned works, it can be seen that in many cases it is quite difficult to fit heavy-tailed distributions to the real data. Therefore, our proposed transformations of heavy-tailed distributions increase the chances of choosing the right distribution. So, in our opinion, it makes sense to continue research on transformations for heavy-tailed distributions. In addition to the randomly stopped structures examined in this paper, moment transformations, random effects, and randomly stopped products can be considered for instance.

Author Contributions: Conceptualization, R.L. and J.Š.; methodology, J.Š.; software, S.D.; validation, R.L, S.D. and J.K.; formal analysis, J.K.; investigation, S.D. and J.K.; resources, J.Š; writing-original draft preparation, S.D.; writing-review and editing, R.L.; visualization, S.D. and J.K.; supervision, J.Š.; project administration, R.L.; funding acquisition, J.Š and J.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding

Institutional Review Board Statement: Not applicable

Conflicts of Interest: The authors declare no conflicts of interest.

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