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Article

Adjacency Matrix for the Graph Exponential

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Abstract: Introduced in 1967 by Lovász, graph exponentiation has received little attention over the past several decades. Since graph exponentiation produces large graphs and any graph can be constructed utilizing its adjacency matrix, we provide a four step linear construction algorithm that generates the adjacency matrix for the graph exponential.

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MSC: Primary 05C76; Secondary 05C75

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1. Introduction

As graphs are relational structures, graph exponentiation and the graph exponential G^K were introduced by Lovász in 1967 [6]. Although Lovász develops G^K based on its relationship to the direct product, he mentions that G^K is interesting in its own right and provides several properties of G^K. Over the past several decades, the G^K product has received very little attention. Any graph can be constructed using its adjacency matrix; and large graphs are generated by G^K. Proposition 4.1 states the construction of the G^K adjacency matrix when K is K_2. This proposition demonstrates the recognized connection of G^K to the direct power of G; and Proposition 4.1 is utilized in the proof of Theorem 4.3 that gives the adjacency matrix for a general K. In Figure 3 we provide a four step linear construction algorithm for the adjacency matrix of the general G^K.

Section 2 discusses notation plus fundamental information on the discussed matrices and on the direct product. As there exists an alternative definition of G^K, Section 3 gives detailed information concerning the form of G^K used in this note. In Section 4, we provide the construction of the adjacency matrix for G^K in the form of a construction algorithm, provide a detailed example plus give the general theorem.

2. Background

Only finite undirected graphs are considered. Basic graph theory knowledge as found in [1] is assumed. The vertex set of graph G is denoted by $V(G)$ while $E(G)$ indicates the edge set. It is assumed that all graphs have the same vertex labeling scheme of $\{0, 1, 2, \cdots, n-1\}$ where n is graph order. Graph order is also indicated with $|G|$. For any vertex $v \in V(G)$, $N(v)$ reflects the open neighborhood of v, and $N[v]$ is used for the closed neighborhood. Vertex adjacency is given by $v_1 \sim v_2$, two graphs being isomorphic is $G \cong H$, and $G + H$ is used for the disjoint union of G and H. For $x \in \mathbb{N}$, xG represents x copies of G. The automorphism group of G is $\mathrm{Aut}(G)$.

Complete graphs are given as K_n and complete graphs with a loop at each vertex are K_n^*. Notation D_n is the disjoint union of n number of K_1 subgraphs and is called the *empty graph*. The *null graph* $\mathbb{O}$ has empty vertex and edge sets. Let $|S|$ be the order of set S.

2.1. Matrices

In this paper, loops in an adjacency matrix are 1s along the diagonal. As will be shown, the direct product $G \times H$ (or more specifically the direct power G^x) has connections to G^K as seen in the adjacency matrices of both products. Let the adjacency matrix of graph G be A_G. Denoted $A_{G\times H}$, the adjacency matrix for the direct product (A_{G^x} is the adjacency matrix for the direct power) is the Kronecker product $A_{G\times H} = A_G \otimes A_H$. The adjacency matrix for G^K is given by A_{G^K} and is discussed

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in Section 4. Denote the square all zero matrix as Z while J is the square matrix of all ones. Matrix multiplication is given by $A_G * A_H$.

2.2. Direct Product

The *direct product* $G \times H$ has vertex set that is the Cartesian product $V(G) \times V(H)$ producing ordered pair vertices (g, h) where $g \in V(G)$ and $h \in V(H)$. An edge $(g, h)(g', h')$ in $E(G \times H)$ is defined when $(g, g') \in E(G)$ and $(h, h') \in E(H)$. The *direct power* G^x is the direct product of G to itself x number of times. Additional information on the direct product can be found in [3].

3. Graph Exponentiation Overview

As there exist multiple definitions for graph exponentiation (see [5] Figure 1 for an alternate definition), we clearly explain our interpretation of this graph product (also see [2], [4], [6], [7]).

Graph exponentiation is a graph product operation where the vertex set of the *graph exponential* G^K is the set of all functions $f : V(K) \to V(G)$ where two functions f_1 and f_2 are adjacent in G^K if $(f_1(k)f_2(k')) \in E(G)$ for all $(k, k') \in E(K)$. Thus $|V(G^K)|$ is $|G|^{|K|}$. Since only undirected G and K are considered, all functions are symmetric and G^K is undirected as shown in [7].

Although K_n indicates a complete graph, in this paper K with no subscript exclusively refers to the graph that is the *exponent* in the graph exponentiation product, G^K. As shown by $K_2^{K_2}$ in Figure 1, even if G and K are loopless, G^K generates two components with loops. Hence, G and K in this note are permitted to have loops thus creating a closed system.

In addition to the notation $|K|$ as graph order, let n_K indicate the order of K. If $V(K) = \{k_1, k_2, \cdots, k_{n_K}\}$ then each function f_i can be represented by n_K-tuple $(g_1, g_2, \cdots, g_{n_K})$ where $g_i \in V(G)$, reflecting that $f(k_i) = g_i$.

Example 1. Suppose K_2 is both K and G where both have vertex set $\{0, 1\}$ as seen in Figure 1. Then $V(K_2^{K_2}) = V(K_2) \times V(K_2) = \{(0,0), (0,1), (1,0), (1,1)\}$ and function (g_1, g_1') is adjacent to function (g_2, g_2') in G^K if and only if edges $((g_1, g_2')$ and $(g_2, g_1'))$ are in $E(G)$ since K is K_2. In $G := K_2$, $N_G(0) = 1$ and $N_G(1) = 0$ so exponential function/vertex $(0,0) \sim (1,1)$ and $(0,1) \sim (0,1)$.

Denote $K_1^* + K_1^*$ by $2K_1^*$. Figure 1 displays $K_2^{K_2}$ and $(2K_1^*)^{K_2}$. Notice that $K_2^{K_2} \cong (2K_1^*)^{K_2}$ although $K_2 \ncong 2K_1^*$.

Figure 1. Two isomorphic graph exponentials $K_2^{K_2}$ and $(K_1^* + K_1^*)^{K_2}$.

3.1. Known Properties of G^K

As K_1^* has one function that maps vertex v to itself then $G^{K_1^*} = G$. Thus K_1^* is the identity for this product.

Given graphs G, H and K plus direct product $G \times H$, the following hold as proved in [6].

- $G^{K_1^*} = G$.
- $G^{x(K_1^*)} = G \times G \times \cdots \times G$ x number of times is G^x,
- $G^{\mathbb{O}} = n_G K_1$
- $G^H \times G^K \cong G^{H+K}$,
- $(G \times H)^K \cong G^K \times H^K$,
- $(G^H)^K \cong G^{H \times K}$.

Any G has a *neighborhood multiset* $\mathcal{N}(G)$ of the open neighborhoods $N(g)$ for $g \in V(G)$. For K_2 with $V(K_2) = \{0, 1\}$, as $N(0) = 1$ and $N(1) = 0$, then $\mathcal{N}(K_2) = \{\{1\}, \{0\}\}$. It is not always true that

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if $\mathcal{N}(G) = \mathcal{N}(H)$ then $G \cong H$. See [2] for a couple of examples. Graph G is said to be *neighborhood reconstructible* if $\mathcal{N}(G) = \mathcal{N}(H)$ implies that $G \cong H$ [2]. In [2] it is shown that G is neighborhood reconstructible if and only if $G^{K_2} \cong H^{K_2}$ implies $G \cong H$ for all H. This appears to be the only instance in which cancellation in G^K is explored.

3.2. Loops in G^K

The generation of loops in G^K happens even when both K and G are loopless. Loops in the graph exponential indicate a homomorphism between the functions of G^K as determined by the edge structure of G. In fact, f is a homomorphism if and only if (f, f) is a loop of G^K [7].

3.3. Acyclic versus Cyclic K

If G is loopless, then any f_i must have no fixed vertex in order to generate an edge in G^K. Ponder when K is an odd tree. For any acyclic graph, the set of all f_i form transpositions that involve an even number of vertices. Thus there is always a fixed vertex in any function set of an odd tree. Based on the definition of G^K and the fact that $V(G^K)$ is a set of n_K-tuple vertices, when K is an odd ordered acyclic graph and G is loopless, $G^K \cong D_{|G|^{|K|}}$. However, given an even ordered acyclic K, there exist transpositions involving all of the digits in the n_K-tuple so $G^K \ncong D_n$. It also holds that when K has a cycle, then $G^K \ncong D_n$ independent of whether G has a loop or not.

4. Adjacency Matrix

As any graph G can be constructed from its A_G, our goal is to find a relatively simple way to construct A_{G^K} for any G and K. We begin with the adjacency matrix for G^K when K is K_2, followed by using this construction in the proof of the general case for K.

4.1. Adjacency Matrix When K_2 is K

When K is K_2, then $V(G^K)$ is Cartesian product $V(G) \times V(G)$ and function f_i is a single transposition. Thus, for the functions (g_1, g_1') and (g_2, g_2') in $V(G^K)$, $(g_1, g_1') \sim (g_2, g_2')$ if and only if edges (g_1, g_2') and (g_1', g_2) are in $E(G)$.

Define a *row reordered* matrix to be a square matrix with its row indices ordered in a pattern that does not match the order of the column indices with the index ordering difference due to either a row permutation or a row labeling (or similarly for columns). For now we consider the impact of row reordering a matrix as applying only to $K := K_2$. Figure 2 shows the adjacency matrix A_{G^K} for $K_3^{K_2}$ along with a colexicographic row reordered matrix, $A^*_{G^K}$ where the column indices of $A^*_{G^K}$ differ from the row indices. As the rows of A_{G^K} are simply permuted in $A^*_{G^K}$, the row-column adjacency structure of G^{K_2} is preserved. Also shown in the top right is the block matrix of $A^*_{G^K}$. Take note that the block form of $A^*_{G^K}$ reveals that $A^*_{G^K} = A_{K_3} \otimes A_{K_3}$; and note that $A_G \otimes A_G$ is the adjacency matrix of the direct power G^2.

In Figure 2, the fact that row reordering A_{G^K} according to colexicographic order produces the matrix $A_G \otimes A_G$ implies that A_{G^K} can be constructed from $A_G \otimes A_G$. Hence, $A_G \otimes A_G$ as $A^*_{G^K}$ with colexicographic row indices, followed by row permutation to lexicographic order that matches column index order, generates A_{G^K} when $K := K_2$. This follows from the fact that $\mathrm{Aut}(K_2)$ contains a single transposition.

Let P be permutation matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ that represents the transposition of K_2 as K for G with order 2; and let AA be the matrix $A_{K_2} \otimes A_{K_2}$ while $A'A' = A_{2K_1^*} \otimes A_{2K_1^*}$. Returning to Figure 1, $P * AA$ gives $A_{(K_2^{K_2})}$ and $P * A'A'$ results in $A_{(2K_1^{*K_2})}$ where $A_{(K_2^{K_2})}$ and $A_{(2K_1^{*K_2})}$ are permutation equivalent showing that $K_2^{(K_2)} \cong 2K_1^{*(K_2)}$.

By choice, Proposition 4.1 is stated utilizing row labeling although a permutation matrix is a clear option.

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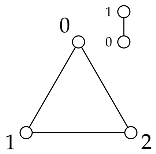
$K_3^{K_2}$:



$$A_{K_3} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{Block } A^*_{G^K} = \begin{bmatrix} Z & A_{K_3} & A_{K_3} \\ A_{K_3} & Z & A_{K_3} \\ A_{K_3} & A_{K_3} & Z \end{bmatrix}$$

$A_{K_3^{K_2}} =$

	00	01	02	10	11	12	20	21	22
00	0	0	0	0	1	1	0	1	1
01	0	1	1	0	0	0	0	1	1
02	0	1	1	0	1	1	0	0	0
10	0	0	0	1	0	1	1	0	1
11	1	0	1	0	0	0	1	0	1
12	1	0	1	1	0	1	0	0	0
20	0	0	0	1	1	0	1	1	0
21	1	1	0	0	0	0	1	1	0
22	1	1	0	1	1	0	0	0	0

A_{G^K} for $K_3^{K_2}$

$A^*_{K_3^{K_2}} =$

	00	01	02	10	11	12	20	21	22
00	0	0	0	0	1	1	0	1	1
10	0	0	0	1	0	1	1	0	1
20	0	0	0	1	1	0	1	1	0
01	0	1	1	0	0	0	0	1	1
11	1	0	1	0	0	0	1	0	1
21	1	1	0	0	0	0	1	1	0
02	0	1	1	0	1	1	0	0	0
12	1	0	1	1	0	1	0	0	0
22	1	1	0	1	1	0	0	0	0

Row reordered matrix $A^*_{G^K}$ for $K_3^{K_2}$

Figure 2. The graph and matrices for Example 2.

Proposition 4.1. *Let G be any graph without multiple edges and let K in G^K be K_2. Construct a block matrix $A^*_{G^K} = A_G \otimes A_G$ using the colexicographic ordering of $V(G^K)$ as row index labels while column indices are lexicographically labeled. Matrix $A^*_{G^K}$ produces A_{G^K} by row permuting to have both row and column indices in lexicographic order.*

Proof. Suppose A_{G^K} is the adjacency matrix for G^{K_2} with arbitrary $(g_i, g_j) \in V(G^{K_2})$ and $g_i, g_j \in V(G)$ where $i = j$ is permitted. Since K is K_2 then $(g_1, g_1') \sim (g_2, g_2')$ if and only if $(g_1, g_2'), (g_2, g_1') \in E(G)$ as reflected in the entries of A_G. Row reorder A_{G^K} via colexicographic order and call the revised matrix $A^*_{G^K}$, thus preserving the adjacency structure of G^{K_2} that is based on that of A_G. $A^*_{G^K}$ has $n_G \times n_G$ blocks where g_i of (g_i, g_j) are the vertices of G in lexicographic order and all g_j are the same. Consider a *block row* to be the collection of rows where all g_j are the same. Let r_{g_j} be the g_j row of A_G. Then any block row reflects the adjacency structure of g_j and is equivalent to the product $r_{g_j} \otimes A_G$ reflecting the n_G number of g_i in (g_i, g_j) for each specific g_j of the block row. Thus, $A^*_{G^K}$ is $A_G \otimes A_G$; and A_{G^K} is found by row permutation back to lexicographic index order.

Now suppose that we have $A^*_{G^K} = A_G \otimes A_G$ which is the adjacency matrix for the direct power $G \times G = G^2$ with vertex set $V(G) \times V(G) = V(G^{K_2})$. Let $(g_i^*, g_j^*) \in V(G \times G)$. In this direct power, $(g_1^*, g_1^{*\prime}) \sim (g_2^*, g_2^{*\prime})$ if and only if $(g_1^*, g_2^*), (g_1^{*\prime}, g_2^{*\prime}) \in E(G)$. Each block row of $A^*_{G^K}$ reflects the adjacency structure of g_i^*. Using $V(G^{K_2})$, assign a colexicographic labeling only to the rows of $A^*_{G^K}$, so that each labeled block row reflects the adjacency structure of g_j^*. Since $(g_1^*, g_1^{*\prime}) \sim (g_2^*, g_2^{*\prime})$ in G^{K_2} if and only if $(g_1^*, g_2^{*\prime}), (g_2^*, g_1^{*\prime}) \in E(G)$, row reordering the colexicographic rows of $A^*_{G^K}$ via lexicographic order produces A_{G^K} for G^{K_2}. $\square$

4.2. General A_{G^K}

The adjacency matrix for the general case of G^K is now addressed. First we give some definitions.

Imagine set $\{f_1, f_2, \cdots, f_k\}$ of n_K-tuple functions where $f_i : V(K) \to V(G)$ and k is the number of f_i that apply to G (i.e. the k number of f_i that generate edges in G^K). Let π_i be a $|G|^{|K|} \times |G|^{|K|}$ permutation matrix for each f_i and $\pi_i \in \Omega_G$. Thus, based on the structure of G, Ω_G contains only the k number of π_i where the f_i produce edges in G^K.

Let $A_{G\otimes}$ be the Kronecker product of A_G over $|K|$ where the Z blocks are maximized by having A_G as the first multiplicand:

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$$A_{G\otimes} = \prod_{i=1}^{n_K - 1} A_G \otimes (A_G)_i \tag{1}$$

Define $\Omega_{A_\wedge}$ as a collection of $|\Omega_G|$ number of $(A_\wedge)_i = \pi_i * A_{G\otimes}$. Thus, each $(A_\wedge)_i$ is associated with a distinct member π_i of Ω_G. Thus each $(A_\wedge)_i$ member of $\Omega_{A_\wedge}$ is a submatrix of A_{G^K} based on a specific π_i, with set $\Omega_{A_\wedge}$ over all π_i in Ω_G.

Define $\sum_\Delta$ as a matrix sum of $(A_\wedge)_i$ such that a_{ij} is changed to 1 for all a_{ij} where $a_{ij} > 1$. This eliminates the miscounting of redundantly generated neighbors in G^K.

Applying $\sum_\Delta$ to $\Omega_{A_\wedge}$ generates A_{G^K} for the specific G and K. Prior to giving proof of the last statement, we provide a linear construction algorithm in Figure 3 and give Example 1 that uses the algorithm in the figure.

Construction algorithm for A_{G^K}

1. Determine the π_i members of Ω_G based on the set of functions $f_i : V(K) \to V(G)$ that apply to G (i.e. only the f_i that generate an edge in G^K).
2. Utilizing A_G and Kronecker product, produce $A_{G\otimes}$ using equation 1.
3. For $|\Omega_G|$ number of $A_{G\otimes}$, build the set $\Omega_{A_\wedge}$ where each $(A_\wedge)_i$ member matrix is $(A_\wedge)_i = \pi_i * A_{G\otimes}$.
4. Apply $\sum_\Delta$ to the $(A_\wedge)_i$ in $\Omega_{A_\wedge}$ to generate A_{G^K} of G^K.

Figure 3. The construction algorithm of A_{G^K}.

The use of Ω for sets Ω_G and $\Omega_{A_\wedge}$ is by design as these sets can be viewed as a single "evolving" set that begins with Ω_G.

Remark 4.2. Because Figure 3 provides a linear construction, the following are true. Notice that $|\Omega_{A_\wedge}| = |\Omega_G|$, and there exists a bijection between the f_i members of Ω_G, the π_i members of Ω_G and the members of $\Omega_{A_\wedge}$.

Example 2: Imagine K_2 with $V(K_2) = \{0, 1\}$ and K_3 with $V(G) = \{v_1, v_2, v_3\}$. Suppose exponential $K_2^{K_3}$ with vertex function set of $\{000, 001, 010, 011, 100, 101, 110, 111\}$ given in shorthand notation. Using the algorithm in Figure 3, we construct A_{G^K} for $K_3^{K_2}$ as given below on the far left. Hence, this A_{G^K} is the goal of our example construction. The other two matrices displayed in Figure 4 are the two members of $\Omega_{A_\wedge}$ whose construction is explained following.

$$A_{G^K} = \begin{bmatrix} 0&0&0&0&0&0&0&1 \\ 0&0&0&1&0&1&0&0 \\ 0&0&0&1&0&0&1&0 \\ 0&1&1&0&0&0&0&0 \\ 0&0&0&0&0&1&1&0 \\ 0&1&0&0&1&0&0&0 \\ 0&0&1&0&1&0&0&0 \\ 1&0&0&0&0&0&0&0 \end{bmatrix} \quad A_{\wedge 1} = \begin{bmatrix} 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \end{bmatrix} \quad A_{\wedge 2} = \begin{bmatrix} 0&0&0&0&0&0&0&1 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 1&0&0&0&0&0&0&0 \end{bmatrix}$$

Figure 4. On the far left, A_{G^K} that is the goal for the Example 1 construction.

1. Although the set of f_i is $\text{Aut}(K_3)$ (the dihedral group of 3 excluding the identity: $(v_2v_3v_1)$, $(v_3v_1v_2)$, $(v_2v_1v_3)$, $(v_1v_3v_2)$, $(v_3v_2v_1)$), since G is a loopless K_2, then any function containing a fixed vertex is disregarded and the set of f_i in Ω_G is $\{(v_2v_3v_1), (v_3v_1v_2)\}$ [4]. Construct two permutation matrices: π_1 for $(v_2v_3v_1)$ and π_2 as $(v_3v_1v_2)$.
2. Given $G := K_2$ and $A_{K_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, using equation 1 produces $A_{G\otimes} := A_{K_2\otimes}$ where $A_{K_2\otimes} = A_{K_2} \otimes A_{K_2} \otimes A_{K_2}$; and $A_{K_2\otimes}$ is a 8×8 permutation matrix of cross-diagonal 1s with indices of $\{000, 001, 010, 011, 100, 101, 110, 111\}$.

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3. Based on the π_i members of Ω_G and using two $A_{K_2\otimes}$ matrices, there are two members of $\Omega_{A_\wedge}$: $A_{\wedge 1} = \pi_1 * A_{K_2\otimes}$ and $A_{\wedge 2} = \pi_2 * A_{K_2\otimes}$. Using the vertices of $A_{K_2\otimes}$, the rows of $A_{K_2\otimes}$ are permuted by π_1 to be the ordered set $(000, 010, 100, 110, 001, 011, 101, 111)$ as shown in the center of Figure 4. Matrix π_2 permutes the rows of $A_{K_2\otimes}$ to be the ordered set $(000, 100, 001, 101, 010, 1101, 011, 111)$ as on the far right in Figure 4.
4. Utilizing $\sum_\Delta$, the 2s generated by the duplicate 000 and 111 entries are changed to 1s with the result being A_{G^K} for $K_2^{K_3}$ as desired.

□

Theorem 4.3. *Given G^K where G and K are graphs without multiple edges, A_{G^K} is the $\sum_\Delta$ sum of the members of $\Omega_{A_\wedge}$.*

Proof. Consider Remark 4.2 and let r_{g_i} be the g_i indexed row of A_G where $g_i \in V(G)$. Although Proposition 4.1 is based on the single transposition of K_2 as K, we know that for all functions $f : V(K) \to V(G)$ each row of A_{G^K} is found utilizing r_{g_j} of A_G and the Kronecker product. Thus, keeping Proposition 4.1 in mind, consider each function/vertex of G^K with a general K. Since $A_{G^{K_2}}$ is based on a permutation of $A_{G\otimes}$ rows, then each row of A_{G^K} must be found via the rows of A_G.

Let $x := (g_1, \cdots, g_{n_K})$ where $g_i \in x$ with $g_i \in V(G)$ and r_x is the row of A_{G^K} indexed by x. The neighbors of x are based on the set of neighbors of $g_i \in V(G)$ as represented by row r_{g_i} of A_G. For any edge $e \in E(G^K)$ where $e := (x, x')$, each element g_i of x must be a neighbor of its corresponding element g'_i of x' in G based on the f_i that generate edges in G^K; so $r_{g_i} \otimes r_{g'_i}$ gives their joint neighborhood. As each g_i is an element of a function x, then for all g_i and all g'_i for specific edge e, $r_{g_i} \otimes r_{g'_i}$, (for that specific π_i in Ω_G), associates x with the specific π_i. Finding $r_{g_i} \otimes r_{g'_i}$ for all π_i in Ω_G gives the complete neighborhood of x in G^K; thus using $\sum_\Delta$ to sum all $r_{g_i} \otimes r_{g'_i}$ for a given x gives row r_x of A_{G^K}. Each row sum represents a basis of the vector space of G^K as the vertices are n_K-tuple functions. So the disjoint union of these rows into a matrix is A_{G^K}. □

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