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A Fractional Differential Operator Based on Quantum Calculus and Bi-Close-to-Convex Functions

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Abstract: This study explores the properties and behavior of bi-close-to-convex functions, introducing new subclasses defined by their relationship with fractional differential operator and bi-univalent functions. Using the Faber polynomial technique, we derive upper bounds for the n^{th} coefficient of functions in these classes. We also investigate the erratic behavior of initial coefficients in bi-close-to-convex functions, as characterized by the $(\lambda, q)-$fractional differintegral operator. Furthermore, we address Fekete-Szegö problems and present notable findings from our investigation, contributing to the continued growth and refinement of geometric function theory, yielding new insights and practical uses.

Keywords: convex functions; starlike functions close-to-convex functions; bi-close-to-convex functions; fractional q-differintegral operator

MSC: Primary 05A30; 30C45; Secondary 11B65; 47B38

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1. Preliminaries and Basic Notations

In mathematics, symmetry is defined as the property of two shapes being identical when one is moved, rotated, or flipped. The open unit disk, denoted by $\Delta = \{\mu : |\mu| < 1\}$, exhibits a rich set of symmetries, consisting of inversion, rotational and reflection symmetry. Specifically, inversion symmetry means that the disk remains unchanged when inverted about a particular point, maintaining its overall appearance and structure. The disk Δ has inversion symmetry about its center (origion), meaning that inverting any complex number μ within the disk about the origin results in another complex number also within the disk, specifically $\frac{1}{\mu}$. This disk exhibits a rich set of symmetries, making it valuable in various mathematical and geometric contexts. Our goal is to explore additional geometric properties within this symmetric domain.

A function is considered starlike (or convex) if it transforms Δ into a star-shaped (or convex) region, centered at a fixed point, through scaling and rotation. This means the function's image is contained within a star-shaped (or convex) domain, formed by connecting the fixed point to all other points with straight lines. Starlike and univalent functions are crucial subclasses of analytic functions with numerous applications and properties. Univalent functions are used in geometric function theory (GFT) for conformal mappings, while starlike functions model phenomena like electrostatics and fluid flow in GFT. Another important subclass of analytic functions is the class of close-to-convex functions. In this article we will focus on the study of bi-close-to-convex functions.

Let $\mathcal{A}$ indicate a collection of all analytic functions $\eta(\mu)$ in the region $\Delta = \{\mu : |\mu| < 1\}$, which are normalized by

$$\eta(0) = 0 \text{ and } \eta'(0) = 1.$$

Thus, every $\eta \in \mathcal{A}$ can be expressed as:

$$\eta(\mu) = \mu + \sum_{n=2}^{\infty} a_n \mu^n. \tag{1.1}$$

Moreover, $\mathcal{S}$ is the subclass of $\mathcal{A}$ whose members in Δ are univalent. Let the class $\mathcal{P}$ be defined by

$$\mathcal{P} = \{p \in \mathcal{A}\colon p(0) = 1 \text{ and } Re(p(\mu)) > 0\}. \tag{1.2}$$

The following are some notable subclasses of the univalent functions in class $\mathcal{S}$:

$$\mathcal{S}^*(\alpha) = \left\{\eta \in \mathcal{A} : \Re\left(\frac{\mu\eta'(\mu)}{\eta(\mu)}\right) > \alpha\right\}, \quad 0 \le \alpha < 1,$$

$$\mathcal{Q}(\alpha) = \left\{\eta \in \mathcal{A} : \Re\left(\frac{\left(\mu\eta'(\mu)\right)'}{\eta'(\mu)}\right) > \alpha\right\}, \, 0 \le \alpha < 1$$

and

$$\mathcal{C}(\alpha) = \left\{\eta \in \mathcal{A}, \, \sqcup \in \mathcal{S}^* : \Re\left(\frac{\mu\eta'(\mu)}{\sqcup(\mu)}\right) > \alpha\right\}, \, 0 \le \alpha < 1.$$

These classes defined in the following articles [1–5]. For $\alpha = 0$, then

$$\mathcal{S}^*(0) = \mathcal{S}^*, \ \mathcal{Q}(0) = \mathcal{Q} \text{ and } \mathcal{C}(0) = \mathcal{C}.$$

For $\eta_1, \eta_2 \in \mathcal{A}$, and η_1 subordinate to η_2 in η, denoted by

$$\eta_1(\mu) \prec \eta_2(\mu), \quad \mu \in \Delta.$$

If we have a function $u_0 \in \mathcal{A}$, satisfy the condition $|u_0(\mu)| < 1$, $u_0(0) = 0$, and

$$\eta_1(\mu) = \eta_2(u_0(\mu)), \ \mu \in \Delta.$$

The inverse of $\eta \in \mathcal{S}$, defined as:

$$\eta^{-1}(\eta(\mu)) = \mu, \quad \mu \in \Delta$$

and

$$\eta(\eta^{-1}(w)) = w, \ |w| < r_0(\eta), \text{ and } r_0(\eta) \ge \frac{1}{4}.$$

The series of $\eta^{-1} = \mathtt{F}$ is given by

$$\begin{aligned} \mathtt{F}(w) &= w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots \\ &= \mu + \sum_{n=2}^{\infty} A_n \mu^n. \end{aligned} \tag{1.3}$$

If $\sqcup \in \mathcal{A}$ and

$$\sqcup(\mu) = \mu + \sum_{n=2}^{\infty} b_n \mu^n \tag{1.4}$$

then the series of $\sqcup^{-1} = G$ is given by

$$G(\mu) = \mu + \sum_{n=2}^{\infty} B_n \mu^n.$$

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If η and $\eta^{-1}$are in $\mathcal{S}$, then η is considered the bi-univalent in Δ and such type of functions are denoted by Σ. For $\eta \in \Sigma$, author in [6] proved that $|a_2| < 1.51$ and after that authors in [7] gave the improvement of $|a_2|$ and proved that $|a_2| \leq \sqrt{2}$. Furthermore, for $\eta \in \Sigma$, Netanyahu [8] proved that $\max|a_2| = \frac{4}{3}$. (see for details [9–11]). Only non-sharp estimates on the initial coefficients were achieved in these recent works.

The Faber polynomials expansion method was first presented by Faber [12], who also utilized this approach to study coefficient bound $|a_n|$ for $n \geq 3$. Gong [13] emphasized the significance of Faber polynomials in mathematical sciences, specifically in the context of GFT. In the cited work [14–17], authors introduced new subcategories of bi-univalent functions and using the Faber polynomials expansion approach to establish coefficient bounds. Additionally, a number of researchers have used the Faber polynomials approach [18–22] to derive some intriguing findings for bi-univalent functions (see for detail [23,24]). In 2014, a group of researchers [25] in the field of geometric function theory developed a new class of functions called bi-close-to-convex functions of order α, where $0 \leq \alpha < 1$. This class is denoted by $C_{\Sigma}(\alpha)$ and was previously discussed in references [26–28].

To express the coefficients of its inverse map $\mathtt{F}$ in terms of the analytic functions η, use the Faber polynomial method (see [19,29]).

$$F(w) = \eta^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{n}(a_2, a_3, ..., a_n) w^n,$$

where

$$\begin{aligned}
K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!(n-3)!} a_2^{n-3} a_3 \\
&+ \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\
&+ \frac{(-n)!}{[2(-n+2)]!(n-5)!} a_2^{n-5} \left[a_5 + (-n+2) a_3^2\right] \\
&+ \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} [a_6 + (-2n+5) a_3 a_4] \\
&+ \sum_{i \geq 7} a_2^{n-i} \mathcal{Q}_i.
\end{aligned}$$

For $7 \leq i \leq n$, $\mathcal{Q}_i$ is a homogeneous polynomial in the variables $a_2, a_3, ..., a_n$. Particularly, the first three terms of K_{n-1}^{-n} are

$$\begin{aligned}
\frac{1}{2} K_1^{-2} &= -a_2, \ \frac{1}{3} K_2^{-3} = 2a_2^2 - a_3, \\
\frac{1}{4} K_3^{-4} &= -(5a_2^3 - 5a_2 a_3 + a_4).
\end{aligned}$$

For integers r, that is, $r = 0, \pm 1, \pm 2, ...,$ and integers $n \geq 2$, the quantity K_{n-1}^{r} (see [19]) admits an expansion of the form:

$$K_{n-1}^{r} = r a_n + \frac{r(r-1)}{2} \mathcal{V}_{n-1}^2 + \frac{r!}{(r-3)!3!} \mathcal{V}_{n-1}^3 + ... + \frac{r!}{(r-n+1)!(n-1)!} \mathcal{V}_{n-1}^{n-1},$$

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where,

$$\mathcal{V}_{n-1}^{r} = \mathcal{V}_{n-1}^{r}(a_2, a_3, ...),$$

and by [29], we have

$$\mathcal{V}_{n-1}^{v}(a_2, ..., a_n) = \sum_{n=1}^{\infty} \frac{v!}{\mu_1!, ..., \mu_n!} a_2^{\mu_1} ... a_n^{\mu_{n-1}}, \text{ for } a_1 = 1 \text{ and } v \leq n.$$

Taking the summation for $\mu_1, ..., \mu_n$ which satisfying

$$\begin{aligned} \mu_1 + \mu_2 + ... + \mu_{n-1} &= v, \\ \mu_1 + 2\mu_2 + ... + (n-1)\mu_{n-1} &= n-1. \end{aligned}$$

Clearly,

$$\mathcal{V}_{n-1}^{n-1}(a_1, ..., a_n) = a_2^{n-1}$$

and

$$\mathcal{V}_{n-1}^{1}(a_1, ..., a_n) = a_{n-1}.$$

In the realm of (GFT), many scholars have built upon the foundational work of Jackson [30], who introduced the $q-$calculus operator in 1909. Ismail et al. [31] subsequently employed this operator to define $q-$starlike functions in Δ. (Recent contributions to this field can be found in references [32–37]). To further expand this research, we must revisit the core definitions and principles of $q-$calculus and fractional $q-$calculus, paving the way for the construction of new subclasses of analytic and bi-univalent functions.

Definition 1.1. For $(0 < q < 1)$, the q-number n is given by

$$\begin{aligned} [n]_q &= \frac{1-q^n}{1-q}, \ n \in \mathbb{C} \\ &= \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + ... + q^{n-1}, \ n \in \mathbb{N} \end{aligned} \tag{1.5}$$

and

$$[0]_q = 0.$$

Definition 1.2. [38]. The q-number shift factorial is given by for $\gamma \in \mathbb{C}$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

$$(\gamma, q)_n = \prod_{k=0}^{n-1} \left(1 - \gamma q^k\right), \quad (n \in \mathbb{N})$$

and

$$(\gamma, q)_0 = 1.$$

In terms of the q-Gamma function

$$(q^{\gamma}, q)_n = \frac{(1-q)^n \Gamma_q(\gamma + n)}{\Gamma_q(\gamma)}, \quad (n \in \mathbb{N}_0),$$

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where the q-gamma function is defined by

$$\Gamma_q(\mu) = \frac{(1-q)^{1-\mu}(q,q)_\infty}{(q^\mu,q)_\infty}, \quad |q| < 1.$$

We note that

$$(\gamma,q)_\infty = \prod_{k=0}^{\infty}\left(1-\gamma q^k\right), \quad |q| < 1.$$

For the q-gamma function $\Gamma_q(\mu)$, it is known that (see [39])

$$\Gamma_q(\mu+1) = [\mu]_q \Gamma_q(\mu).$$

Jackson [40] introduced the q-difference operator for analytic functions as follows:

Definition 1.3. [40]. For $\eta \in \mathcal{A}$, the q-difference operator is defined as:

$$\begin{aligned}\partial_q \eta(\mu) &= \frac{\eta(q\mu)-\eta(\mu)}{\mu(q-1)}, \quad \mu \in \Delta,\ \mu \neq 0,\ q \in (0,1) \\ &= 1 + \sum_{n=1}^{\infty}[n]_q a_n \mu^{n-1}\end{aligned}$$

and

$$\partial_q(\mu^n) = [n]_q \mu^{n-1}, \text{ and } \partial_q\left(\sum_{n=1}^{\infty} a_n \mu^n\right) = \sum_{n=1}^{\infty}[n]_q a_n \mu^{n-1}.$$

where $[n]_q$ given by (1.5) and

$$\lim_{q\to 1-} \partial_q \eta(\mu) = \eta'(\mu).$$

The q-analogous of the class of starlike functions was first introduced by Ismail et al. in [31] by means of the q-difference operator $\partial_q \eta(\mu)$, $(0 < q < 1)$ and the q-integral is defined by

$$\int_0^\mu \eta(t) d_q t = \mu(1-q)\sum_{n=0}^{\infty} q^n \eta(\mu q^n).$$

Remark 1.4.

$$\lim_{q\to 1-}\int_0^\mu \eta(t) d_q t = \int_0^\mu \eta(t) dt.$$

Definition 1.5. Fractional q-integral operator, (see [41], page 57, Definition 1) the Fractional q-integral operator $I^\lambda_{q,\mu}$ of order λ is defined by (see also [42], page 257)

$$I^\lambda_{q,\mu}\eta(\mu) = \frac{1}{\Gamma_q(\lambda)}\int_0^\mu (\mu - tq)_{\lambda-1}\eta(t) d_q(t), \quad \lambda > 0, \tag{1.6}$$

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where, $\eta(\mu)$ is analytic in a simply connected region of the μ-plane containing the origin and the q-binomial function $(\mu - tq)_{\lambda-1}$ is defined by

$$(\mu - tq)_{\lambda-1} = \prod_{n=0}^{\infty}\left(\frac{1-\left(\frac{qt}{\mu}\right)q^n}{1-\left(\frac{qt}{\mu}\right)q^{\lambda+n-1}}\right)$$
$$= \mu^{\lambda-1}{}_1\Phi_0\left(q^{-\lambda+1}, -, q, \frac{tq^\lambda}{\mu}\right).$$

The definition of series ${}_1\Phi_0$ is

$${}_1\Phi_0(a, -, q, \mu) = 1 + \sum_{n=1}^{\infty}\frac{(a,q)_n}{(q,q)_n}\mu^n, \quad (|q| < 1, |\mu| < 1).$$

The last equation is known as the q-binomial theorem (see [43] for more information). The series ${}_1\Phi_0(a, -, q, \mu)$ is single valued, when $|\arg(\mu)| < \pi$ and $|\mu| < 1$, (see for detail [39], pages 104–106) and $(\mu - tq)_{\lambda-1}$in (1.6) is single valued, when $\left|\arg(-tq^\lambda/\mu)\right| < \pi, \left|(tq^\lambda/\mu)\right| < \pi$ and $|\arg(\mu)| < \pi$.

Definition 1.6. [41]. The fractional q-derivative operator D_q of order λ is defined by (see also [42], page 257, Definition 1.2)

$$D_{q,\mu}^{\lambda}\eta(\mu) = D_{q,\mu}\left(I_{q,\mu}^{1-\lambda}\eta(\mu)\right)$$
$$= \frac{1}{\Gamma_q(1-\lambda)}\partial_q \int_0^{\mu}(\mu - tq)_{-\lambda}\eta(t)d_q(t), \quad (0 \le \lambda < 1),$$

where $\eta(\mu)$ is suitably constrained and the multiplicity of $(\mu - tq)_{-\lambda}$ is removed as in Definition 1.

Definition 1.7. For m be the smallest integer. The extended fractional q-derivative (D_q^λ) of order λ defined by

$$D_q^\lambda \eta(\mu) = D_q^m I_{q,\mu}^{m-\lambda}\eta(\mu). \tag{1.7}$$

We find from (1.7) that

$$D_q^\lambda \mu^n = \frac{\Gamma_q(n+1)}{\Gamma_q(n+1-\lambda)}\mu^{n-\lambda}, \quad (0 \le \lambda,\ n > -1).$$

Note that: When $-\infty < \lambda < 0$, then D_q^λ represents a fractional q-integral of η of order λ. For $0 \le \lambda < 2$, then D_q^λ represents a fractional q-derivative of η of order λ.

Definition 1.8. ([44]). The (λ, q)-fractional-differintegral operator $D_q^\lambda : \mathcal{A} \to \mathcal{A}$ defined as follows:

$$D_q^\lambda \eta(\mu) = \frac{\Gamma_q(2-\lambda)}{\Gamma_q(2)}\mu^\lambda D_q^\lambda \eta(\mu)$$
$$= \mu + \sum_{n=2}^{\infty}\frac{\Gamma_q(2-\lambda)\Gamma_q(n+1)}{\Gamma_q(2)\Gamma_q(n+1-\lambda)}a_n\mu^n$$
$$= \mu + \sum_{n=2}^{\infty}\mathcal{L}_n a_n \mu^n, \quad \mu \in \Delta, \tag{1.8}$$

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where,

$$\mathcal{L}_n = \frac{\Gamma_q(2-\lambda)\Gamma_q(n+1)}{\Gamma_q(2)\Gamma_q(n+1-\lambda)} \tag{1.9}$$

and

$$\lambda < 2,\ 0 < q < 1.$$

Note that:

i.

$$\lim_{\lambda \to 1} D_q^{\lambda}\eta(\mu) = D_q\eta(\mu) = \mu\partial_q\eta(\mu).$$

ii.

$$D_q^{\lambda}\left(D_q^{\delta}\eta(\mu)\right) = D_q^{\delta}\left(D_q^{\lambda}\eta(\mu)\right) = \mu + \sum_{n=2}^{\infty} \frac{\Gamma_q(2-\lambda)\Gamma_q(2-\delta)\left(\Gamma_q(n+1)\right)^2}{\Gamma_q(2)\Gamma_q(n+1-\lambda)\Gamma_q(n+1-\delta)} a_n\mu^n$$

and

$$\begin{aligned}\frac{D_q\left(D_q^{\lambda}\eta(\mu)\right)}{D_q^{\lambda}\eta(\mu)} &= \frac{D_q\eta(\mu)}{\eta(\mu)} = \frac{\mu\partial_q\eta(\mu)}{\eta(\mu)}, \text{ for } \lambda = 0\\ &= \frac{D_q(D_q\eta(\mu))}{D_q\eta(\mu)} = \left(1 + \frac{\mu\partial_q^2\eta(\mu)}{\partial_q\eta(\mu)}\right) \text{ for } \lambda = 1.\end{aligned}$$

First, we define a class of q-starlike functions of order β associated with (λ, q)-fractional-differintegral operator D_q^{λ} and then we define the class of close-to-convex function by using the same operator. We start by creating a class of q-starlike functions of order β that are linked to the (λ, q)-fractional-differintegral operator D_q^{λ}. Then, we use the same operator to create a class $C_{\Sigma}^{\lambda,q}(\alpha)$ of close-to-convex functions.

Definition 1.9. Let $\sqcup$ be of the form (1.4) be in the class $\mathcal{S}^*(q, \beta, \lambda)$, if

$$\Re\left(\frac{D_q\left(D_q^{\lambda}\sqcup(\mu)\right)}{\sqcup(\mu)}\right) > \beta,$$

where $0 \leq \beta < 1, 0 < q < 1$.

Definition 1.10. Let η be of the form (1.1). Then $\eta \in C_{\Sigma}^{\lambda,q}(\alpha)$ if there is a function $\sqcup \in \mathcal{S}^*(q, \beta, \lambda)$ satisfying

$$\Re\left(\frac{D_q\left(D_q^{\lambda}\eta(\mu)\right)}{\sqcup(\mu)}\right) > \alpha$$

and

$$\Re\left(\frac{D_q\left(D_q^{\lambda}F(w)\right)}{G(w)}\right) > \alpha,$$

where, $0 \leq \alpha < 1, 0 < q < 1, 0 \leq \lambda < 2, \mu, w \in \Delta$.

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Remark 1.11. For $\lambda = 0$, we have a new class $\mathcal{C}_\Sigma(\alpha, q)$ of bi-close-to-convex functions and be defined as:

$$\Re\left(\frac{\mu\partial_q\eta(\mu)}{\sqcup(\mu)}\right) > \alpha$$

and

$$\Re\left(\frac{\mu\partial_q F(w)}{G(w)}\right) > \alpha,$$

where, $0 \le \alpha < 1, 0 < q < 1, \mu, w \in \eta, \eta^{-1} = F$ and $\sqcup^{-1} = G$.

Remark 1.12. For $\lambda = 0$, and $q \to 1-$, then we have the known class of bi-close-to-convex functions investigated by Bulut in [45].

2. Set of Lemmas:

To show our primary points, we need the following Lemmas:

Lemma 2.1. [46]. *If $p \in \mathcal{P}$ and*

$$p(\mu) = 1 + \sum_{n=1}^{\infty} c_n \mu^n,$$

then

$$|c_n| \le 2.$$

Lemma 2.2. [1]. *If $p \in \mathcal{P}$ and $\mu \in \mathbb{C}$, then*

$$\left|c_2 - \mu c_2^2\right| \le 2\max\{1, |2\mu - 1|\}.$$

The aim of this work is to investigate a new subclass of the bi-univalent functions defined by (λ, q)-fractional operator and by using the Faber Polynomials technique to obtain coefficients $|a_n|$, and initial coefficients bounds $|a_2|$ and $|a_3|$.

3. Main Results

Theorem 3.1. *Let an analytic function $\sqcup$ be of the form (1.4). If $\sqcup \in \mathcal{S}^*(q, \beta, \lambda)$, then*

$$|b_n| \le \frac{2(1-\beta)}{\left(\mathcal{L}_n[n]_q - 1\right)} \prod_{j=1}^{n-2}\left(1 + \frac{2(1-\beta)}{\left(\mathcal{L}_{j+1}[j+1]_q - 1\right)}\right), \quad n \ge 3 \tag{3.1}$$

and

$$|b_2| \le \frac{2(1-\beta)}{\left(\mathcal{L}_2[2]_q - 1\right)}, \tag{3.2}$$

where $\mathcal{L}_n$ is given by (1.9) and $0 \le \beta < 1, 0 < q < 1, 0 \le \lambda < 1$.

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Proof. Suppose $\sqcup \in \mathcal{S}^*(q, \beta, \lambda)$ then

$$\Re\left(\frac{D_q\left(D_q^{\lambda}\sqcup(\mu)\right)}{\sqcup(\mu)}\right) > \beta$$

Thus, by setting

$$\frac{\frac{D_q\left(D_q^{\lambda}\sqcup(\mu)\right)}{\sqcup(\mu)} - \beta}{1-\beta} = p(\mu)$$

or, equivalently

$$D_q\left(D_q^{\lambda}\sqcup(\mu)\right) = ((1-\beta)p(\mu)+\beta)\sqcup(\mu)$$
$$\left(\mu + \sum_{n=2}^{\infty} \mathcal{L}_n[n]_q b_n \mu^n\right) = \left(1 + (1-\beta)\sum_{n=1}^{\infty} c_n\mu^n\right)\left(\mu + \sum_{n=2}^{\infty} b_n\mu^n\right)$$
$$\sum_{n=2}^{\infty}\left(\mathcal{L}_n[n]_q - 1\right)b_n\mu^n = (1-\beta)\sum_{n=1}^{\infty} c_n\mu^{n+1} + (1-\beta)\sum_{n=2}^{\infty}\left(\sum_{j=1}^{n-1} c_j b_{n-j}\right)\mu^n.$$

Comparing the μ^n both side, we have

$$\left(\mathcal{L}_n[n]_q - 1\right)b_n = (1-\beta)\sum_{j=1}^{n-1} c_j b_{n-j}$$

and

$$\left|\left(\mathcal{L}_n[n]_q - 1\right)b_n\right| = (1-\beta)\sum_{j=1}^{n-1} |c_j||b_{n-j}|.$$

Using the Lemma 2, we have

$$\left|\left(\mathcal{L}_n[n]_q - 1\right)b_n\right| \leq 2(1-\beta)\sum_{j=1}^{n-1} |b_j|. \tag{3.3}$$

So for $n = 2$, in (3.3), we have

$$|b_2| \leq \frac{2(1-\beta)}{\left(\mathcal{L}_2[2]_q - 1\right)}, \tag{3.4}$$

This confirms that equation (3.1) is true for the base case $n = 2$. To establish the general validity of equation (3.1), we employ mathematical induction. In the next step, we consider the case $n = 3$, and from equation (3.3), we obtain

$$|b_3| \leq \frac{2(1-\beta)}{\left(\mathcal{L}_3[3]_q - 1\right)}(1 + |b_2|)$$
$$\leq \frac{2(1-\beta)}{\left(\mathcal{L}_3[3]_q - 1\right)}\left(1 + \frac{2(1-\beta)}{\left(\mathcal{L}_2[2]_q - 1\right)}\right),$$

This verifies that equation (3.1) is true for $n = 3$. Moving on to the case $n = 4$, we can see from equation (3.3) that

$$|b_4| \leq \frac{2(1-\beta)}{\left(\mathcal{L}_4[4]_q - 1\right)}(1 + |b_2| + |b_3|)$$

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or this can be written as:

$$|b_4| \leq \frac{2(1-\beta)}{\left(\mathcal{L}_4[4]_q - 1\right)}\left(1 + \frac{2(1-\beta)}{\left(\mathcal{L}_2[2]_q - 1\right)}\right)\left(1 + \frac{2(1-\beta)}{\left(\mathcal{L}_3[3]_q - 1\right)}\right),$$

This demonstrates that equation (3.1) is valid for $n = 3$. Next, we assume that equation (3.1) is true for all n less than or equal to t that is

$$|b_t| \leq \frac{2(1-\beta)}{\left(\mathcal{L}_t[t]_q - 1\right)} \prod_{j=1}^{t-2}\left(1 + \frac{2(1-\beta)}{\left(\mathcal{L}_{j+1}[j+1]_q - 1\right)}\right).$$

Consider

$$\begin{aligned}
|b_{t+1}| &\leq \frac{2(1-\beta)}{\left(\mathcal{L}_{t+1}[t+1]_q - 1\right)}(1 + |b_2| + |b_3| + ... + |b_t|) \\
&\leq \frac{2(1-\beta)}{\left(\mathcal{L}_{t+1}[t+1]_q - 1\right)} \\
&\left[\begin{array}{c}
1 + \frac{2(1-\beta)}{\left(\mathcal{L}_2[2]_q-1\right)} + \frac{2(1-\beta)}{\left(\mathcal{L}_3[3]_q-1\right)}\left(1 + \frac{2(1-\beta)}{\left(\mathcal{L}_2[2]_q-1\right)}\right) \\
+ \frac{2(1-\beta)}{\left(\mathcal{L}_4[4]_q-1\right)}\left(1 + \frac{2(1-\beta)}{\left(\mathcal{L}_2[2]_q-1\right)}\right)\left(1 + \frac{2(1-\beta)}{\left(\mathcal{L}_3[3]_q-1\right)}\right) \\
+ \frac{2(1-\beta)}{\left(\mathcal{L}_2[2]_q-1\right)} \prod_{j=1}^{t-2}\left(1 + \frac{2(1-\beta)}{\left(\mathcal{L}_{j+1}[j+1]_q-1\right)}\right)
\end{array}\right] \\
&= \frac{2(1-\beta)}{\left(\mathcal{L}_{t+1}[t+1]_q - 1\right)} \prod_{j=1}^{t-1}\left(1 + \frac{2(1-\beta)}{\left(\mathcal{L}_{j+1}[j+1]_q - 1\right)}\right).
\end{aligned}$$

Therefore, the result holds for $n = t + 1$. Thus, by mathematical induction, we have established that equation (3.1) is true for all integers n greater than or equal to 2. This concludes the proof. □

Theorem 3.2. *If* $\sqcup$ *is of the form* (1.4) *and* $\sqcup \in \mathcal{S}^*(q, \beta, \lambda)$*, then*

$$\left|b_3 - \mu b_2^2\right| \leq \frac{2(1-\beta)}{\mathcal{L}_3[3]_q - 1} \max\left(1, \left|1 + \frac{2(1-\beta)}{\mathcal{L}_2[2]_q - 1}\left(1 - \mu\frac{\left(\mathcal{L}_3[3]_q - 1\right)}{\mathcal{L}_2[2]_q - 1}\right)\right|\right)$$

where

$$\mathcal{L}_3 = \left(\frac{\Gamma_q(2-\lambda)\Gamma_q(4)}{\Gamma_q(2)\Gamma_q(4-\lambda)}[3]_q - 1\right)$$
$$\mathcal{L}_2 = \left(\frac{\Gamma_q(2-\lambda)\Gamma_q(3)}{\Gamma_q(2)\Gamma_q(3-\lambda)}[2]_q - 1\right)$$

and $0 \leq \beta < 1, 0 < q < 1, 0 \leq \lambda < 2, \mu \in \mathbb{C}$.

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Proof. If $\sqcup \in \mathcal{S}^*(q, \beta, \lambda)$, then we have

$$\Re\left(\frac{D_q\left(D_q^\lambda \sqcup(\mu)\right)}{\sqcup(\mu)}\right) > \beta.$$

Then there exist a positive real part function $p(\mu) = 1 + \sum_{n=1}^{\infty} c_n \mu^n \in \mathcal{P}$, such that

$$\begin{aligned}\frac{D_q\left(D_q^\lambda \sqcup(\mu)\right)}{\sqcup(\mu)} &= \beta + (1-\beta)p(\mu) \\ &= 1 + (1-\beta)\sum_{n=1}^{\infty} c_n \mu^n. \end{aligned} \tag{3.5}$$

From (3.5), we have

$$b_2 = \frac{(1-\beta)c_1}{\mathcal{L}_2[2]_q - 1} \tag{3.6}$$

and

$$b_3 = \frac{(1-\beta)}{\mathcal{L}_3[3]_q - 1}\left(c_2 + \frac{(1-\beta)c_1^2}{\mathcal{L}_2[2]_q - 1}\right). \tag{3.7}$$

From (3.6) and (3.7)

$$\left|b_3 - \mu b_2^2\right| = \frac{(1-\beta)}{\mathcal{L}_3[3]_q - 1}\left|c_2 - v c_1^2\right|,$$

where

$$v = -\frac{(1-\beta)}{\mathcal{L}_2[2]_q - 1}\left(1 - \mu\frac{\mathcal{L}_3[3]_q - 1}{\mathcal{L}_2[2]_q - 1}\right).$$

Our result is a direct consequence of Lemma 2 This completes the proof. □

Theorem 3.3. *Let $\eta \in \mathcal{C}_{\Sigma}^{\lambda,q}(\alpha)$ be given by* (1.1), if $a_i = 0, 2 \le i \le n-1$. *Then for $n \ge 3$*

$$\begin{aligned}|a_n| \le \frac{1}{\mathcal{L}_n[n]_q}&\left\{\frac{2(1-\beta)}{\left(\mathcal{L}_n[n]_q - 1\right)}\prod_{j=1}^{n-2}\left(1 + \frac{2(1-\beta)}{\left(\mathcal{L}_{j+1}[j+1]_q - 1\right)}\right)\right. \\ &+ 2(1-\alpha) + \sum_{l=1}^{n-2}\left(\frac{2(1-\beta)}{\left([n-l]_q\mathcal{L}_{n-l} - 1\right)}\right. \\ &\left.\left.\times \prod_{j=1}^{n-l-2}\left(1 + \frac{2(1-\beta)}{\left(\mathcal{L}_{j+1}[j+1]_q - 1\right)}\right)\right)\Upsilon(l)\right\},\end{aligned}$$

where

$$\Upsilon(l) = \min\left(\left|K_l^{-1}(b_2, b_3, ..., b_{l+1})\right|, \left|K_l^{-1}(B_2, B_3, ..., B_{l+1})\right|\right).$$

Proof. Let $\eta \in \mathcal{C}_{\Sigma}^{\lambda,q}(\alpha)$. Then there is a function $\sqcup(\mu) = \mu + \sum_{n=2}^{\infty} b_n \mu^n$. The Faber polynomial expansion of $\frac{D_q(D_q^\lambda \eta(\mu))}{\sqcup(\mu)}$ is

$$\frac{D_q\left(D_q^\lambda \eta(\mu)\right)}{\sqcup(\mu)} = 1 + \sum_{n=2}^{\infty}\left[\begin{array}{c}(\mathcal{L}_n[n]_q a_n - b_n) + \sum_{l=1}^{n-2} K_l^{-1}(b_2, b_3, ...b_{l+1}) \\ (([n]_q - l)\mathcal{L}_n a_{n-l} - b_{n-l})\end{array}\right]\mu^{n-1}. \tag{3.8}$$

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For $F = \eta^{-1}$ and $G = \sqcup^{-1}$, we get

$$\frac{D_q\left(D_q^{\lambda}F(w)\right)}{G(w)} = 1 + \sum_{n=2}^{\infty}\left[\begin{array}{c}(\mathcal{L}_n[n]_q A_n - B_n) + \sum_{l=1}^{n-2} K_l^{-1}(B_2, B_3, ...B_{l+1}) \\ (([n]_q - l)\mathcal{L}_n A_{n-l} - B_{n-l})\end{array}\right] w^{n-1}. \tag{3.9}$$

Since $\Re\frac{D_q\left(D_q^{\lambda}\eta(\mu)\right)}{\sqcup(\mu)} > \alpha$ in Δ, there is a function with a positive real part

$$p(\mu) = 1 + \sum_{n=1}^{\infty} c_n \mu^n \in \mathcal{P},$$

such that

$$\begin{aligned}\frac{D_q\left(D_q^{\lambda}\eta(\mu)\right)}{\sqcup(\mu)} &= 1 + (1-\alpha)p(\mu) \\ &= 1 + (1-\alpha)\sum_{n=1}^{\infty} c_n \mu^n.\end{aligned} \tag{3.10}$$

Similarly $\Re\frac{D_q\left(D_q^{\lambda}F(w)\right)}{G(w)} > \alpha$ in Δ, there is a function with a positive real part

$$q(w) = 1 + \sum_{n=1}^{\infty} d_n w^n \in \mathcal{P},$$

so that

$$\begin{aligned}\frac{D_q\left(D_q^{\lambda}F(w)\right)}{G(w)} &= 1 + (1-\alpha)q(w) \\ &= 1 + (1-\alpha)\sum_{n=1}^{\infty} d_n w^n.\end{aligned} \tag{3.11}$$

Evaluating the coefficients of the equations.(3.8) and (3.10), for any $n \geq 2$, yields

$$\left\{\begin{array}{c}([n]_q\mathcal{L}_n a_n - b_n) + \sum_{l=1}^{n-2} K_l^{-1}(b_2, b_3, ...b_{l+1}) \\ \times(\mathcal{L}_{n-l}([n]_q - l)a_{n-l} - b_{n-l})\end{array}\right\} = (1-\alpha)c_{n-1}, \tag{3.12}$$

Evaluating the coefficients of the equations (3.9) and (3.11), for any $n \geq 2$, yields

$$\left\{\begin{array}{c}([n]_q\mathcal{L}_n A_n - B_n) + \sum_{l=1}^{n-2} K_l^{-1}(B_2, B_3, ...B_{l+1}) \\ \\ \times(\mathcal{L}_{n-l}([n]_q - l)A_{n-l} - B_{n-l})\end{array}\right\} = (1-\alpha)d_{n-1}. \tag{3.13}$$

But under the assumption, $2 \leq i \leq n-1$, and $a_i = 0$; respectively, we find from (3.12) and (3.13) that

$$(\mathcal{L}_{n-l}[n]_q a_n - b_n) - \sum_{l=1}^{n-2} b_{n-l}K_l^{-1}(b_2, b_3, ...b_{l+1}) = (1-\alpha)c_{n-1}, \tag{3.14}$$

$$-(\mathcal{L}_{n-l}[n]_q A_n - B_n) - \sum_{l=1}^{n-2} B_{n-l}K_l^{-1}(B_2, B_3, ...B_{l+1}) = (1-\alpha)d_{n-1}. \tag{3.15}$$

Also the equality $a_i = 0$, $(2 \leq i \leq n-1)$, implies that

$$A_n = -a_n$$

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Thus (3.14) and (3.15) gives

$$\mathcal{L}_n[n]_q a_n = b_n + (1-\alpha)c_{n-1} + \sum_{l=1}^{n-2} b_{n-l} K_l^{-1}(b_2, b_3, ...b_{l+1})$$

and

$$-\mathcal{L}_n[n]_q a_n = B_n + (1-\alpha)d_{n-1} + \sum_{l=1}^{n-2} B_{n-l} K_l^{-1}(B_2, B_3, ...B_{l+1}),$$

respectively. Taking the absolute value (or modulus) of both sides, we get

$$\left|\mathcal{L}_n[n]_q\right||a_n| = |b_n| + (1-\alpha)|c_{n-1}| + \sum_{l=1}^{n-2}\left|b_{n-l} K_l^{-1}(b_2, b_3, ...b_{l+1})\right|$$

and

$$\left|\mathcal{L}_n[n]_q\right||a_n| = |B_n| + (1-\alpha)|d_{n-1}| + \sum_{l=1}^{n-2}\left|B_{n-l} K_l^{-1}(B_2, B_3, ...B_{l+1})\right|.$$

Since $\sqcup, G \in \mathcal{S}^*(q, \beta, \lambda)$, therefore, we use the Theorem 3, to get

$$\begin{aligned}
&\left|[n]_q \mathcal{L}_n\right||a_n| \leq \frac{2(1-\beta)}{\left(\mathcal{L}_n[n]_q - 1\right)} \\
&\times \prod_{j=1}^{n-2}\left(1 + \frac{2(1-\beta)}{\left(\mathcal{L}_{j+1}[j+1]_q - 1\right)}\right) + (1-\alpha)|c_{n-1}| \\
&+ \sum_{l=1}^{n-2}\left(\frac{2(1-\beta)}{\left(\mathcal{L}_{n-l}[n-l]_q - 1\right)} \prod_{j=1}^{n-l-2}\left(1 + \frac{2(1-\beta)}{\left(\mathcal{L}_{j+1}[j+1]_q - 1\right)}\right)\right) \\
&\times \left|K_l^{-1}(b_2, b_3, ...b_{l+1})\right|
\end{aligned}$$

and

$$\begin{aligned}
&\left|\mathcal{L}_n[n]_q\right||a_n| \\
&\leq \frac{2(1-\beta)}{\left(\mathcal{L}_n[n]_q - 1\right)} \prod_{j=1}^{n-2}\left(1 + \frac{2(1-\beta)}{\left(\mathcal{L}_{j+1}[j+1]_q - 1\right)}\right) + (1-\alpha)|d_{n-1}| \\
&+ \sum_{l=1}^{n-2}\left(\frac{2(1-\beta)}{\left(\mathcal{L}_{n-l}[n-l]_q - 1\right)} \prod_{j=1}^{n-l-2}\left(1 + \frac{2(1-\beta)}{\left(\mathcal{L}_{j+1}[j+1]_q - 1\right)}\right)\right) \\
&\times \left|K_l^{-1}(B_2, B_3, ...B_{l+1})\right|.
\end{aligned}$$

Using the Lemma 2.1, we obtain

$$\begin{aligned} &\left|[n]_q\mathcal{L}_n\right||a_n| \\ &\leq \frac{2(1-\beta)}{\left(\mathcal{L}_n[n]_q-1\right)}\prod_{j=1}^{n-2}\left(1+\frac{2(1-\beta)}{\left(\mathcal{L}_{j+1}[j+1]_q-1\right)}\right)+2(1-\alpha) \\ &+\sum_{l=1}^{n-2}\left(\frac{2(1-\beta)}{\left(\mathcal{L}_{n-l}[n-l]_q-1\right)}\prod_{j=1}^{n-l-2}\left(1+\frac{2(1-\beta)}{\left(\mathcal{L}_{j+1}[j+1]_q-1\right)}\right)\right) \\ &\times\left|K_l^{-1}(b_2,b_3,...b_{l+1})\right|. \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} &\left|[n]_q\mathcal{L}_n\right||a_n| \\ &\leq \frac{2(1-\beta)}{\left(\mathcal{L}_n[n]_q-1\right)}\prod_{j=1}^{n-2}\left(1+\frac{2(1-\beta)}{\left(\mathcal{L}_{j+1}[j+1]_q-1\right)}\right)+2(1-\alpha) \\ &+\sum_{l=1}^{n-2}\left(\frac{2(1-\beta)}{\left(\mathcal{L}_{n-l}[n-l]_q-1\right)}\prod_{j=1}^{n-l-2}\left(1+\frac{2(1-\beta)}{\left(\mathcal{L}_{j+1}[j+1]_q-1\right)}\right)\right) \\ &\times\left|K_l^{-1}(B_2,B_3,...B_{l+1})\right|. \end{aligned} \tag{3.17}$$

By comparing (3.16) and (3.17), we obtain $|a_n|$ as asserted in Theorem 3.3. □

For $\lambda=0$, and $q\to 1-$, we obtain a recognized corollary in Theorem 3.3 that was proved in [47].

Corollary 3.4. [47]. *Let $\eta\in\mathfrak{C}_\Sigma(\alpha,\beta)$. If $a_i=0, 2\leq i\leq n-1$, then for $n\geq 3$*

$$\begin{aligned} |a_n| \leq \Bigg\{ &\frac{1}{n!}\prod_{j=0}^{n-2}(j+2(1-\beta)) \\ &+\frac{2(1-\alpha)}{n}+\frac{1}{n}\sum_{l=1}^{n-2}\left(\frac{1}{(n-l-1)!}\prod_{j=0}^{n-l-2}(j+2(1-\beta))\right)\Upsilon(l)\Bigg\}, \end{aligned}$$

where

$$\Upsilon(l)=\min\left(\left|K_l^{-1}(b_2,b_3,...b_{l+1})\right|,\left|K_l^{-1}(B_2,B_3,...B_{l+1})\right|\right).$$

For $\lambda=0$, $\beta=0$ and $q\to 1-$, we obtain a recognized corollary in Theorem 3.3 that was proved in [48].

Corollary 3.5. [48]. *Let $\eta\in\mathfrak{C}_\Sigma(\alpha)$,* if $a_i=0, 2\leq i\leq n-1$. *Then*

$$\begin{aligned} |a_n| &\leq 1+\frac{2(1-\alpha)}{n} \\ &+\frac{1}{n}\sum_{l=1}^{n-2}(n-l)\min\left(\left|K_l^{-1}(b_2,b_3,...b_{l+1})\right|,\left|K_l^{-1}(B_2,B_3,...B_{l+1})\right|\right). \end{aligned}$$

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Theorem 3.6. *Let $\eta \in \mathfrak{C}_{\Sigma}(m,\alpha,q)$ be given by (1.1). Then*

$$|a_2| \leq \min\left\{ \begin{array}{c} \frac{2(1-\alpha)}{[2]_q\mathcal{L}_2} + \frac{2(1-\beta)}{[2]_q\mathcal{L}_2\left(\mathcal{L}_2[2]_q-1\right)}, \\ \sqrt{\frac{2(1-\alpha)}{[3]_q\mathcal{L}_3} + \frac{4q(1-\beta)}{[3]_q\mathcal{L}_3\left(\mathcal{L}_2[2]_q-1\right)}\left(\frac{(1-\alpha)}{1} + \frac{(1-\beta)}{\left(\mathcal{L}_2[2]_q-1\right)}\right)} \end{array} \right\} \tag{3.18}$$

and

$$|a_3| \leq \min\left\{ \begin{array}{c} \frac{(1-\alpha)}{[3]_q\mathcal{L}_3}[1+\max(1,|V_1|)] + \frac{(1-\beta)}{[3]_q\mathcal{L}_3\left([3]_q\mathcal{L}_3-1\right)}\max(1,|V_2|) + \frac{8(1-\alpha)(1-\beta)}{[2]_q^2\mathcal{L}_2^2\left([2]_q\mathcal{L}_2-1\right)}, \\ \frac{2(1-\alpha)}{[3]_q\mathcal{L}_3} + \frac{4q(1-\alpha)(1-\beta)}{[3]_q\mathcal{L}_3\left([2]_q\mathcal{L}_2-1\right)} + \frac{4(q-1)(1-\beta)^2}{[3]_q\mathcal{L}_3\left([2]_q\mathcal{L}_2-1\right)^2} + \frac{2(1-\beta)}{[3]_q\mathcal{L}_3\left([3]_q\mathcal{L}_3-1\right)}\left(1+\frac{2(1-\beta)}{[2]_q\mathcal{L}_2-1}\right). \end{array} \right\} \tag{3.19}$$

Proof. Substituting $n=2$ in equation (3.12) and $n=3$ in equation (3.13), we obtain

$$a_2 = \frac{(1-\alpha)}{[2]_q\mathcal{L}_2}c_1 + \frac{b_2}{[2]_q\mathcal{L}_2}, \tag{3.20}$$

$$a_3 = \frac{(1-\alpha)}{[3]_q\mathcal{L}_3}c_2 + \frac{q(1-\alpha)}{[3]_q\mathcal{L}_3}b_2c_1 + \frac{q-1}{[3]_q\mathcal{L}_3}b_2^2 + \frac{b_3}{[3]_q\mathcal{L}_3}, \tag{3.21}$$

$$-a_2 = \frac{(1-\alpha)}{[2]_q\mathcal{L}_2}d_1 - \frac{b_2}{[2]_q\mathcal{L}_2}, \tag{3.22}$$

$$2a_2^2 - a_3 = \frac{(1-\alpha)}{[3]_q\mathcal{L}_3}d_2 - \frac{q(1-\alpha)}{[3]_q\mathcal{L}_3}b_2d_1 + \frac{q+1}{[3]_q\mathcal{L}_3}b_2^2 - \frac{b_3}{[3]_q\mathcal{L}_3}. \tag{3.23}$$

From (3.20) and (3.22), we find

$$c_1 = -d_1. \tag{3.24}$$

On the other hand, from (3.21) and (3.23), we obtain

$$a_2^2 = \frac{(1-\alpha)}{2[3]_q\mathcal{L}_3}(c_2+d_2) + \frac{q(1-\alpha)}{2[3]_q\mathcal{L}_3}b_2(c_1-d_1) + \frac{q}{[3]_q\mathcal{L}_3}b_2^2. \tag{3.25}$$

Therefore by applying triangle inequality to (3.20) and (3.25), using Lemma 2.1, we get

$$|a_2| \leq \frac{2(1-\alpha)}{[2]_q\mathcal{L}_2} + \frac{|b_2|}{[2]_q\mathcal{L}_2} \tag{3.26}$$

and

$$|a_2|^2 \leq \frac{2(1-\alpha)}{[3]_q\mathcal{L}_3} + \frac{2q(1-\alpha)}{[3]_q\mathcal{L}_3}|b_2| + \frac{q}{[3]_q\mathcal{L}_3}|b_2|^2. \tag{3.27}$$

Using inequality (3.2), we have

$$|a_2| \leq \frac{2(1-\alpha)}{[2]_q\mathcal{L}_2} + \frac{2(1-\beta)}{[2]_q\mathcal{L}_2\left(\mathcal{L}_2[2]_q-1\right)} \tag{3.28}$$

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and

$$|a_2| \leq \sqrt{\frac{2(1-\alpha)}{[3]_q\mathcal{L}_3} + \frac{4q(1-\beta)}{[3]_q\mathcal{L}_3\left(\mathcal{L}_2[2]_q - 1\right)}\left(\frac{(1-\alpha)}{1} + \frac{(1-\beta)}{\left(\mathcal{L}_2[2]_q - 1\right)}\right)}. \tag{3.29}$$

We get the required bound $|a_2|$ as asserted in (3.18). Now we subtract (3.23) from (3.21), we thus get

$$2a_3 - 2a_2^2 = \frac{(1-\alpha)}{[3]_q\mathcal{L}_3}(c_2 - d_2) - \frac{2}{[3]_q\mathcal{L}_3}b_2^2 + \frac{2}{[3]_q\mathcal{L}_3}b_3 + \frac{q(1-\alpha)}{[3]_q\mathcal{L}_3}(c_1 + d_1)b_2.$$

By (3.24), we obtain

$$a_3 = a_2^2 + \frac{(1-\alpha)}{2[3]_q\mathcal{L}_3}(c_2 - d_2) + \frac{1}{[3]_q\mathcal{L}_3}\left(b_3 - b_2^2\right). \tag{3.30}$$

If we set the value of a_2^2 from (3.20) in (3.30), then we have

$$\begin{aligned} a_3 = & \frac{(1-\alpha)}{2[3]_q\mathcal{L}_3}\left(c_2 - \left(\frac{-2[3]_q\mathcal{L}_3(1-\alpha)}{[2]_q^2\mathcal{L}_2^2}\right)c_1^2\right) \\ & + \frac{1}{[3]_q\mathcal{L}_3}\left(b_3 - \left(1 - \frac{[3]_q\mathcal{L}_3}{[2]_q^2\mathcal{L}_2^2}\right)b_2^2\right) \\ & + \frac{2(1-\alpha)}{[2]_q^2\mathcal{L}_2^2}c_1 b_2 - \frac{1-\alpha}{2[3]_q\mathcal{L}_3}d_2. \end{aligned}$$

So using g Lemma 2.2, Theorem 3.2, (2.1) and (3.2), we get

$$\begin{aligned} |a_3| \leq & \frac{(1-\alpha)}{[3]_q\mathcal{L}_3}[1 + \max(1, |V_1|)] + \frac{(1-\beta)}{[3]_q\mathcal{L}_3\left([3]_q\mathcal{L}_3 - 1\right)}\max(1, |V_2|) \\ & + \frac{8(1-\alpha)(1-\beta)}{[2]_q^2\mathcal{L}_2^2\left([2]_q\mathcal{L}_2 - 1\right)}, \end{aligned} \tag{3.31}$$

where

$$\begin{aligned} V_1 &= 1 + \frac{4[3]_q\mathcal{L}_3(1-\alpha)}{[2]_q^2\mathcal{L}_2^2}, \\ V_2 &= 1 + \frac{2(1-\beta)}{[2]_q\mathcal{L}_2 - 1}\left(1 - \left(1 - \frac{[3]_q\mathcal{L}_3}{[2]_q^2\mathcal{L}_2^2}\right)\left(\frac{[3]_q\mathcal{L}_3 - 1}{[2]_q\mathcal{L}_2 - 1}\right)\right). \end{aligned}$$

If we set the value of a_2^2 from (3.20) in (3.30), then we have

$$a_3 = \frac{(1-\alpha)}{[3]_q\mathcal{L}_3}c_2 + \frac{q(1-\alpha)}{[3]_q\mathcal{L}_3}b_2 c_1 + \frac{q-1}{[3]_q\mathcal{L}_3}b_2^2 + \frac{1}{[3]_q\mathcal{L}_3}b_3.$$

Using the Lemma 2.2, and Theorem 3.2, we get

$$\begin{aligned} |a_3| \leq & \frac{2(1-\alpha)}{[3]_q\mathcal{L}_3} + \frac{4q(1-\alpha)(1-\beta)}{[3]_q\mathcal{L}_3\left([2]_q\mathcal{L}_2 - 1\right)} + \frac{4(q-1)(1-\beta)^2}{[3]_q\mathcal{L}_3\left([2]_q\mathcal{L}_2 - 1\right)^2} \\ & + \frac{2(1-\beta)}{[3]_q\mathcal{L}_3\left([3]_q\mathcal{L}_3 - 1\right)}\left(1 + \frac{2(1-\beta)}{[2]_q\mathcal{L}_2 - 1}\right). \end{aligned} \tag{3.32}$$

Hence (3.31) and (3.32) give the required estimate $|a_3|$ as asserted in (3.19). □

As a consequence of Theorem 3.6, we recover the well-known corollary established in [47] when $\lambda = 0$ and q approaches $1-$.

Corollary 3.7. [47]. *Let $\eta \in \mathfrak{C}_\Sigma(\alpha)$ be given by (1.1). Then*

$$|a_2| \leq \min\left\{ \begin{array}{c} 2-\alpha-\beta, \\ \\ \sqrt{\frac{1}{3}\{4(1-\beta)(2-\alpha-\beta)+2(1-\alpha)\}} \end{array} \right\}$$

and

$$|a_3| \leq \frac{1}{3}\left\{ \begin{array}{ll} (3-2\beta)(3-2\alpha-\beta), & 0 \leq \alpha \leq \frac{2+\beta}{3} \\ \\ (1-\alpha)(5-3\alpha)+(1-\beta)(2-\beta)+6(1-\alpha)(1-\beta), & \frac{2+\beta}{3} \leq \alpha < 1. \end{array} \right\}$$

4. Conclusion

Modern research has been significantly influenced by fractional calculus, which has many uses in many areas of science and engineering. It also has implications on many areas of mathematics. For example, it is used in a wide range of complex analysis studies, and it has resulted in some interesting new findings in studies involving analytic functions theory. In this studied, we used the concepts of q-calculus and fractional differential operators on analytic functions theory and defined two new subclasses of close-to-convex functions in the open unit disk. Then, we used the Faber polynomial expansion technique to determine the upper bound of the n^{th} coefficient for functions belonging to these newly defined classes.We also investigated Fekete-Szegö problems and gave some well-known results.

Recent studies, like [49], have shown that the classes could be studied further when the strong starlikeness of order α of the operator (λ, q)-fractional differintegral operator can be taken into consideration.

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